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Homogeneous complex hypersurfaces

By Brian SMYTH*)

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In this paper we classify those complex hypersurfaces of a complex space form which are homogeneous spaces with respect to the induced Kähler structure. This is achieved in Theorem 2 and we may observe that the local classification is the same as that obtained for complex hypersurfaces with parallel Ricci tensor (see Theorem 4 [**4**]). In fact, Theorem 1 is a local result from which both classifications follow immediately. In proving Theorem 1 we need only draw on some of the basic properties of complex hypersurfaces, as developed in [**5**], and the results on the holonomy of complex hypersurfaces of § 2 [**4**].

While Theorem 1 contains the classification theorems of Chern [**1**], Nomizu and Smyth [**4**], and a result of Takahashi [**6**], it should be noted that Kobayashi [**3**] recently obtained a stronger result in the case where the ambient space is complex projective space, to wit: any complete complex hypersurface of constant scalar curvature in $P^{n+1}(C)$ is a projective hyperplane or a quadric.

The questions examined here arose from discussions with Professor K. Nomizu, for whose suggestions I am very grateful.

Let M be a complex n-dimensional manifold and let ϕ be a complex immersion of M in a Kähler manifold $\tilde{M}$ of complex dimension $n+1$ and constant holomorphic sectional curvature $\tilde{c}$. The Riemannian metric g induced on M by ϕ is a Kähler metric and all metric properties of M refer to this metric. M will be called *homogeneous* (Riemannian) if the group of isometries of M acts transitively on M; we remark that it will not be necessary to assume that M is homogeneous Kählerian to obtain Theorem 2. To each field ξ of unit vectors normal to M (with respect to the immersion ϕ) on a neighborhood $U(x_0)$ of a point $x_0 \in M$ there is associated a symmetric tensor field A of type (1, 1) on $U(x_0)$; A^2 is independent of the choice of ξ [**5**]. We shall use the same notation as in [**5**].

LEMMA 1. *The characteristic roots of A^2 are constant in value and multiplicity on M if either*

a) *M is homogeneous*

or b) *the Ricci tensor of M is parallel.*

PROOF. a) The Ricci tensor S of M is given by

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$$S(X, Y)=-2g(A^2X, Y)+(n+1)\frac{\tilde{c}}{2}g(X, Y)$$

(see Corollary 3 [5]) and since $S(f_*X, f_*Y)=S(X, Y)$ for every isometry f of M it follows that $f_*^{-1}A^2f_*=A^2$ on M. Since M is homogeneous this proves the first part of the lemma.

b) In this case A^2 must also be parallel, in view of the above expression for the Ricci tensor, and the result is immediate.

In the sequel, M will be any complex hypersurface of $\tilde{M}$ on which the characteristic roots of A^2 are constant in value and multiplicity. A will be the second fundamental form of M corresponding to the unit normal field ξ on $U(x_0)$ and λ, ν will denote non-negative characteristic roots of A. For each $x\in U(x_0)$ we set

$$T_\lambda^+(x)=\{X\in T_x(M)\,|\,AX=\lambda X\},$$

$$T_\lambda^-(x)=\{X\in T_x(M)\,|\,AX=-\lambda X\},$$

$$T_\lambda(x)=T_\lambda^+(x)\oplus T_\lambda^-(x), \qquad \text{where } \lambda>0$$

and

$$T_0(x)=\{X\in T_x(M)\,|\,AX=0\}.$$

In the lemmas that follow we will examine these distributions. X, Y and Z will denote vector fields on $U(x_0)$. The components of a vector field X in the distribution T_λ^+, T_λ^- $(\lambda\neq 0)$ and T_λ are denoted by X_λ^+, X_λ^- and X_λ respectively.

