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An investigation on degrees of unsolvability

Dedicated to Professor Motokiti Kondô on his sixtieth birthday anniversary

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§ 0. Introduction.

By *degree*, we mean the degree of recursive unsolvability as defined by S. C. Kleene and E. L. Post in [2]. For notations not explained here, see [1], [2] and [5].

For each degree $\boldsymbol{d}$, let $R_{\boldsymbol{d}}$ denote the set of all degrees greater than or equal to $\boldsymbol{d}$, recursively enumerable in $\boldsymbol{d}$ and less than or equal to $\boldsymbol{d}'$ (the completion of $\boldsymbol{d}$).

R. M. Friedberg has shown that degree $\boldsymbol{d}'$ does not have a unique preimage in $R_{\boldsymbol{d}}$. G. E. Sacks [4] proved that if $\boldsymbol{a} \in R_{\boldsymbol{b}'}$, then there exists a degree $\boldsymbol{c}$ such that $\boldsymbol{c} \in R_{\boldsymbol{b}}$ and $\boldsymbol{c}' = \boldsymbol{a}$.

The main result of the present paper is that if $\boldsymbol{a} \in R_{\boldsymbol{b}'}$, then for any positive integer n, there exist independent degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ such that $\boldsymbol{c}_i \in R_{\boldsymbol{b}}$ and $\boldsymbol{c}_i' = \boldsymbol{a}$ for $i = 1, 2, \cdots, n$. Thus the degrees which lie between $\boldsymbol{b}'$ and $\boldsymbol{b}''$ and are recursively enumerable in $\boldsymbol{b}'$ can be viewed as the completions of the independent degrees which lie between $\boldsymbol{b}$ and $\boldsymbol{b}'$ and are recursively enumerable in $\boldsymbol{b}$. This shall be proved as a corollary of the following 'main theorem'. The methods used here are those developed in [2], [3] and [4].

We shall denote by $\boldsymbol{a} \upharpoonleft \boldsymbol{b}$ the relation between degrees $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a}$ is recursively enumerable in $\boldsymbol{b}$.

MAIN THEOREM. *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be degrees such that:*

(I) $\boldsymbol{a} \nleqq \boldsymbol{b}$

(II) $\boldsymbol{a} \leqq \boldsymbol{b}' \leqq \boldsymbol{c}$

(III) $\boldsymbol{c} \upharpoonleft \boldsymbol{b}'$

Then for any positive integer n, there exist degrees $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ such that:

(i) $\boldsymbol{b} \leqq \boldsymbol{d}_i$ *for* $i = 0, 1, \cdots, n-1$,

(ii) $\boldsymbol{d}_i \upharpoonleft \boldsymbol{b}$ *for* $i = 0, 1, \cdots, n-1$,

(iii) $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ *are independent,*

(iv) $\boldsymbol{a} \nleqq \boldsymbol{d}_i$ *for* $i = 0, 1, \cdots, n-1$,

(v) $\boldsymbol{d}_i' = \boldsymbol{c}$ *for* $i = 0, 1, \cdots, n-1$.

§ 1. Definitions.

Let A, B, C and B' be the sets of degrees $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{b}'$ satisfying the assumptions (I), (II) and (III) of the above.

Let $\alpha_0(x)$, $\beta(x)$, $\gamma(x)$ and $\beta'(x)$ be the representing function of A, B, C and B', respectively.

We put

$$\alpha(x) = \alpha_0(x) + 1 .$$

Let $\phi(x)$ be a function recursive in $\beta'(x)$ which enumerates C, and $\psi(x)$ be a function recursive in $\beta(x)$ which enumerates B'.

In the following lines, we shall define the functions $\beta^{\#}(x, s)$, $\phi^{\#}(x, s)$ and $\alpha^{\natural}(x, s)$. First we set

$$\beta^{\#}(x, s) = \begin{cases} 0 & \text{if } (Ek)_{k<s}(\psi(k) = x) , \\ 1 & \text{otherwise} . \end{cases}$$

It is clear that $\beta^{\#}(x, s)$ is a function recursive in $\beta(x)$, and that for each x, $\lim_s \beta^{\#}(x, s)$ exists and

$$\lim_s \beta^{\#}(x, s) = \beta'(x) .$$

By the definition of $\phi(x)$ and $\alpha(x)$, there exist Gödel numbers e_1 and e_2 of ϕ and α from β' respectively:

$$\phi(x) = \{e_1\}\beta'(x) ,$$

$$\alpha(x) = \{e_2\}\beta'(x) .$$

By using e_1 and e_2, we set

$$\phi^{\#}(x, s) = \begin{cases} U(\mu y T_1^1(\tilde{\beta}^{\#}(y ; s), e_1, x, y)) & \\ \qquad \text{if } (Ey)_{y<s}(T_1^1(\tilde{\beta}^{\#}(y ; s), e_1, x, y)) , & \\ s+1 \quad \text{otherwise} . & \end{cases}$$

$$\alpha^{\natural}(x, s) = \begin{cases} U(\mu y T_1^1(\tilde{\beta}^{\#}(y ; s), e_2, x, y)) & \\ \qquad \text{if } (Ey)_{y<s}(T_1^1(\tilde{\beta}^{\#}(y ; s), e_2, x, y)) , & \\ 2 \quad \text{otherwise} . & \end{cases}$$

Clearly, $\phi^{\#}(x, s)$ and $\alpha^{\natural}(x, s)$ are recursive in $\beta(x)$, and $\lim_s \phi^{\#}(x, s)$ and $\lim_s \alpha^{\natural}(x, s)$ exist and equal to $\phi(x)$ and $\alpha(x)$, respectively.

By induction on s, we shall define the functions $\tau(x, s)$, $\kappa(x, s)$, $\eta(x, e, i, s)$, $\nu(x, e, i, s)$, $\xi(e, i, s)$, $\theta(z, e, m, i, s)$ and $\delta(x, i, s)$, and furthermore the predicate $\Gamma(z, e, m, i, s)$ simultaneously for all $i < n$, z, x, e and m. By the definitions, it is clear that these are all recursive in $\beta(x)$.

Stage $s=0$. We set as follows:

$$\tau(x, 0)=\eta(x, e, i, 0)=\nu(x, e, i, 0)=0\,.$$

$$\kappa(x, 0)=1\,.$$

$$\xi(e, i, 0)=e+1\,.$$

$$\theta(z, e, m, i, 0)=2^z\cdot 5\,.$$

$$\delta(x, i, 0)=\begin{cases}\beta(m) & \text{if } x=2\cdot 3\cdot 5^{m+1}\,,\\ 1 & \text{otherwise}\,.\end{cases}$$

$$\Gamma(z, e, m, i, 0)\equiv 0=1\,.$$

Stage $s>0$. We set as follows:

$$\eta(x, e, i, s)=\begin{cases}\mu y T_1^1(\tilde{\delta}(y\,;i, s-1), e, x, y) & \\ \qquad \text{if } x\geqq e \;\&\; (Ey)_{y<s}T_1^1(\tilde{\delta}(y\,;i, s-1), e, x, y)\,, & \\ 0 \qquad \text{otherwise}\,. & \end{cases}$$

We define $\xi(e, i, s)$ by three mutually exclusive cases.

Case 1:

$$\eta(e, e, i, s)=0\,.$$

We set

$$\xi(e, i, s)=e+1\,.$$

Case 2:

$$\eta(e, e, i, s)>0 \quad \& \quad (Ex)[e<x<\xi(e, i, s-1)$$
$$\& \quad \eta(x, e, i, s)\neq\eta(x, e, i, s-1) \quad \& \quad \alpha^{\natural}(x, s)\neq U(\eta(x, e, i, s))]\,.$$

We set

$$\xi(e, i, s)=\mu x_{e<x<\xi(e,i,s-1)}[\eta(x, e, i, s)\neq\eta(x, e, i, s-1) \quad \&$$
$$\alpha^{\natural}(x, s)\neq U(\eta(x, e, i, s))]\,.$$

Case 3: Otherwise. We set

$$\xi(e, i, s)=\mu x[\xi(e, i, s-1)\leqq x<2\cdot\xi(e, i, s-1)+s \quad \&$$
$$(Et)[e<t\leqq x \quad \& \quad \alpha^{\natural}(t, s)\neq U(\eta(t, e, i, s))]]\,.$$

We now define $\tau(x, s)$.

$$\tau(x, s)=\mu r_{r<s}(\phi^{\#}(r, s)=x)\,.$$

We set

$$\kappa(x, s)=\kappa(x, s-1)+sg(|\tau(x, s)-\tau(x, s-1)|)\,.$$

$$\nu(x, e, i, s)=\begin{cases} 0 & \text{if } (Ek)_{k<s}(Et)_{t<s}(Er)_{r<s}(Eu)_{u<s}[(k<e\leqq t\leqq x \vee k<u) \\ & \quad \& \ (\theta(u, k, r, i, s-1)<\eta(t, e, i, s))], \\ 1 & \text{otherwise}. \end{cases}$$

We shall abbreviate $T_1^{1\cdots 1}(\tilde{\delta}(y;0, s-1), \tilde{\delta}(y, 1, s-1), \cdots, \tilde{\delta}(y;i-1, s-1), \tilde{\delta}(y;i+1, s-1), \cdots, \tilde{\delta}(y;n-1, s-1), e, x, y)$ as $T_1^{1\cdots 1}(\langle\tilde{\delta}(y;\hat{i}, s-1)\rangle, e, x, y)$. (In the following, we shall use the notation $\langle A(\hat{j})\rangle$ instead of the sequence $A(0), A(1), \cdots, A(j-1), A(j+1), \cdots, A(n-1)$, as the above, where $A(x)$ is an expression containing the letter x.)

$$\begin{aligned} \Gamma(z, e, m, i, s)\equiv (Ey)_{y<s}[&T_1^{1\cdots 1}(\langle\delta(y;\hat{i}, s-1)\rangle, z, \theta(z, e, m, i, s-1), y) \\ &\& \ U(y)\neq 0] \ \& \ \delta(\theta(z, e, m, i, s-1), i, s-1)=1 \\ &\& \ m<\kappa(e, s) \ \& \ (e^*)(r)[[(e^*\leqq e \ \& \ z\leqq e \\ &\& \ e^*\leqq r<\xi(e^*, i, s))\rightarrow\theta(z, e, m, i, s-1)\geqq\eta(r, e^*, i, s)] \\ &\& \ (Es')_{s'\leqq s}[(e^*\leqq e \ \& \ z>e \ \& \ e^*\leqq r<\xi(e^*, i, s'))\rightarrow \\ &(\nu(r, e^*, i, s')=0 \vee \theta(z, e, m, i, s'-1)\geqq\eta(r, e^*, i, s'))]]. \end{aligned}$$

We define $\theta(z, e, m, i, s)$ and $\delta(x, i, s)$ by n cases, corresponding to the values of remainder $rm(s, n)$.

