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# The prolongation of the holonomy group

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In a series of recent papers [2], Kobayashi and Yano [2; I] have defined a mapping from the tensor algebra of a manifold $M$ into the tensor algebra of its tangent bundle $T(M)$. This mapping they called the "complete lift". They have also defined the complete lift of a connection on $M$ to a connection on $T(M)$. In [2; III], they have shown that the holonomy group of the connection on $T(M)$ is the tangent group of the holonomy group of the connection on $M$. They mention that it should be possible to prove this in the spirit of [2; I]. The purpose of this paper is to compare the infinitesimal holonomy groups of $M$ and $T(M)$ (see Nijenhuis [3] for definition and properties).

We will suppose that the manifold $M$ is connected and analytic and also that the connection is analytic. In this case, Nijenhuis [3] has shown that the dimension of the infinitesimal holonomy group is constant on $M$ and thus the infinitesimal holonomy group is equal to the restricted holonomy group of $M$. The main theorem of this paper then tells us that if the dimension of the Lie algebra of the holonomy group of $M$ is $r$, then the dimension of the Lie algebra of the holonomy group of $T(M)$ is $2r$ and furthermore, it has an abelian ideal of dimension $r$. The result of [2; III] for $M$ can easily be seen by the constructions contained here.

## § 1. Preliminaries.

Let $M$ be a connected, analytic manifold of dimension $n$ and $\mathfrak{X}(M)$ the module of vector fields on $M$. The connection will be denoted by $\nabla$ and the covariant derivative operator by $\nabla_X (X \in \mathfrak{X}(M))$. Let $R$ denote the curvature tensor of $\nabla$. $\nabla$ is assumed to be analytic. If $(x^i)$ is a local coordinate system on $M$, let the corresponding coordinate system on $T(M)$ (the tangent bundle of $M$) be denoted by $(x^i, y^i)$. Here we have $i=1, \cdots, n$.

Let $\pi : T(M) \to M$ be the natural projection map. Then, following Kobayashi and Yano [2], we define two mappings from the tensor algebra of $M$ into the tensor algebra of $T(M)$. The first is called the "vertical lift", and is characterized by

$1_v$) $(S\otimes T)^v = S^v\otimes T^v$, where $S$ and $T$ are tensor fields on $M$ and $S^v$ and $T^v$ their images under the mapping,

$2_v$) if $\varphi$ is a function on $M$,

$$\varphi^v = \varphi\circ\pi\,,$$

$3_v$) if $X = X^k\dfrac{\partial}{\partial x^k}$, then

$$X^v = X^k\frac{\partial}{\partial y^k}\,,$$

$4_v$) if $\omega = \omega_k dx^k$, then

$$\omega^v = \omega_k dx^k\,.$$

The second mapping is called the "complete lift" and it is characterized by

$1_c$) $$(S\otimes T)^c = S^c\otimes T^v + S^v\otimes T^c\,,$$

$2_c$) if $\varphi$ is a function on $M$,

$$\varphi^c = y^i\frac{\partial\varphi}{\partial x^i}\,,$$

$3_c$) if $X = X^k\dfrac{\partial}{\partial x^k}$, then

$$X^c = X^k\frac{\partial}{\partial x^k} + y^i\frac{\partial X^k}{\partial x^i}\frac{\partial}{\partial y^k}\,,$$

$4_c$) if $\omega = \omega_k dx^k$, then

$$\omega^c = y^i\frac{\partial\omega_k}{\partial x^i}dx^k + \omega_k dy^k\,.$$

It may then be shown that a unique connection $\nabla^c$ on $T(M)$ is determined by defining $\nabla^c_{X^c}Y^c = (\nabla_X Y)^c$ for $X, Y\in\mathfrak{X}(M)$. The curvature tensor of $\nabla^c$ is then found to be $R^c$. Also, for any tensor $T$ on $M$, $\nabla^c T^c = (\nabla T)^c$.

Let $p$ be a fixed point of $M$ and let $W_0 = \{R(X, Y)(p)\,|\,X, Y\in\mathfrak{X}(M)\}$. Here $R(X, Y)(p)$ denotes $R(X, Y)$ evaluated at $p$. Similarly, let

$$W_\infty = \{(\nabla_{X_\alpha}\cdots\nabla_{X_1}R)(X, Y)(p)\,|\,\alpha = 1, 2, \cdots, X_1, X_2, \cdots, X_\alpha\in\mathfrak{X}(M)\}\,.$$

Then, if we let $\mathfrak{G}$ be the linear span of $W_0 + W_\infty$, $\mathfrak{G}$ is a Lie algebra (under the usual bracket product). $\mathfrak{G}$ is the Lie algebra of the infinitesimal holonomy group of $M$ at $p$ (see Nijenhuis [3]). Nijenhuis has proved that if $M$ and the connection are analytic, then the dimension of $\mathfrak{G}$ is constant on $M$. It can be shown in this case that $R$ can be locally decomposed as $R = L_a\otimes M^a$, $a = 1, 2, \cdots, r$ $(=\dim\mathfrak{G})$, where the $L_a(p)$ form a basis of $\mathfrak{G}$. We can also show that $(\nabla_{X_\alpha}\cdots\nabla_{X_1}R) = L_a\otimes N^a$(i.e. $N^a = N^a(X_1, \cdots, X_\alpha)$).

## § 2. Main Theorem.

Henceforth, by holonomy group we mean the infinitesimal holonomy group at a fixed point of $M$.