LEMMA 2. *If $X\in T_\lambda$ and $Y\in T_\nu$ then ∇_XY is orthogonal to T_λ, provided $\lambda\neq\nu$.*

PROOF. We first suppose $\lambda>0$ and $\nu>0$. If $X\in T_\lambda^+$ and $Y\in T_\nu^+$, Codazzi's equation

$$\nabla_X(AY)-\nabla_Y(AX)-A([X, Y])-s(X)JAY+s(Y)JAX=0$$

(see Corollary 3 [5]) becomes

$$\nu\nabla_XY-\lambda\nabla_YX-A(\nabla_XY-\nabla_YX)-\nu s(X)JY+\lambda s(Y)JX=0.$$

Considering the T_λ^+-component of this equation we find $(\nabla_XY)_\lambda^+=0$. Similarly we obtain $(\nabla_XY)_\lambda^+=0$ if $X\in T_\lambda^+$ and $Y\in T_\nu^-$. It follows that $(\nabla_XY)_\lambda^+=0$ when $X\in T_\lambda^+$ and $Y\in T_\nu$ and consequently that $(\nabla_XY)_\lambda^-=-J(\nabla_X(JY))_\lambda^+=0$ also. Thus $(\nabla_XY)_\lambda=0$ when $X\in T_\lambda^+$ and $Y\in T_\nu$. The same reasoning shows that $(\nabla_XY)_\lambda=0$ when $X\in T_\lambda^-$ and $Y\in T_\nu$ and the lemma is proved when $\lambda, \nu>0$. If either λ or ν is zero the same argument works with minor modifications.

LEMMA 3. *If $X, Y\in T_\lambda$ then $\nabla_XY\in T_\lambda$. In particular T_λ is involutive.*

PROOF. It suffices to show that $g(\nabla_XY, Z)=0$ when $Z\in T_\nu$ and $\nu\neq\lambda$. But then $g(Y, Z)=0$ and this implies that

$$g(\nabla_XY, Z)+g(Y, \nabla_XZ)=0.$$

Now $g(Y, \nabla_X Z)=0$ in view of Lemma 2, and so Lemma 3 is proved.

LEMMA 4. *If* $X \in T_\lambda$, $Y \in T_\lambda^+$ *and* $\lambda \neq 0$ *then* $(\nabla_X Y)_\lambda^- = \frac{1}{2} s(X) JY$.

PROOF. Suppose $X \in T_\lambda^-$, then Codazzi's equation becomes

$$\lambda \nabla_X Y + \lambda \nabla_Y X - A[X, Y] - \lambda s(X) JY - \lambda s(Y) JX = 0 .$$

It follows that $(\nabla_X Y)_\lambda^- = \frac{1}{2} s(X) JY$ and $(\nabla_Y X)_\lambda^+ = \frac{1}{2} s(Y) JX$ when $X \in T_\lambda^-$ and $Y \in T_\lambda^+$. Using the fact that J is parallel we can easily infer from the latter equation that $(\nabla_X Y)_\lambda^- = \frac{1}{2} s(X) JY$ when $X \in T_\lambda^+$ and $Y \in T_\lambda^+$, and the lemma is proved.

LEMMA 5. *If* $Z \in T_\lambda^+$ *is a unit vector field and* $\lambda \neq 0$ *then* $K(Z) = ds(JZ, Z)$, *where* $K(Z)$ *is the sectional curvature in* M *of the holomorphic plane generated by* Z.

PROOF. If $X, Y \in T_\lambda$ then, using Lemmas 3 and 4 repeatedly, we obtain

$$\begin{aligned} \nabla_X \nabla_Y Z &= \nabla_X \Big(\frac{1}{2} s(Y) JZ + (\nabla_Y Z)_\lambda^+ \Big) \\ &= \frac{1}{2} X(s(Y)) JZ + \frac{1}{2} s(Y) J \nabla_X Z + \nabla_X ((\nabla_Y Z)_\lambda^+) \\ &= \frac{1}{2} X(s(Y)) JZ - \frac{1}{4} s(X) s(Y) Z + \frac{1}{2} s(Y) J (\nabla_X Z)_\lambda^+ \\ &\quad + \frac{1}{2} s(X) J (\nabla_Y Z)_\lambda^+ + (\nabla_X (\nabla_Y Z)_\lambda^+)_\lambda^+ . \end{aligned}$$

Since Z is a unit vector in T_λ^+ this equation yields

$$g(\nabla_X \nabla_Y Z, JZ) = \frac{1}{2} X(s(Y)) .$$

By virtue of Lemma 3 the distribution T_λ is involutive, so that $[X, Y] \in T_\lambda$ and from Lemma 4 we obtain

$$g(\nabla_{[X,Y]} Z, JZ) = \frac{1}{2} s([X, Y]) .$$

Thus

$$R(X, Y, JZ, Z) = \frac{1}{2} (X(s(Y)) - Y(s(X)) - s([X, Y])) = ds(X, Y) .$$

In particular, $K(Z) = ds(JZ, Z)$.