Case l: $rm(s, n)=l$.

$$\theta(z, e, m, i, s)=\begin{cases} 2^z\cdot 3^e\cdot 5^m\cdot 7^s & \text{if } [z\geqq lh(s) \ \& \ e<s \ \& \ m<s \\ & \quad \& \ i>l \ \& \ (Ee')_{e'<s}(Em')_{m'<s}[\Gamma(lh(s), e', m', l, s)]] \\ & \quad \vee [z>lh(s) \ \& \ e<s \ \& \ m<s \ \& \ i<l \\ & \quad \& \ (Ee')_{e'<s}(Em')_{m'<s}[\Gamma(lh(s), e', m', l, s)]], \\ \theta(z, e, m, i, s-1) & \text{otherwise}. \end{cases}$$

$$\delta(x, i, s)=\begin{cases} 0 & \text{if } x=\theta(lh(s), e, m, l, s-1) \ \& \ i=l \\ & \quad \& \ \Gamma(lh(s), e, m, l, s), \\ \beta(m) & \text{if } x=2\cdot 3\cdot 5^{m+1}, \\ \delta(x, i, s-1) & \text{otherwise}. \end{cases}$$

This completes the definitions of all auxiliary functions and predicate.

By the definition of $\delta(x, i, s)$, $\lim_s \delta(x, i, s)$ exists and is less than 2 for each $i<n$ and each x. For each $i<n$, and each x, let $\delta(x, i)$ be $\lim_s \delta(x, i, s)$ and D_i be the sets whose representing functions are $\delta(x, i)$. Let $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ be the degrees of $D_0, D_1, \cdots, D_{n-1}$ respectively.

We shall show that the degrees $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-2}$ and $\boldsymbol{d}_{n-1}$ satisfy the conclusions (i), (ii), (iii), (iv) and (v) of the main theorem.

§ 2. Plan of the proof.

By the definition of $\delta(x, i, s)$, $\delta(2\cdot 3\cdot 5^{m+1}, i, s)=\beta(m)$ for each m, i, s. Thus we have the conclusion (i) of the main theorem:

$$\boldsymbol{b}\leqq \boldsymbol{d}_i \qquad \text{for } i=0, 1, \cdots, n-1\,.$$

It follows from the definition of $\delta(x, i)$ that $\delta(x, i)=0$ or 1 according as $(Es)\ [\delta(x, i, s)=0]$ or not. And $\delta(x, i, s)$ is recursive in $\beta(x)$ for each $i<n$. Thus $\delta(x, i)$ is recursively enumerable in $\beta(x)$, that is

$$\boldsymbol{d}_i \restriction \boldsymbol{b} \qquad \text{for } i=0, 1, \cdots, n-1\,.$$

This is the conclusion (ii). In order to prove the conclusion (iii), we shall show Lemma 1, Lemma 2 and Lemma 3.

LEMMA 1. *For any given z, the set $\{\theta(z, e, m, i, s)\,|\,e\geqq 0\ \&\ m\geqq 0\ \&\ i<n\ \&\ s\geqq 0\}$ is finite.*

PROOF. We use the induction on z.

Suppose that the lemma holds for $z<\bar{z}$ and fails for $\bar{z}$. That is, $\{\theta(\bar{z}, e, m, i, s)\,|\,e\geqq 0\ \&\ m\geqq 0\ \&\ i<n\ \&\ s\geqq 0\}$ is infinite. We set

$$\bar{i}=\mu i\ [\{\theta(\bar{z}, e, m, i, s)\,|\,e\geqq 0\ \&\ m\geqq 0\ \&\ s\geqq 0\} \text{ is infinite}]\,.$$

From the definition of $\theta(z, e, m, i, s)$, there exists l_0 such that $\theta(\bar{z}, e, m, \bar{i}, s)$ changes its value infinitely many times in case l_0, i. e.

$$(1)\qquad \theta(\bar{z}, e, m, \bar{i}, s)=2^{\bar{z}}\cdot 3^e\cdot 5^m\cdot 7^s \qquad \text{for } [\bar{z}\geqq lh(s)\ \&\ e<s$$
$$\&\ m<s\ \&\ \bar{i}>l_0\ \&\ (Ee')_{e'<s}(Em')_{m'<s}[\Gamma(lh(s), e', m', l_0, s)]]$$
$$\vee\,[\bar{z}>lh(s)\ \&\ e<s\ \&\ m<s$$
$$\&\ \bar{i}<l_0\ \&\ (Ee')_{e'<s}(Em')_{m'<s}[\Gamma(lh(s), e', m', l_0, s)]]$$

occurs for infinitely many s.

Thus we have

$$\Gamma(lh(s), e', m', l_0, s)\quad \&\quad lh(s)\leqq \bar{z}$$

for infinitely many s. Then,

$$\delta(\theta(lh(s), e', m', l_0, s-1), l_0, s)=0$$

for s satisfying (1). And this requires infinitely many changes of $\theta(lh(s), e', m', l_0, s-1)$.

By the hypothesis of our induction, it is not the case $lh(s)<\bar{z}$. But, $lh(s)=\bar{z}$ is contrary to the definition of $\bar{i}$, since $\bar{i}>l_0$.

We set

$$x(z, e, m, i)=\max\,\{\theta(z, e, m, i, s)\,|\,s\geqq 0\}\,.$$

LEMMA 2.

$$(z)(i)_{i<n}[\delta(x(z, 0, 0, i), i)=0$$
$$\rightarrow (Ey)(T_1^{1\cdots 1}(\langle \tilde{\delta}(y\,;\hat{i})\rangle, z, x(z, 0, 0, i), y) \ \&\ U(y)\neq 0)].$$

PROOF. Fix $z, i<n$. From the assumption of the lemma, we have

$$(Es)\ [\delta(\theta(z, 0, 0, i, s-1), i, s)=0$$
$$\&\quad z=lh(s)\quad \&\quad \theta(z, 0, 0, i, s-1)=x(z, 0, 0, i)].$$

Let

$$s_0=\mu s[\delta(\theta(z, 0, 0, i, s-1), i, s)=0$$
$$\&\quad z=lh(s)\quad \&\quad \theta(z, 0, 0, i, s-1)=x(z, 0, 0, i)].$$

By the definition of $\delta(x, i, s)$, $\Gamma(z, 0, 0, i, s_0)$ holds, where $z=lh(s_0)$. Thus we have a $y<s_0$ such that

$$[T_1^{1\cdots 1}(\langle \tilde{\delta}(y\,;\hat{i}, s-1)\rangle, z, x(z, 0, 0, i), y)\quad \&\quad U(y)\neq 0].$$

Then, the proof will be complete, if we can show

(1) $$(x)_{x<y}(j)_{j\neq i\ \&\ j<n}(s)_{s>s_0-1}[\delta(x, j, s)=\delta(x, j, s_0-1)].$$

From $\Gamma(lh(s_0), 0, 0, i, s_0)$, we know that

(2) $$(e')_{e'<s_0}(m')_{m'<s_0}(z')_{z'\geqq lh(s_0)}(k)_{k>i}[\theta(z', e', m', k, s_0)=2^{z'}\cdot 3^{e'}\cdot 5^{m'}\cdot 7^{s_0}]$$

and

(3) $$(e')_{e'<s_0}(m')_{m'<s_0}(z')_{z'>lh(s_0)}(k)_{k<i}[\theta(z', e', m', k, s_0)=2^{z'}\cdot 3^{e'}\cdot 5^{m'}\cdot 7^{s_0}].$$

We shall prove (1) by means of a reductio ad absurdum argument. That is, we shall start from the hypothesis, there exist x^*, j^* and s^* such that

(*) $$x^*<y\quad \&\quad j^*\neq i\quad \&\quad j^*<n\quad \&\quad s^*>s_0-1\quad \&$$
$$\delta(x^*, j^*, s^*-1)=1\quad \&\quad \delta(x^*, j^*, s^*)=0.$$

By the definition of $\delta(x, i, s)$ and by (*), we obtain for some z'', e'' and m'',

$$j^*=rm(s^*, n),$$
$$x^*=\theta(z'', e'', m'', j^*, s^*-1),$$
$$x^*<y<s_0<s^*\ \text{[1]},$$

and then by (*),

$$\Gamma(lh(s^*), e'', m'', j^*, s^*)\quad \text{and}\quad lh(s^*)=z''.$$

Thus we have

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1) By the definitions of the number s_0 and of the function δ, we have $rm(s_0, n)=i$. This implies $s_0<s^*$, since $i\neq j^*$ and $s_0\leqq s^*$.