THEOREM. *If the dimension of the holonomy group $G$ of $M$ is $r$, then the dimension of the holonomy group $G^c$ of $T(M)$ is $2r$. Moreover, the Lie algebra $\mathfrak{G}^c$ of $G^c$ has an abelian ideal of dimension $r$.*

Let $\{L_a | a=1, \cdots, r\}$ be a basis of $\mathfrak{G}$. If we could show that $\{L_a^v, L_a^c\}$ is a basis for $\mathfrak{G}^c$ and that $[L_a^v, L_b^v]=0$ and $[L_a^v, L_b^c]=0$ for all $a$ and $b$, the proof of the theorem would be finished. Instead of doing this for a general connection, we will consider the special case where the curvature tensor is recurrent (i. e. there is a 1-form $\eta$ such that $\nabla R=\eta \otimes R$) and merely note that the proof will carry over to the general case.

In order to state the following proposition, we need a definition due to Hlavaty [**1**].

DEFINITION. The holonomy group is called *perfect* if $\mathfrak{G}=W_0$.

PROPOSITION. *Suppose the curvature tensor of $M$ is recurrent. Then $\mathfrak{G}^c$ is perfect and satisfies the conclusions of the theorem.*

PROOF. It is clear that since $R$ is recurrent it is perfect and we can locally decompose $R$ as $R=L_a \otimes M^a$ $(a=1, \cdots, r)$, where $\{L_a\}$ is a basis for $\mathfrak{G}$ $(=W_0)$ and the $M^a$ are linearly independent. By $1_c$ we have that $R^c=L_a^c \otimes M^{av}+L_a^v \otimes M^{ac}$. If we let the components of $M^a$ be denoted by $M_{ij}^a$, then the components of $M^{ac}$ are $\begin{pmatrix} \frac{\partial M_{ij}^a}{\partial x^k} y^k & M_{ij}^a \\ M_{ij}^a & 0 \end{pmatrix}$ and those of $M^{av}$ are $\begin{pmatrix} M_{ij}^a & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to see that $\{M^{av}, M^{bc}\}$ is a linearly independent set of tensors and that $W_0^c=sp\{L_a^v(P), L_a^c(P)\}$. Here, $W_0^c$ is formed from $R^c$ in the same manner as $W_0$ was formed from $R$. $P$ is a point of $T(M)$. Similarly, if $L_{aj}^i$ are the components of $L_a$, then we have that $L_a^v$ : $\begin{pmatrix} 0 & 0 \\ L_{aj}^i & 0 \end{pmatrix}$ and $L_a^c$ : $\begin{pmatrix} L_{aj}^i & 0 \\ y^k \frac{\partial L_{aj}^i}{\partial x^k} & L_{aj}^i \end{pmatrix}$, and thus $\{L_a^v(P), L_b^c(P)\}$ form a basis for $W_0^c$.

Now suppose that $\nabla R=\eta \otimes R$. Then, we see that $\nabla^c R^c=(\nabla R)^c=(\eta \otimes R)^c =\eta^c \otimes R^v+\eta^v \otimes R^c$. Therefore, if $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(T(M))$, then

$$\begin{aligned}(\nabla_{\tilde{X}}^c R^c)(\tilde{Y}, \tilde{Z}) &= \eta^c(\tilde{X}) L_a^v M^{av}(\tilde{Y}, \tilde{Z}) \\ &\quad +\eta^v(\tilde{X}) L_a^c M^{av}(\tilde{Y}, \tilde{Z})+\eta^v(\tilde{X}) L_a^v M^{ac}(\tilde{Y}, \tilde{Z}),\end{aligned}$$

which, when evaluated at $P$, is in $W_0^c$. Continuing, we obtain that $W_\infty^c \subseteqq W_0^c$. This shows that $\mathfrak{G}^c=W_0^c$ and $\dim W_0^c=2r$.

The components of $[L_a(p), L_b(p)]$ are given by $L_{ai}^k(p) L_{bk}^j(p)-L_{bi}^k(p) L_{ak}^j(p)$.

Suppose that $[L_a(p), L_b(p)] = C^d_{ab}(p) L_d(p)$ (since the $L_a(p)$'s are a basis of the Lie algebra $\mathfrak{G}$). This formula is valid in a neighborhood of $p$. A simple calculation making use of the components of the $L^v_a$'s and $L^c_a$'s then shows that $[L^v_a, L^v_b] = 0$ for all $a$ and $b$. Also, we find that $[L^v_a, L^c_b] = C^d_{ab} L^v_d$ for all $a$ and $b$. This shows that the linear span of the $L^v_a(P)$'s form an abelian ideal of $\mathfrak{G}^c$.

We can easily go a step further and show that $[L^c_a, L^c_b] = C^d_{ab} L^c_d + (C^d_{ab})^c L^v_d$. Therefore we have computed all of the structure constants for $\mathfrak{G}^c$. The above procedure is extended to a general connection by noting that $R$ can be locally decomposed as $R = L_a \otimes M^a$, where the non-zero $M^a$'s are linearly independent. We then pick a similar decomposition for the covariant derivatives of $R$.

## § 3. Concluding remarks.

The results in this paper remain true if we replace the analyticity requirement by $C^\infty$, understand that we mean infinitesimal holonomy groups and assume that the dimension of the holonomy group is constant.

Y. C. Wong [4] has given a characterization of recurrent tensors. He assumes that the manifold and tensors are $C^\infty$. Using this characterization and the proof presented here, it can be easily shown that the proposition above is true on a $C^\infty$ manifold. Likewise the result of [2; III] can be seen by this method for this special case.

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