THEOREM 1. *Let* M *be a complex hypersurface of complex dimension* n *in a space* $\tilde{M}$ *of constant holomorphic curvature* $\tilde{c}$ *and let the characteristic roots of* A^2 *be constant in value and multiplicity on* M. *Then either* M *is of constant holomorphic curvature* $\tilde{c}$ *and totally geodesic in* $\tilde{M}$, *or* M *is locally holomorphically isometric to the complex quadric* Q^n *in* $P^{n+1}(C)$, *the latter case arising* *only when* $\tilde{c}>0$.

PROOF. If $A^2=0$ then M is totally geodesic in $\tilde{M}$ and of constant holomorphic curvature $\tilde{c}$ by virtue of Corollary 2 [**5**]. We may therefore assume that the second fundamental form A on $U(x_0)$ has at least one positive characteristic root λ, say. Let Z be a unit vector field in T_λ^+. In view of Lemma 5 and Corollary 2 [**5**] we have

$$ds(JZ, Z)=K(Z)=-2\lambda^2+\tilde{c}\,. \tag{1}$$

However Corollary 3 [**5**] and Proposition 4 [**5**] yield

$$S(Z, Z)=-2\lambda^2+(n+1)\frac{\tilde{c}}{2},$$

and

$$S(Z, Z)=(n+2)\frac{\tilde{c}}{2}-2ds(JZ, Z),$$

so that

$$ds(JZ, Z)=\lambda^2+\frac{\tilde{c}}{4}\,. \tag{2}$$

It follows from (1) and (2) that $\lambda^2=\frac{\tilde{c}}{4}$, which is impossible if $\tilde{c}<0$ and gives a contradiction when $\tilde{c}=0$.

If $\tilde{c}>0$ then all nonzero characteristic roots of A^2 must equal $\frac{\tilde{c}}{4}$. Assuming A^2 is nonsingular we have $A^2=\frac{\tilde{c}}{4}I$ on M and so M is Einstein (see Corollary 3 [**5**]). However the complex quadric Q^n in $P^{n+1}(C)$ (with the Fubini-Study metric of constant holomorphic curvature $\tilde{c}$) is Einstein but not totally geodesic and therefore $A^2=kI$ on Q^n for some positive constant k; from the previous remark we see that $k=\frac{\tilde{c}}{4}$. The argument used in Proposition 11 [**5**] may now be applied locally to show that M is locally holomorphically isometric to Q^n. We now assume that A^2 is singular. A^2 has then precisely two characteristic roots, 0 and $\frac{\tilde{c}}{4}$. In view of Lemmas 2 and 3 the distributions T_0 and $T_{\sqrt{\frac{\tilde{c}}{4}}}$ are parallel, so that M is locally reducible. It follows from Theorem 2 [**4**] that n must equal 2 and that M is locally holomorphically isometric to Q^2. But then $A^2=\frac{\tilde{c}}{4}I$ on M, which contradicts the assumption that A^2 is singular.

If M is complete and locally holomorphically isometric to Q^n, its Ricci tensor is positive definite so that M is compact. By Kobayashi's Theorem [**2**] M is also simply connected and is therefore holomorphically isometric to Q^n. Combining Lemma 1 with Theorem 1 we obtain

THEOREM 2. *Let M be a complex hypersurface in a space $\tilde{M}$ of constant* *holomorphic sectional curvature $\tilde{c}$. If M is homogeneous (resp. if M has parallel Ricci tensor) then either M is of constant holomorphic sectional curvature $\tilde{c}$ and totally geodesic in $\tilde{M}$, or M is globally (resp. locally) holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$, the latter case arising only when $\tilde{c}>0$.*

Since M is complete if it is homogeneous we have the following analogue of Theorem 5 [4].

THEOREM 3. i) *$P^n(C)$ and the complex quadric Q^n are the only homogeneous complex hypersurfaces in $P^{n+1}(C)$.*

ii) *D^n (resp. C^n) is the only homogeneous complex hypersurface in D^{n+1} (resp. C^{n+1}).*

University of Notre Dame

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