(4) $$(z''')_{z'''\geqq z''}(e''')_{e'''<s^*}(m''')_{m'''<s^*}(k')_{k'>j^*}[\theta(z''', e''', m''', k', s^*) = 2^{z'''}\cdot 3^{e'''}\cdot 5^{m'''}\cdot 7^{s^*}].$$

(5) $$(z''')_{z'''>z''}(e''')_{e'''<s^*}(m''')_{m'''<s^*}(k')_{k'<j^*}[\theta(z''', e''', m''', k', s^*) = 2^{z'''}\cdot 3^{e'''}\cdot 5^{m'''}\cdot 7^{s^*}].$$

Since $s^* > s_0$, by the definitions of θ and x, we have

$$\theta(lh(s_0), 0, 0, i, s^*) = x(lh(s_0), 0, 0, i) = x(z, 0, 0, i).$$

By (4) and (5), this means that

$$i > j^* \to z = lh(s_0) < z'' = lh(s^*)$$

and

$$i < j^* \to z = lh(s_0) \leqq z'' = lh(s^*).$$

Then, we obtain by (3)

$$i > j^* \to \theta(lh(s^*), e'', m'', j^*, s_0) = 2^{lh(s^*)}\cdot 3^{e''}\cdot 5^{m''}\cdot 7^{s_0}$$

and by (2)

$$i < j^* \to \theta(lh(s^*), e'', m'', j^*, s_0) = 2^{lh(s^*)}\cdot 3^{e''}\cdot 5^{m''}\cdot 7^{s_0}.$$

Consequently $\theta(lh(s^*), e'', m'', j^*, s_0) > s_0$. But this is absurd, since

$$s_0 > x^* = \theta(lh(s^*), e'', m'', j^*, s^*-1) \geqq \theta(lh(s^*), e'', m'', j^*, s_0) > s_0.$$

Thus, we have shown that (1) holds.

LEMMA 3.

$$(z)(i)_{i<n}[(Ey)T_1^{1,1}(\langle\tilde{\delta}(y\,;\hat{i})\rangle, z, x(z, 0, 0, i), y) \ \&\ U(y) \neq 0) \to \delta(x(z, 0, 0, i), i) = 0].$$

PROOF. Fix z, i. By the assumption of this lemma, we have

$$T_1^{1,1}(\langle\tilde{\delta}(y\,;\hat{i})\rangle, z, x(z, 0, 0, i), y) \quad \& \quad U(y) \neq 0$$

for some y. We set

$$s_0 = \mu s(j)_{j\neq i\ \&\ j<n}(x)_{x<y}[\delta(x, j, s) = \delta(x, j)].$$

Let

$$s_1 = \mu s[\theta(z, 0, 0, i, s) = x(z, 0, 0, i)],$$

and

$$s^* = \mu s[s \geqq s_0 \quad \& \quad s \geqq s_1 \quad \& \quad lh(s) = z].$$

Then we have

(1) $$T_1^{1,1}(\langle\tilde{\delta}(y\,;\hat{i}, s^*-1)\rangle, lh(s^*), \theta(lh(s^*), 0, 0, i, s^*-1), y)\ \&\ U(y) \neq 0.$$

Since $\kappa(0, s) \geqq 1$ for all s,

(2) $$0 < \kappa(0, s^*).$$

If e is not the Gödel number of a system of equations, then $\eta(x, e, i, s)=0$ for all x and s. And 0 should not be any Gödel number of a system of equations. Then

$$\eta(x, 0, i, s)=0 \qquad \text{for all } x \text{ and } s\,.$$

Thus

$$\text{(3)} \qquad \theta(lh(s^*), 0, 0, i, s-1) \geqq \eta(x, 0, i, s) \qquad \text{for all } x \text{ and } s\,.$$

If $\delta(\theta(lh(s^*), 0, 0, i, s^*-1), i, s^*-1)=0$, then by the definition, evidently

$$\delta(\theta(lh(s^*), 0, 0, i, s^*-1), i, s^*)=0\,.$$

Now suppose that

$$\delta(\theta(lh(s^*), 0, 0, i, s^*-1), i, s^*-1)=1\,,$$

then it follows from (1), (2) and (3) that $\Gamma(lh(s^*), 0, 0, i, s^*)$ holds. Hence, we have

$$\delta(\theta(lh(s^*), 0, 0, i, s^*-1), i, s^*)=0\,.$$

Thus, by the definition of s^*, we obtain

$$\delta(x(z, 0, 0, i), i)=0\,.$$

From Lemma 2 and Lemma 3, we have

$$(z)(i)_{i<n}[[(Ey)T_1^{1\cdots1}(\langle\tilde{\delta}(y\,;\hat{i})\rangle, z, x(z, 0, 0, i), y)$$
$$\& \quad U(y) \neq 0] \equiv \delta(x(z, 0, 0, i), i)=0]\,.$$

Hence $\delta(x, i)$ can not be recursive in $\delta(x, 0), \delta(x, 1), \cdots, \delta(x, i-1), \delta(x, i+1), \cdots, \delta(x, n-1)$ for all $i<n$. Thus the independency of degrees $\boldsymbol{d}_0, \boldsymbol{d}_1, \cdots, \boldsymbol{d}_{n-1}$ is proved.

Following G. E. Sacks [4], for each $e \geqq 0$, we say e is stable if for all $x \geqq e$, $\lim_s \eta(x, e, i, s)$ exists and is positive.

We introduce two predicates:

$\Delta(e, i)$: if e is stable, then the set $\{\xi(e, i, s) \mid s \geqq 0\}$ is finite.

$\Lambda(e, i)$: there are numbers z, m and c such that

$\delta(\theta(z, e, m, i, c), i)$ is equal to $1-\gamma(e)$ [2].

PROPOSITION 1. $\qquad (e)(i)_{i<n}\Delta(e, i)\,.$

PROPOSITION 2. $\qquad (e)(i)_{i<n}\Lambda(e, i)\,.$

The proof of Proposition 1 and 2 will be given in the next section. First,

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2) These are found by procedure recursive in $\boldsymbol{d}_i'$.

we will prove the conclusion (iv) from Proposition 1. That is, we shall show that $\alpha(x)$ is not recursive in $\delta(x, i)$ for all $i<n$.

We suppose that $\alpha(x)$ is recursive in $\delta(x, i)$ for some i. That is, we suppose there exists a Gödel number e such that

$$\alpha(x)=U(\mu y T_1^1(\tilde{\delta}(y;i), e, x, y))$$

for all x and some i, and then show $\Delta(e, i)$ is false.

First, we must show e is stable. Fix $x \geqq e$; let

$$y'=\mu y T_1^1(\tilde{\delta}(y;i), e, x, y).$$

Let s' be so large that $s'>y'$ and

$$\delta(m, i, s)=\delta(m, i)$$

whenever $s \geqq s'$ and $m<y'$. Then

$$\eta(x, e, i, s)=y' \quad \& \quad U(y')=\alpha(x)$$

for all $s \geqq s'$ and $y'>0$, since $U(0)=0$ and $\alpha(x)>0$.

Thus, $\lim_s \eta(x, e, i, s)$ exists and is positive for all $x \geqq e$. That is, e is stable.

Then, if we show the set $\{\xi(e, i, s) \mid s \geqq 0\}$ is infinite, the proof is complete.

We fix $e'>e$ and look for an s'' such that

$$\xi(e, i, s'')>e'.$$

Let s be so large that $s>e'$ and

$$\begin{aligned}\alpha^{\natural}(t, s)=\alpha(t)&=U(\mu y T_1^1(\tilde{\delta}(y;i), e, t, y))\\ &=U(\eta(t, e, i, s))\end{aligned}$$

for all t such that $e \leqq t \leqq e'$.

If $\xi(e, i, s-1)>e'$, then $s-1$ is the desired s''. Now suppose $\xi(e, i, s-1) \leqq e'$. This means

$$\alpha^{\natural}(t, s)=U(\eta(t, e, i, s))$$

for all t such that $e \leqq t \leqq \xi(e, i, s-1)$; in addition, $\eta(e, e, i, s)>0$, since $\alpha^{\natural}(e, s) \geqq 1$ and $U(0)=0$. Then Case 3 of the definition of $\xi(e, i, s)$ holds, and

$$\xi(e, i, s)=2 \cdot \xi(e, i, s-1)+s.$$

It follows that

$$\xi(e, i, s)>e'$$

since $s>e'$. That is, s is the desired s''. Thus the set $\{\xi(e, i, s) \mid s \geqq 0\}$ is infinte. Hence $\alpha(x)$ is not recursive in $\delta(x, i)$ for all $i<n$. That is,

$$\boldsymbol{a} \not\leqq \boldsymbol{d}_i \qquad \text{for } i=0, 1, \cdots, n-1.$$

Now, we will show from Proposition 2 $\boldsymbol{c} \leqq \boldsymbol{d}_i'$ for all $i<n$, that is, the half

of one conclusion (v).

For each e and i, we have z, e, m and c by procedure recursive in $\boldsymbol{d}_i'$ such that $\gamma(e)=1-\delta(\theta(z, e, m, i, c), i)$ as an immediate consequence of $(e)(i)_{i<n}\Lambda(e, i)$. Then,

$$\boldsymbol{c} \leqq \boldsymbol{d}_i'$$

for all $i<n$.

Thus, our proof is complete, if we can show Proposition 1, Proposition 2 and $\boldsymbol{c} \geqq \boldsymbol{d}_i'$ for all $i<n$.

In § 3, Proposition 1 and 2 will be proved. In § 4, $\boldsymbol{c} \geqq \boldsymbol{d}_i'$ will be proved and the proof of the conclusion (v) will be complete.

§ 3. The proof of Proposition 1 and 2.

We will prove $(e)(i)_{i<n}\Delta(e, i)$ and $(e)(i)_{i<n}\Lambda(e, i)$ by means of a simultaneous induction on e.

Fix $e^* \geqq 0$ and suppose $(e)(i)_{i<n}[e<e^* \rightarrow \Delta(e, i)\ \&\ \Lambda(e, i)]$.

LEMMA 4. *For any* $i<n$, *let* $\eta(x, e^*, i, s)>0$ *and* $\xi(e^*, i, s)>x \geqq e^*$. *Let* $\delta(\theta(u, k, r, i, s-1), i, s)=\delta(\theta(u, k, r, i, s-1), i, s-1)$ *for all* u, t, k *and* r *such that* $(k<e^* \leqq t \leqq x \vee k<u)\ \&\ \theta(u, k, r, i, s-1)<\eta(t, e^*, i, s)$. *Then* $\eta(x, e^*, i, s) =\eta(x, e^*, i, s+1)$.

PROOF. Since $\eta(x, e^*, i, s)>0$, we have

$$\eta(x, e^*, i, s)=\mu y_{y<s} T_1^1(\tilde{\delta}(y ; i, s-1), e^*, x, y) .$$

Suppose that $\eta(x, e^*, i, s) \neq \eta(x, e^*, i, s+1)$. From the definition of $\eta(x, e, i, s)$, we have

$$(Ej)_{j<\eta(x, e^*, i, s)}[\delta(j, i, s) \neq \delta(j, i, s-1)] .$$

This means

$$\begin{aligned}(Ez)(Ee)(Er)\ [z=lh(s)\ \&\ &\delta(\theta(z, e, r, i, s-1), i, s)\\ &\neq \delta(\theta(z, e, r, i, s-1), i, s-1)\ \&\ \theta(z, e, r, i, s-1)\\ &<\eta(x, e^*, i, s)] .\end{aligned}$$

Thence using the assumption of the lemma, $e \geqq e^*$ and $z \leqq e$. Thus we have

(1) $\quad e^* \leqq e\ \&\ z \leqq e\ \&\ e^* \leqq x<\xi(e^*, i, s)\ \&\ \theta(z, e, r, i, s-1)<\eta(x, e^*, i, s) .$

And

(2) $\quad z=lh(s)\ \&\ \delta(\theta(z, e, r, i, s-1), i, s) \neq \delta(\theta(z, e, r, i, s-1), i, s-1) .$

It follows from the definition of $\delta(x, i, s)$ and (2) that $\Gamma(z, e, r, i, s)$ holds. But this is contrary to (1).

LEMMA 5.

$$(x)(e)(i)_{i<n}(s)\ [(\eta(x, e, i, s)=0 \quad \& \quad x>e)\rightarrow\xi(e, i, s)\leqq x].$$

PROOF. We use the induction on s.

If $s=0$, then the lemma is clear.

Let s be such that $s>0$ and

$$(x)(e)(i)_{i<n}[(\eta(x, e, i, s-1)=0 \quad \& \quad x>e)\rightarrow\xi(e, i, s-1)\leqq x].$$

Let x and e be such that

$$\eta(x, e, i, s)=0 \quad \& \quad x>e.$$

Then we have

$$U(\eta(x, e, i, s))=U(0)=0 \quad \& \quad \alpha^{\natural}(x, s)\geqq 1.$$

Hence

(1) $$\alpha^{\natural}(x, s)\neq U(\eta(x, e, i, s)).$$

First we suppose $x<\xi(e, i, s-1)$. Then it follows, as a consequence of the induction hypothesis that

(2) $$\eta(x, e, i, s-1)>0.$$

From (1), (2) and the assumption of the lemma at the induction step s, either Case 1 or Case 2 of the definition of $\xi(e, i, s)$ holds. If Case 1 holds, then

$$\xi(e, i, s)=e+1\leqq x.$$

If Case 2 holds, then

$$\xi(e, i, s)=\mu t_{e<t}[\eta(t, e, i, s)\neq\eta(t, e, i, s-1)$$
$$\& \quad \alpha^{\natural}(t, s)\neq U(\eta(t, e, i, s))]\leqq x.$$

Next we suppose $x\geqq\xi(e, i, s-1)$. From the definition of $\xi(e, i, s)$, if Case 1 holds, then

$$\xi(e, i, s)=e+1\leqq x.$$

If Case 2 holds, then

$$\xi(e, i, s)\leqq\xi(e, i, s-1)\leqq x.$$

If Case 3 holds and $x<2\cdot\xi(e, i, s-1)+s$, then by (1)

$$\xi(e, i, s)\leqq x.$$

If Case 3 holds and $x\geqq 2\cdot\xi(e, i, s-1)+s$, then

$$\xi(e, i, s)\leqq 2\cdot\xi(e, i, s-1)+s\leqq x.$$

LEMMA 6. *For any* $i<n$, *let* $\eta(x, e^{*}, i, s)>0$ *and* $\xi(e^{*}, i, s)>x>e^{*}$. *Let* $\delta(\theta(u, k, r, i, s-1), i, s)=\delta(\theta(u, k, r, i, s-1), i, s-1)$ *for all* $u, t, k,$ *and* r *such that* $k<e^{*}\leqq t\leqq x$ *and* $\theta(u, k, r, i, s-1)<\eta(t, e^{*}, i, s)$. *Then* $\xi(e^{*}, i, s+1)>x$.

PROOF. It follows from $\xi(e^{*}, i, s)>x>e^{*}$, Lemma 5 and Case 1 of the definition of $\xi(e, i, s)$ that

(1) $$\eta(t, e^*, i, s) > 0$$

for all t such that $e^* \leqq t \leqq x$.

By Lemma 4, we have

(2) $$\eta(t, e^*, i, s) = \eta(t, e^*, i, s+1)$$

for all t such that $e^* \leqq t \leqq x$. Suppose

$$\xi(e^*, i, s+1) \leqq x .$$

From the assumption of the lemma,

$$\xi(e^*, i, s+1) < \xi(e^*, i, s) .$$

Consequently, Case 2 of the definition of $\xi(e, i, s)$ holds, since (1) holds. This means there is a t such that

$$e^* < t = \xi(e^*, i, s+1) \leqq x \quad \& \quad \eta(t, e^*, i, s) \neq \eta(t, e^*, i, s+1)$$

But this last is absurd, since (2) holds.

LEMMA 7. *$\Delta(e^*, i)$ for all $i < n$.*

PROOF. By the assumption of main theorem, $\alpha(x)$ is not recursive in $\beta(x)$. We suppose $\Delta(e^*, i)$ is false for some $i < n$ and show $\alpha(x)$ is recursive in $\beta(x)$. Thus, for each $x \geqq e^*$, $\lim_s \eta(x, e^*, i, s)$ exists and is positive and the set $\{\xi(e^*, i, s) \mid s \geqq 0\}$ is infinite.

Let $\Pi(x, i, s)$ denote the predicate

$$\xi(e^*, i, s) > x \quad \& \quad (u)(e)(t)(y)[(x(u, e, i) < \eta(t, e^*, i, s)$$
$$\& \quad e < u < e^* \leqq t \leqq x \quad \& \quad y \leqq x(u, e, i)) \rightarrow \delta(y, i, s-1) = \delta(y, i)] ,$$

where

$$x(u, e, i) = \max \{\theta(u, e, m, i, s) \mid m \geqq 0 \quad \& \quad s \geqq 0\} .$$

We define a function $\omega(y, u, e, i)$ recursive in B as follows:

$$\omega(y, u, e, i) = \begin{cases} \delta(y, i) & \text{if } y \leqq x(u, e, i) \ \& \ e < u < e^* \ \& \ i < n , \\ 1 & \text{otherwise} . \end{cases}$$

The predicate $\Pi(x, i, s)$ can be now rewritten as

$$\xi(e^*, i, s) > x \ \& \ (u)(e)(t)(y)[(x(u, e, i) < \eta(t, e^*, i, s)$$
$$\& \ e < u < e^* \leqq t \leqq x \ \& \ y \leqq x(u, e, i)) \rightarrow \delta(y, i, s-1) = \omega(y, u, e, i)] .$$

It is clear that the predicate $\Pi(x, i, s)$ is recursive in $\beta(x)$.

We claim $(x)(i)_{i<n}(Es)\ \Pi(x, i, s)$. Fix x and $i < n$. Since $\lim_s \eta(x, e^*, i, s)$ exists for all $x \geqq e^*$, there is a y such that

$$(t)(s)[e^* \leqq t \leqq x \rightarrow y \geqq \eta(t, e^*, i, s)] .$$

Let s' be a number such that

$$(w)(s)[(w<y \ \& \ s \geqq s') \rightarrow \delta(w, i, s-1) = \delta(w, i)] .$$

Since the set $\{\xi(e^*, i, s) | s \geqq 0\}$ is infinite, we have

$$(x)(Es)_{s \geqq s'}[\xi(e^*, i, s) > x] .$$

But then $(x)(i)_{i<n}(Es)\Pi(x, i, s)$ holds. We define

$$w(x, i) = \mu s \Pi(x, i, s) ;$$

the function $w(x, i)$ is recursive in $\beta(x)$. We now show

$$\eta(x, e^*, i, w(x, i)) = \lim_s \eta(x, e^*, i, s)$$

for all $x > e^*$ and $i < n$.

Fix $x > e^*$ and $i < n$. We prove by induction on s that $\eta(x, e^*, i, w(x, i)) = \eta(x, e^*, i, s)$ for all $s \geqq w(x, i)$.

Let s be such that $s \geqq w(x, i)$ and

$$\eta(x, e^*, i, w(x, i)) = \eta(x, e^*, i, s) \ \& \ \Pi(x, i, s) .$$

Since $\Pi(x, i, s)$ holds, we have

(1) $$\xi(e^*, i, s) > x > e^* ;$$

it follows from Lemma 5 and Case 1 of the definition of $\xi(e, i, s)$ that

(2) $$(t)[e^* \leqq t \leqq x \rightarrow \eta(t, e^*, i, s) > 0] .$$

Since $\Pi(x, i, s)$ holds, we have

(3) $$(u)(e)(t)(y)[(x(u, e, i) < \eta(t, e^*, i, s) \ \& \ e < u < e^* \leqq t \leqq x \ \& \ y \leqq x(u, e, i)) \rightarrow \delta(y, i, s-1) = \delta(y, i)] .$$

From (1), (2), (3) and Lemma 4, we have

$$\eta(t, e^*, i, s) = \eta(t, e^*, i, s+1)$$

for all t such that $e^* \leqq t \leqq x$.

It follows from Lemma 6 that

$$\xi(e^*, i, s+1) > x .$$

Then,

$$\eta(x, e^*, i, w(x, i)) = \eta(x, e^*, i, s+1) \ \& \ \Pi(x, i, s+1) .$$

Thus

$$\eta(x, e^*, i, w(x, i)) = \eta(x, e^*, i, s) \qquad \text{for all } s \geqq w(x, i) .$$

Finally, we show by means of a reductio ad absurdum argument that

$$\alpha(x) = U(\eta(x, e^*, i, w(x, i)))$$

for all $x>e^*$. Fix $x>e^*$ and suppose

$$\alpha(x)\neq U(\eta(x, e^*, i, w(x, i)))\,.$$

Since $\eta(x, e^*, i, w(x, i))=\lim_s \eta(x, e^*, i, s)$ and $\alpha(x)=\lim_s \alpha^{\natural}(x, s)$, there exists s^* such that

$$(s)[s\geqq s^*\rightarrow(\alpha(x)=\alpha^{\natural}(x, s)\ \&\ U(\eta(x, e^*, i, s)) = U(\eta(x, e^*, i, w(x, i))))]\,.$$

That is,

$$\alpha^{\natural}(x, s)\neq U(\eta(x, e^*, i, s)) \qquad \text{for all } s\geqq s^*\,. \tag{4}$$

We show that

$$(s)_{s\geqq s^*}[\xi(e^*, i, s)\leqq \xi(e^*, i, s^*)+x+e^*+1]\,.$$

We use the induction on $s\geqq s^*$. Let $s>s^*$ and suppose

$$\xi(e^*, i, s-1)\leqq \xi(e^*, i, s^*)+x+e^*+1\,.$$

If either Case 1 or Case 2 of the definition of $\xi(e, i, s)$ holds, then

$$\begin{aligned}\xi(e^*, i, s)&\leqq \max\{e^*+1, \xi(e^*, i, s-1)\}\\ &\leqq \xi(e^*, i, s^*)+x+e^*+1\,.\end{aligned}$$

If Case 3 holds and $x<2\cdot\xi(e^*, i, s-1)+s$, then

$$\xi(e^*, i, s)\leqq x\leqq \xi(e^*, i, s^*)+x+e^*+1\,,$$

since (4) holds. If Case 3 holds and $x\geqq 2\cdot\xi(e^*, i, s-1)+s$, then

$$\xi(e^*, i, s)=2\cdot\xi(e^*, i, s-1)+s\leqq x\leqq \xi(e^*, i, s^*)+x+e^*+1\,.$$

Thus we have

$$(s)_{s\geqq s^*}[\xi(e^*, i, s)\leqq \xi(e^*, i, s^*)+x+e^*+1]\,.$$

But this last is absurd, since the set $\{\xi(e^*, i, s)\,|\,s\geqq 0\}$ is infinite.

Then, we obtain

$$\alpha(x)=U(\eta(x, e^*, i, w(x, i))) \qquad \text{for all } x>e^*\,.$$

That is, $\alpha(x)$ is recursive in $\beta(x)$.

We define

$$\begin{cases} e_0 \ =\mu e[e \text{ is not stable}]\,. \\ e_{j+1}=\mu e[e>e_j \text{ and } e \text{ is not stable}]\,. \end{cases}$$

Let x_j^i be the least $x\geqq e_j$ such that $\lim_s \eta(x, e_j, i, s)$ does not exist or is equal to 0.

LEMMA 8.

$$(i)_{i<n}(k)(v)(Es)_{s\geqq v}(j)_{j<k}[\xi(e_j, i, s)\leqq x_j^i$$
$$\vee \nu(x_j^i, e_j, i, s)=0 \vee \eta(x_j^i, e_j, i, s)=0].$$

PROOF. Fix i, k and v. We suppose there does not exist s with the properties required by the lemma, and then show it is possible to define an infinite, descending sequence of natural numbers.

We shall define two functions, $\chi(t)$ and $\lambda(t)$, simultaneously by induction.

$$\chi(0)=\mu s(s\geqq v).$$
$$\lambda(t)=\mu j[j<k \ \& \ x_j^i<\xi(e_j, i, \chi(t)) \ \&$$
$$\nu(x_j^i, e_j, i, \chi(t))=1 \ \& \ \eta(x_j^i, e_j, i, \chi(t))>0]$$
$$\chi(t+1)=\mu s(Em)[s\geqq \chi(t) \ \& \ m<\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, \chi(t))$$
$$\& \ \delta(m, i, s)\neq\delta(m, i, \chi(t)-1)].$$

We shall show that $\chi(t)$ is well-defined and $\chi(t)\geqq v$. Clearly $\chi(0)$ is well-defined and $\chi(0)\geqq v$. Suppose $t\geqq 0$ and $\chi(t)$ is well-defined and $\chi(t)\geqq v$.

We have supposed the lemma to be false, so $\lambda(t)$ is well-defined and $\lambda(t)<k$. Thus

$$\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, \chi(t))>0.$$

Since $e_{\lambda(t)}$ is not stable, there must be an $s>\chi(t)$ such that

$$\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, s)\neq\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, \chi(t)).$$

It follows there is an $s>\chi(t)$ and an m such that

$$m<\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, \chi(t)) \ \& \ \delta(m, i, s-1)\neq\delta(m, i, \chi(t)-1).$$

Then $\chi(t+1)$ is well-defined.

For each $t\geqq 0$, let

$$x^*(t)=\mu m[\delta(m, i, \chi(t+1))\neq\delta(m, i, \chi(t)-1)].$$

Now we show $x^*(t)<x^*(t-1)$ for all $t>0$. Fix $t>0$. By the definitions of $\chi(t)$ and $x^*(t)$, we have

$$x^*(t)<\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, \chi(t)). \tag{1}$$

Then it is sufficient to show that

$$\eta(x_{\lambda(t)}^i, e_{\lambda(t)}, i, \chi(t))\leqq x^*(t-1). \tag{2}$$

Since

$$\delta(x^*(t-1), i, \chi(t))\neq\delta(x^*(t-1), i, \chi(t)-1)$$

as a consequence of the definitions of $\chi(t)$ and $x^*(t)$, we have

$$(Ee')(Er)[\delta(x^*(t-1), i, \chi(t)) \neq \delta(x^*(t-1), i, \chi(t)-1)$$
$$\& \; x^*(t-1) = \theta(lh(\chi(t)), e', r, i, \chi(t)-1)] .$$

First we suppose $e' < e_{\lambda(t)} \vee e' < lh(\chi(t))$. Then we have

$$(e' < e_{\lambda(t)} \vee e' < lh(\chi(t))) \; \& \; \nu(x^i_{\lambda(t)}, e_{\lambda(t)}, i, \chi(t)) = 1 .$$

But then it follows from the definition of $\nu(x, e, i, s)$ that

$$x^*(t-1) = \theta(lh(\chi(t)), e', r, i, \chi(t)-1) \geqq \eta(x^i_{\lambda(t)}, e_{\lambda(t)}, i, \chi(t)) .$$

Now we suppose $e' \geqq e_{\lambda(t)} \; \& \; e' \geqq lh(\chi(t))$. Then we have

$$e_{\lambda(t)} \leqq e' \; \& \; lh(\chi(t)) \leqq e' \; \& \; e_{\lambda(t)} \leqq x^i_{\lambda(t)} < \xi(e_{\lambda(t)}, i, \chi(t))$$
$$\& \; \delta(\theta(lh(\chi(t), e', r, i, \chi(t)-1), i, \chi(t))$$
$$\neq \delta(\theta(lh(\chi(t)), e', r, i, \chi(t)-1), i, \chi(t)-1) .$$

Since $\Gamma(lh(\chi(t)), e', r, i, \chi(t))$ holds, it follows that

$$x^*(t-1) = \theta(lh(\chi(t)), e', r, i, \chi(t)-1) \geqq \eta(x^i_{\lambda(t)}, e_{\lambda(t)}, i, \chi(t)) .$$

Thus we have shown that (2) holds.

LEMMA 9. *If* $\gamma(e^*) = 0$, *then*

$$(i)_{i<n}(Em')(m)_{m \geqq m'}(z)(s)[\delta(\theta(z, e^*, m, i, s), i) = 1] .$$

PROOF. We shall define

$$t(e) = \mu r[\phi(r) = e] ,$$
$$(*) \qquad s'(e) = \mu s(r)_{r \leqq t(e)}[\phi^\#(r, s) = \phi(r) \; \& \; s > t(e)] .$$

Then $t(e^*)(=t)$ and $s'(e^*)(=s')$ are defined, because $\gamma(e^*) = 0$. By the definition of $\tau(x, s)$, we have

$$\tau(e^*, s) = t \qquad \text{for each } s \geqq s' .$$

Then

$$\kappa(e^*, s) = \kappa(e^*, s') \qquad \text{for each } s \geqq s' .$$

It follows from the definition of $\kappa(x, s)$ that $\kappa(x, s_1) \geqq \kappa(x, s_2)$ for all s_1 and s_2 such that $s_1 > s_2$. Thus we obtain from the definition of $\Gamma(z, e, m, i, s)$ that

$$(z)(m)(i)_{i<n}(s)_{s>0}[m \geqq \kappa(e^*, s') \rightarrow$$
$$\delta(\theta(z, e^*, m, i, s-1), i, s) = \delta(\theta(z, e^*, m, i, s-1), i, s-1)] .$$

Then

$$(i)_{i<n}(Em')(m)_{m \geqq m'}(z)(s)[\delta(\theta(z, e^*, m, i, s), i) = 1] .$$

LEMMA 10.

$$(x)(i)_{i<n}(e)(s)[\nu(x, e, i, s) = 0 \rightarrow \nu(x+1, e, i, s) = 0] .$$

PROOF. This lemma is easily deduced from the definition.

LEMMA 11. *If* $\gamma(e^*)=1$, *then*

$$(i)_{i<n}(m)(Es)[\delta(\theta(lh(s), e^*, m, i, s-1), i)=0].$$

PROOF. Fix $i<n$. First we show that the set $\{\tau(e^*, s)\mid s\geqq 0\}$ is infinite.

Suppose $\tau(e^*, s)\leqq t$ for all s. Let s' be so large that $s'>t$ and $\phi^\#(r, s)=\phi(r)$ for all s and r such that $s\geqq s'$ and $r\leqq t$. Then we have

$$\phi^\#(\tau(e^*, s'), s')=\phi(\tau(e^*, s'))=e^*,$$

since $\tau(e^*, s')\leqq t<s'$.

This is impossible because $\gamma(e^*)=1$. Thus

$$\{\tau(e^*, s)\mid s\geqq 0\} \quad \text{is infinite.}$$

Then

$$\{\kappa(e^*, s)\mid s\geqq 0\} \quad \text{is infinite.} \tag{1}$$

By Lemma 7, we know $\Delta(e, i)$ holds for all $e\leqq e^*$. This means that if $e\leqq e^*$ and e is stable, then the set $\{\xi(e, i, s)\mid s\geqq 0\}$ is finite.

We define $\xi^*(e, i)$ for all $e\leqq e^*$ by two cases.

Case 1: $e\leqq e^*$ and e is stable. We set

$$\xi^*(e, i)=\max\{\xi(e, i, s)\mid s\geqq 0\}.$$

Case 2: $e\leqq e^*$ and e is not stable. We set

$$\xi^*(e, i)=x^i_j, \text{ where } j \text{ is such that } e=e_j.$$

If $e\leqq e^*$ and $e\leqq q<\xi^*(e, i)$, then $\lim_s \eta(q, e, i, s)$ exists. Then there exists a y_0, such that

$$(s)(e)_{e\leqq e^*}(q)_{e\leqq q<\xi^*(e,i)}[y_0\geqq \eta(q, e, i, s)]. \tag{2}$$

We fix an m for the rest of discussion. The proof will complete, if we can show

$$(Es)[\delta(\theta(lh(s), e^*, m, i, s-1), i, s)=0]. \tag{3}$$

By (1), there exists an s_1 such that

$$(s)_{s\geqq s_1}[\kappa(e^*, s)>m], \tag{4}$$

since $\kappa(e^*, s)$ is a nondecreasing function of s.

Let k be such that if $e\leqq e^*$ and e is not stable, then $e=e_j$ for some $j<k$. By Lemma 8, there is an $s_2\geqq s_1$ such that

$$(j)_{j<k}[\xi(e_j, i, s_2)\leqq x^i_j \vee \nu(x^i_j, e_j, i, s_2)=0 \vee \eta(x^i_j, e_j, i, s_2)=0]. \tag{5}$$

Let s^* be such that $lh(s^*)>\max\{e^*, y_0\}$, $s^*\geqq s_2$ and

(6) $$(Ey)_{y<s^*}[T_1^{1\cdots 1}(\langle\tilde{\delta}(y\,;\hat{i},s^*-1)\rangle, lh(s^*), \theta(lh(s^*), e^*, m, i, s^*-1), y)\ \&\ U(y)\neq 0].$$

We shall show $\delta(\theta(lh(s^*), e^*, m, i, s^*-1), i, s^*)=0$. If $\delta(\theta(lh(s^*), e^*, m, i, s^*-1), i, s^*-1)=0$, then by the definition evidently $\delta(\theta(lh(s^*), e^*, m, i, s^*-1), i, s^*)=0$.

Now we suppose $\delta(\theta(lh(s^*), e^*, m, i, s^*-1), i, s^*-1)=1$. Then it will suffice to show

(i) $$(Ey)_{y<s^*}[T_1^{1\cdots 1}(\langle\tilde{\delta}(y\,;\hat{i},s^*-1)\rangle, lh(s^*), \theta(lh(s^*), e^*, m, i, s^*-1), y)\ \&\ U(y)\neq 0],$$

(ii) $$m<\kappa(e^*, s^*),$$

and

(iii) $$\begin{aligned}&(e)(q)[[(e\leqq e^*\ \&\ lh(s^*)\leqq e^*\ \&\ e\leqq q<\xi(e, i, s^*))\\&\to\theta(lh(s^*), e^*, m, i, s^*-1)\geqq\eta(q, e, i, s^*)]\\&\&\ (Es')_{s'\leqq s^*}[(e\leqq e^*\ \&\ lh(s^*)>e^*\\&\&\ e\leqq q<\xi(e, i, s'))\to(\nu(q, e, i, s')=0\\&\vee\ \theta(lh(s^*), e^*, m, i, s'-1)\geqq\eta(q, e, i, s'))]].\end{aligned}$$

Since $s^*\geqq s_1$, from (6) and (4), (i) and (ii) evidently hold. Since $lh(s^*)>e^*$ and $s_2\leqq s^*$, we have only to show that

$$\begin{aligned}&(e\leqq e^*\ \&\ lh(s^*)>e^*\ \&\ e\leqq q<\xi(e, i, s_2))\to(\nu(q, e, i, s_2)=0\\&\vee\ \theta(lh(s^*), e^*, m, i, s_2-1)\geqq\eta(q, e, i, s_2)).\end{aligned}$$

Fix e and q so that $e\leqq e^*$ and $e\leqq q<\xi(e, i, s_2)$. Suppose e is stable, then

$$\xi^*(e, i)\geqq\xi(e, i, s_2)>q\geqq e.$$

Consequently, by using (2),

$$y_0\geqq\eta(q, e, i, s_2).$$

Since $lh(s^*)>y_0$, we obtain

$$\theta(lh(s^*), e^*, m, i, s_2-1)\geqq 2^{lh(s^*)}>y_0\geqq\eta(q, e, i, s_2).$$

Now suppose e is not stable, then by the definition of $\xi^*(e, i)$, $e=e_j$, where $j<k$, and $\xi^*(e, i)=x_j^i$. If $q<x_j^i$, then $e_j=e\leqq q<x_j^i=\xi^*(e, i)$. Then we have

$$\theta(lh(s^*), e^*, m, i, s_2-1)\geqq 2^{lh(s^*)}>y_0\geqq\eta(q, e, i, s_2).$$

If $q\geqq x_j^i$, then $\xi(e_j, i, s_2)>q\geqq x_j^i=\xi^*(e, i)$. By (5), this means that either

$$\nu(x_j^i, e_j, i, s_2)=0\quad\text{or}\quad\eta(x_j^i, e_j, i, s_2)=0.$$

Suppose $\nu(x_j^i, e_j, i, s_2)=0$, then by Lemma 10, $\nu(q, e, i, s_2)=0$, since $q\geqq x_j^i$ and $e=e_j$. If $\eta(x_j^i, e_j, i, s_2)=0$, then by Lemma 5,

$$x_j^i \leqq e = e_j\,,$$

since $x_j^i \leqq q < \xi(e, i, s_2)$. By the definition of x_j^i, we know

$$x_j^i \geqq e_j = e\,.$$

Thus we have

$$x_j^i = e_j = e\,.$$

Then we have

$$\eta(e, e, i, s_2) = \eta(e_j, e_j, i, s_2) = \eta(x_j^i, e_j, i, s_2) = 0\,,$$

and it follows that $\xi(e, i, s_2) = e+1$.

Hence we obtain

$$x_j^i = e \leqq q < \xi(e, i, s_2) = e+1, \text{ that is,}$$

$$q = e\,.$$

Thus

$$\eta(q, e, i, s_2) = \eta(e, e, i, s_2) = \eta(x_j^i, e_j, i, s_2) = 0\,.$$

Consequently

$$\theta(lh(s^*), e^*, m, i, s_2-1) \geqq \eta(q, e, i, s_2)\,.$$

Then (iii) holds. That is, we have shown that (3) holds.

LEMMA 12. *$\Lambda(e^*, i)$ for all $i < n$.*

PROOF. We put $\tilde{s}$ and t as follows:

$$\tilde{s} = \mu s(x)_{x \leqq e^*}[\beta^\#(x, s) = \beta'(x) \ \&\ e^* < s]\,.$$

$$t = \begin{cases} \mu x((x)_0 \geqq \tilde{s} \ \&\ (x)_1 < (x)_0 \ \&\ \phi^\#((x)_1, (x)_0) = e^*))_1 \\ \qquad\qquad \text{if } (Es)_{s \geqq \tilde{s}}(Er)_{r<s}[\phi^\#(r, s) = e^*]\,, \\ \tilde{s}+1 \quad \text{otherwise.} \end{cases}$$

Letting

$$s' = \mu s(r)_{r \leqq t}[\phi^\#(r, s) = \phi(r) \ \&\ s > t]\,,$$

$$m^\natural = \kappa(e^*, s')$$

and

$$s^\natural = \begin{cases} \mu s[\delta(\theta(lh(s), e^*, m^\natural, i, s-1), i) = 0] \\ \qquad \text{if } (Es)[\delta(\theta(lh(s), e^*, m^\natural, i, s-1), i) = 0]\,, \\ 1 \quad \text{otherwise,} \end{cases}$$

we obtain

$$r(e^*) = 1 - \delta(\theta(lh(s^\natural), e^*, m^\natural, i, s^\natural - 1), i)$$

from the proof of Lemma 9 and Lemma 11. And $m^\natural$, $s^\natural$ are obtained by the procedure recursive in $\boldsymbol{d}_i'$. This constitutes a proof of $\Lambda(e^*, i)$.

Thus we have accomplished the proof of Proposition 1 and 2.

§4. The proof of $\boldsymbol{c} \geqq \boldsymbol{d}_i'$ $(i=0, 1, \cdots, n-1)$.

In this section, we shall prove $\boldsymbol{c} \geqq \boldsymbol{d}_i'$ for all $i<n$.

We shall define a function $\sigma(x, e, i)$ which is recursive in $\gamma(x)$, and satisfies the following (1).

$$(1) \qquad (x)(e)(i)_{i<n}[\sigma(x, e, i)=0 \leftrightarrow (x \geqq e \ \& \ (w)(Es)(s>w \ \& \ \xi(e, i, s)>x) \\ \& \ (m)(x \geqq m \geqq e \rightarrow (Ey)T_1^1(\tilde{\delta}(y\,; i), e, m, y)))] .$$

From the definition of $\xi(e, i, s)$, we have $(e)(i)_{i<n}(s)[\xi(e, i, s)>e]$. It follows immediately from (1) that

$$(e)(i)_{i<n}[\sigma(e, e, i)=0 \leftrightarrow (Ey)T_1^1(\tilde{\delta}(y\,; i), e, e, y)] .$$

Then, if $\sigma(x, e, i)$ is recursive in $\gamma(x)$, we have

$$\boldsymbol{c} \geqq \boldsymbol{d}_i' \qquad \text{for all } i<n .$$

Thus we have only to define $\sigma(x, e, i)$ recursively in $\gamma(x)$, and satisfying the property (1).

First, we define $\pi(e, s)$ as follows:

$$\pi(e, s)=\begin{cases} \kappa(e, s'(e)) & \text{if } \gamma(e)=0 , \\ s & \text{otherwise,} \end{cases}$$

where $s'(e)$ is the function defined by (*) in the proof of Lemma 9. Then we easily see that this function satisfies the following:

$$(2) \qquad (z)(e)(m)(i)_{i<n}[m \geqq \pi(e, s) \rightarrow \delta(\theta(z, e, m, i, s), i) \\ =\delta(\theta(z, e, \pi(e, i, s), i, s), i)] .$$

In fact, if $\gamma(e)=0$, the property (2) follows from the proof of Lemma 9; otherwise by the definition of $\theta(z, e, m, i, s)$, we have

$$(z)(e)(m)(i)_{i<n}(s)[m \geqq s \rightarrow \theta(z, e, m, i, s)=2^z \cdot 5] ,$$

from which (2) also follows.

By the definition, $\pi(e, s)$ is recursive in $\gamma(e)$.

Now, we will define $\sigma(x, e, i)$ by induction on e.

Let $\Sigma(e)$ denote the following predicate:

$$\Sigma(e) \equiv (x)(j)_{j<e}(i)_{i<n}[(\sigma(x, j, i) \text{ has been defined}) \\ \& \ (\sigma(x, j, i)=0 \leftrightarrow (x \geqq j \ \& \ (w)(Es)(s>w \ \& \ \xi(j, i, s)>x) \\ \& \ (m)(x \geqq m \geqq j \rightarrow (Ey)T_1^1(\tilde{\delta}(y\,; i), j, m, y))))] .$$

We suppose that $\Sigma(e^*)$ holds. Let $\Sigma(e^*+1, x)$ denote the following predicate:

$$\Sigma(e^*+1, x) \equiv (t)_{t<x}(i)_{i<n}[(\sigma(t, e^*, i) \text{ has been defined})$$
$$\& \ (\sigma(t, e^*, i)=0 \leftrightarrow (t \geqq e^* \ \& \ (w)(Es)(s>w \ \& \ \xi(e^*, i, s)>t)$$
$$\& \ (m)(t \geqq m \geqq e^* \rightarrow (Ey)T_1^1(\tilde{\delta}(y\,;i), e^*, m, y))))] .$$

To verify $\Sigma(e^*+1)$, it suffices to prove $\Sigma(e^*+1, x)$ for all x.

We shall define $\sigma(x, e^*, i)$ and prove $\Sigma(e^*+1, x)$ for all x by means of an induction on x. We fix x^* and suppose $\Sigma(e^*+1, x^*)$ holds. We define $\sigma(x^*, e^*, i)$ and then prove $\Sigma(e^*+1, x^*+1)$.

Case 1: $x^*<e^*$. We set

$$\sigma(x^*, e^*, i)=1 \qquad \text{for all } i<n .$$

Case 2: $x^* \geqq e^*$.

Subcase 2.1: $(Et)(Ei)_{i<n}[e^* \leqq t < x^* \ \& \ \sigma(t, x^*, i) \neq 0]$. We set

$$\sigma(x^*, e^*, i)=1 \qquad \text{for all } i<n \text{ such that}$$
$$(Et)[e^* \leqq t<x^* \ \& \ \sigma(t, e^*, i) \neq 0] .$$

Subcase 2.2: Otherwise. It follows from $\Sigma(e^*+1, x^*)$ that

$$(Et)T_1^1(\tilde{\delta}(y\,;i), e^*, m, y) \qquad \text{for all } i<n, m \text{ such that } x^*>m \geqq e^* .$$

For each m and i such that $x^*>m \geqq e^*$ and $i<n$, let

$$\eta(m, i)=\mu y T_1^1(\tilde{\delta}(y\,;i), e^*, m, y) .$$

Let

$$y^*=\max \{\{\eta(m, i) \mid x^*>m \geqq e^* \ \& \ i<n\} \cup \{0\}\}$$

and

$$s^*=\mu s[(i)_{i<n}(j)(j<y^* \rightarrow \delta(j, i, s-1)=\delta(j, i)) \ \& \ s>y^*] .$$

It follows from the definition of $\eta(m, i)$ and from the fact that 0 is not the Gödel number of any deduction that

(3) $$(m)(i)_{i<n}(s)[(s>s^* \ \& \ x^*>m \geqq e^*) \rightarrow \eta(m, e^*, i, s)=\eta(m, i)>0] .$$

We define

$$\sigma(x^*, e^*, i)=\begin{cases} 0 & \text{if } (Es)[\eta(x^*, e^*, i, s)>0 \ \& \ \xi(e^*, i, s)>x^* \ \& \ s>s^* \\ & \& \ (z)_{z<\eta(x^*,e^*,i,s)}(e)_{e<\eta(x^*,e^*,i,s)}(m)_{m<\eta(x^*,e^*,i,s)} \\ & (\theta(z, e, m, i, s-1)=x(z, e, m, i)) \\ & \& \ (z)_{z<\eta(x^*,e^*,i,s)}(e)_{e^* \leqq e<\eta(x^*,e,i,s)}(m)_{m<\eta(x^*,e^*,i,s)} \\ & (\theta(z, e, m, i, s-1)<\eta(x^*, e^*, i, s) \rightarrow \delta(\theta(z, r, m, i, s-1), i, s-1) \\ & =\delta(\theta(z, e, m, i, s-1), i)) \\ & \& \ (z)_{z<\eta(x^*,e^*,i,s)}(e)_{e<e^*}((m)(m<\pi(e, s) \rightarrow \\ & \delta(\theta(z, e, m, i, s-1), i, s-1)=\delta(\theta(z, e, m, i, s-1), i)) \\ & \& \ (m)(\pi(e, s) \leqq m<\eta(x^*, e^*, i, s) \rightarrow \delta(\theta(z, e, m, i, s-1), i, s-1) \\ & =\delta(\theta(z, e, \pi(e, s), i, s-1), i)))] , \\ 1 & \text{otherwise,} \end{cases}$$

for all $i<n$.

To verify $\Sigma(e^*+1, x^*+1)$, it suffices to prove

$$\sigma(x^*, e^*, i)=0 \leftrightarrow [x^* \geqq e^* \ \& \ (w)(Es)(s>w \ \& \ \xi(e^*, i, s)>x^*) \\ \& \ (m)(x^* \geqq m \geqq e^* \rightarrow (Ey)T_1^1(\tilde{\delta}(y\,; i), e^*, m, y))] .$$

Suppose $\sigma(x^*, e^*, i)=0$. Then Subcase 2.2 of the definition of $\sigma(x, e, i)$ must hold. Let $\tilde{s}$ be the nutural number whose existence is required by $\sigma(x^*, e^*, i)=0$. Thus, by the definition of $\sigma(x^*, e^*, i)=0$, we have

$$x^* \geqq e^* \ \& \ \eta(x^*, e^*, i, \tilde{s})>0 \ \& \ \xi(e^*, i, \tilde{s})>x^* \ \& \ \tilde{s}>s^* .$$

We shall prove

$$\eta(x^*, e^*, i, s)=\eta(x^*, e^*, i, \tilde{s}) \ \& \ \xi(e^*, i, s)>x^*$$

for all s such that $s>\tilde{s}$ by the induction on s.

Fix $s \geqq \tilde{s}$. Suppose

$$\eta(x^*, e^*, i, s)=\eta(x^*, e^*, i, \tilde{s}) \ \& \ \xi(e^*, i, s)>x^* \\ \& \ \eta(x^*, e^*, i, s) \neq \eta(x^*, e^*, i, s+1) .$$

From the definition of $\eta(x, e, i, s)$, we have

$$(Ew)[\delta(w, i, s) \neq \delta(w, i, s-1) \ \& \ w<\eta(x^*, e^*, i, s)] .$$

Then, there exist $\bar{z}$, $\bar{e}$ and $\bar{m}$ such that

$$\text{(4)} \qquad \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s) \neq \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s-1) \\ \& \ \bar{m}<\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1)<\eta(x^*, e^*, i, s) \ \& \ \bar{z}=lh(s) .$$

Suppose $\bar{e}<e^*$. Then, by the definition of $\sigma(x^*, e^*, i)=0$, $s \geqq \tilde{s}$ and the second member of conjunction (4), $\tilde{s}$ has the property that

$$\delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, \tilde{s}-1)=\begin{cases} \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i) & \text{if } \bar{m}<\pi(\bar{e}, \tilde{s}) , \\ \delta(\theta(\bar{z}, \bar{e}, \pi(\bar{e}, \tilde{s}), i, s-1), i) & \text{otherwise.} \end{cases}$$

It follows from (2) and $\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1)=x(\bar{z}, \bar{e}, \bar{m}, i)$ that

$$\delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i)=\delta(\theta(\bar{z}, \bar{e}, \pi(\bar{e}, \tilde{s}), i, s-1), i) \qquad \text{for } \bar{m} \geqq \pi(\bar{e}, \tilde{s}) .$$

Consequently

$$\delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s)=\delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s-1) ,$$

which contradicts the first member of conjunction (4). Thus we have shown $\bar{e} \geqq e^*$.

Since $\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1)=x(\bar{z}, \bar{e}, \bar{m}, i)=\theta(\bar{z}, \bar{e}, \bar{m}, i, \tilde{s}-1)<\eta(x^*, e^*, i, s)=\eta(x^*, e^*, i, \tilde{s})$, it follows from the definition of $\sigma(x^*, e^*, i)=0$ that

$$\delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, \tilde{s}-1), i, \tilde{s}-1) = \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, \tilde{s}-1), i)$$
$$= \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s-1) = \delta(\theta(\bar{z}, \bar{e}, \bar{m}, i, s-1), i, s),$$

which contradicts the first member of conjunction (4). Thus we have

$$(w)_{w<\eta(x^*, e^*, i, s)}[\delta(w, i, s) = \delta(w, i, s-1)].$$

That is,

$$\eta(x^*, e^*, i, \tilde{s}) = \eta(x^*, e^*, i, s) = \eta(x^*, e^*, i, s+1).$$

Since $s \geqq \tilde{s} > s^*$, we now obtain from (3) that

$$(m)(i)_{i<n}[x^* \geqq m \geqq e^* \rightarrow \eta(m, e^*, i, s) = \eta(m, e^*, i, s+1) > 0].$$

Then, either Case 2 or Case 3 of the definition of $\xi(e^*, i, s+1)$ holds. If Case 2 holds, then

$$\xi(e^*, i, s+1) = \xi(e^*, i, s) > x^*.$$

If Case 3 holds, then

$$\xi(e^*, i, s+1) \geqq \xi(e^*, i, s) > x^*.$$

Thus we have shown that

$$(s)_{s \geqq \tilde{s}}[\eta(x^*, e^*, i, s) = \eta(x^*, e^*, i, \tilde{s}) > 0 \ \& \ \xi(e^*, i, s) > x^*].$$

It follows immediately that

(5) $$(Ey)T_1^1(\tilde{\delta}(y; i), e^*, x^*, y) \ \& \ (w)(Es)[s > w \ \& \ \xi(e^*, i, s) > x^*].$$

Since $\sigma(x^*, e^*, i) = 0$, $\Sigma(e^*+1, x^*)$ implies

(6) $$(m)[x^* > m \geqq e^* \rightarrow (Ey)T_1^1(\tilde{\delta}(y; i), e^*, m, y)].$$

By (5) and (6), we obtain

(7) $$x^* \geqq e^* \ \& \ (w)(Es)[s > w \ \& \ \xi(e^*, i, s) > x^*]$$
$$\& \ (m)[x^* \geqq m \geqq e^* \rightarrow (Ey)T_1^1(\tilde{\delta}(y; i), e^*, m, y)].$$

Now we suppose (7), and then show $\sigma(x^*, e^*, i) = 0$. By $\Sigma(e^*+1, x^*)$, we have Subcase 2.2 of the definition of $\sigma(x, e, i)$.

Let

$$\tilde{\eta}(m, i) = \mu y T_1^1(\tilde{\delta}(y; i), e^*, m, y)$$

for all $i < n$ and all m such that $x^* \geqq m \geqq e^*$. We put

$$\tilde{\eta} = \max\{\tilde{\eta}(m, i) \mid i < n \ \& \ x^* \geqq m \geqq e^*\},$$

and

$$\bar{s} = \mu w[w > \max\{\tilde{\eta}, s^*\} \ \& \ (z)_{z<\tilde{\eta}}(e)_{e<\tilde{\eta}}(m)_{m<\tilde{\eta}}(\theta(z, e, m, i, w) = x(z, e, m, i))$$
$$\& \ (z)_{z<\tilde{\eta}}(e)_{e<\tilde{\eta}}(t)(t < x(z, e, i) \rightarrow \delta(t, i, w) = \delta(t, i)) \ \& \ \xi(e^*, i, w) > x^*].$$

It follows from the definition of $\sigma(x^*, e^*, i)$ that $\bar{s}$ has the properties required to conclude $\sigma(x^*, e^*, i)=0$. Thus we have shown $\Sigma(e^*+1, x^*+1)$. That is, (1) holds.

From the definition of $\sigma(x, e, i)$, $\sigma(x, e, i)$ is recursive in $\gamma(x)$, $\delta(x, i)$ and $\beta'(x)$. Then $\sigma(x, e, i)$ is recursive in γ, since $\boldsymbol{d}_i < \boldsymbol{c}$ and $\boldsymbol{b}' \leqq \boldsymbol{c}$. Hence we have

$$\boldsymbol{c} \geqq \boldsymbol{d}_i' \qquad \text{for all } i < n$$

Thus the conclusion (v) of main theorem is complete, and the proof of main theorem has been accomplished.

§ 5. Theorems.

THEOREM 1. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are degrees, the following conditions* (i), (ii), (iii), (iv) *and* (v) *are equivalent:*

(i) $\boldsymbol{a}' \leqq \boldsymbol{b} \leqq \boldsymbol{a}''$ & $\boldsymbol{b} \upharpoonleft \boldsymbol{a}'$;

(ii) *there exists a $\boldsymbol{c}$ such that* $\boldsymbol{a} \leqq \boldsymbol{c} \leqq \boldsymbol{a}'$ & $\boldsymbol{c}' = \boldsymbol{b}$;

(iii) *for any positive integer n, there exist independent degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ such that* $\boldsymbol{a} < \boldsymbol{c}_i < \boldsymbol{a}'$ & $\boldsymbol{c}_i' = \boldsymbol{b}$ *for* $i = 1, 2, \cdots, n$;

(iv) *there exists a $\boldsymbol{c}$ such that* $\boldsymbol{a} \leqq \boldsymbol{c} \leqq \boldsymbol{a}'$ & $\boldsymbol{c} \upharpoonleft \boldsymbol{a}$ & $\boldsymbol{c}' = \boldsymbol{b}$;

(v) *for any positive integer n, there exist independent degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ such that* $\boldsymbol{a} < \boldsymbol{c}_i < \boldsymbol{a}'$ & $\boldsymbol{c}_i \upharpoonleft \boldsymbol{a}$ & $\boldsymbol{c}_i' = \boldsymbol{b}$ *for* $i = 1, 2, \cdots, n$.

PROOF. It is clear that (v)→(iv), (v)→(iii), (iv)→(ii), (iii)→(ii) and (ii)→(i). (i)→(v) is easily deduced from Main Theorem.

G. E. Sacks proved the equivalency of (i), (ii) and (iv) in [4].

THEOREM 2. *For any degree $\boldsymbol{a}$ and any positive integer n, there exist degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ such that:*

(1) $\boldsymbol{c}_i \upharpoonleft \boldsymbol{a}$ *for* $i = 1, 2, \cdots, n$,

(2) $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n$ *are independent,*

(3) $\boldsymbol{a} < \boldsymbol{c}_i < \boldsymbol{a}' < \boldsymbol{a}'' = \boldsymbol{c}_i'$ *for* $i = 1, 2, \cdots, n$.

PROOF. Apply Theorem 1 ((i)→(v)) with $\boldsymbol{b} = \boldsymbol{a}''$.

By Theorem 2, we can easily see that for any degree $\boldsymbol{a}$ and any partially ordered set T whose cardinarity is finite, there exists a set U of degrees such that T is imbeddable in U and U has the following properties: $\boldsymbol{u} \in U \rightarrow (\boldsymbol{u} \upharpoonleft \boldsymbol{d}$ & $\boldsymbol{d} < \boldsymbol{u} < \boldsymbol{d}'$ & $\boldsymbol{u}' = \boldsymbol{d}'')$.

THEOREM 3. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be degrees and n be any positive integer such that:*

(1) $\boldsymbol{a}^{(n)} \leqq \boldsymbol{b}$,

(2) $\boldsymbol{b} \upharpoonleft \boldsymbol{a}^{(n)}$.

Then, for any positive integer m, there exist degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_m$ such that:

(i) $\boldsymbol{a} < \boldsymbol{c}_i < \boldsymbol{a}'$ *for* $i = 1, 2, \cdots, m$,

(ii) $\boldsymbol{c}_i \upharpoonleft \boldsymbol{a}$ *for* $i = 1, 2, \cdots, m$,

(iii) $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_m$ *are independent,*

(iv) $\boldsymbol{c}_i^{(n)} = \boldsymbol{b}$ *for* $i = 1, 2, \cdots, m$.

PROOF. By Theorem 1, there exists a degree $\boldsymbol{d}_1$ such that $\boldsymbol{d}_1 | \boldsymbol{a}^{(n-1)}$ & $\boldsymbol{a}^{(n-1)} < \boldsymbol{d}_1 < \boldsymbol{a}^{(n)}$ & $\boldsymbol{d}_1' = \boldsymbol{b}$.

By making $n-1$ further applications of Theorem 1, we obtain degrees $\boldsymbol{d}_2, \boldsymbol{d}_3, \cdots, \boldsymbol{d}_{n-1}$ such that $\boldsymbol{a}^{(n-j)} < \boldsymbol{d}_j < \boldsymbol{a}^{(n-j+1)}$ & $\boldsymbol{d}_j | \boldsymbol{a}^{(n-j)}$ & $\boldsymbol{d}_j' = \boldsymbol{d}_{j-1}$ for $j=1, 2, \cdots, n-1$ and obtain independent degrees $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_m$ such that $\boldsymbol{a} < \boldsymbol{c}_i < \boldsymbol{a}'$ & $\boldsymbol{c}_i | \boldsymbol{a}$ & $\boldsymbol{c}_i' = \boldsymbol{d}_{n-1}$ for $i = 1, 2, \cdots, m$.

THEOREM 4. *A degree* $\boldsymbol{a}$ *is the completion of a infinite recursively enumerable degrees if and only if* $\boldsymbol{a} \geqq \boldsymbol{o}'$ *and* $\boldsymbol{a} | \boldsymbol{o}'$.

PROOF. It is easily deduced from Theorem 1 ((i)$\leftrightarrow$(v)).

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