

## On explicit formulas for the norm residue symbol

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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(Received Dec. 1, 1966)

Let  $p$  be an odd prime and let  $\mathbf{Z}_p$  and  $\mathbf{Q}_p$  denote the ring of  $p$ -adic integers and the field of  $p$ -adic numbers respectively. For each integer  $n \geq 0$ , let  $q_n = p^{n+1}$  and let  $\Phi_n$  denote the local cyclotomic field of  $q_n$ -th roots of unity over  $\mathbf{Q}_p$ . We fix a primitive  $q_n$ -th root of unity  $\zeta_n$  in  $\Phi_n$  so that  $\zeta_{n+1}^p = \zeta_n$ , and put  $\pi_n = 1 - \zeta_n$ ;  $\pi_n$  generates the unique prime ideal  $\mathfrak{p}_n$  in the ring  $\mathfrak{o}_n$  of local integers in  $\Phi_n$ . Let  $(\alpha, \beta)_n$  denote Hilbert's norm residue symbol in  $\Phi_n$  for the power  $q_n$  and let

$$(\alpha, \beta)_n = \zeta_n^{[\alpha, \beta]_n}$$

with  $[\alpha, \beta]_n$  in  $\mathbf{Z}_p$ , well determined mod  $q_n$ . The classical formulas for  $[\alpha, \beta]_n$  state that

$$\begin{aligned} [\zeta_n, \beta]_n &= q_n^{-1} T_n(\log \beta), & \beta \in 1 + \mathfrak{p}_n, \quad n \geq 0, \\ [\pi_n, \beta]_n &= -q_n^{-1} T_n(\zeta_n \pi_n^{-1} \log \beta), & \beta \in 1 + \mathfrak{p}_n, \quad n \geq 0, \\ [\alpha, \beta]_0 &= -q_0^{-1} T_0(\zeta_0 \alpha^{-1} \frac{d\alpha}{d\pi_0} \log \beta), & \alpha \in 1 + \mathfrak{p}_0, \quad \beta \in 1 + \mathfrak{p}_0^2, \end{aligned}$$

where  $T_n$  denotes the trace from  $\Phi_n$  to  $\mathbf{Q}_p$ <sup>1)</sup>.

In a previous note [7], we have announced formulas for  $[\alpha, \beta]_n$  which generalize the above formulas of Artin-Hasse. In the present paper, we shall prove those formulas and then discuss some related results.

As in the above, we retain most of the notations introduced in our earlier paper [6]. In particular, we denote by  $N_n$  the norm from  $\Phi_n$  to  $\mathbf{Q}_p$ , and by  $T_{n,m}$  and  $N_{n,m}$  the trace and the norm, respectively, from  $\Phi_m$  to  $\Phi_n$ ,  $m \geq n \geq 0$ ; for an automorphism  $\sigma$  of the union  $\Phi$  of all  $\Phi_n$ ,  $n \geq 0$ , we denote by  $\kappa(\sigma)$  the

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<sup>\*)</sup> The present research was supported in part by the National Science Foundation grant NSF-GP-4361.

1) See [1], [2], [5]. For the general theory of the norm residue symbol needed in the following, see [2], Chap. 12 or [4], II, § 11, § 19. It is to be noted that the symbol  $(\alpha, \beta)_n$  in [2] is the inverse of the same symbol in [4]. Here we follow the definition of  $(\alpha, \beta)_n$  in [2] as we did in [7].

unique  $p$ -adic unit such that  $\zeta_n^\sigma = \zeta_n^{\kappa(\sigma)}$  for every  $n \geq 0^{(2)}$ .

1. We shall first prove several elementary lemmas. For  $n \geq 0$ , we denote by  $\nu_n$  the normalized valuation on  $\Phi_n$  such that  $\nu_n(\pi_n) = 1$ .

LEMMA 1. Let  $\mathfrak{d}_n$  be the different of  $\Phi_n/\mathbf{Q}_p$ :  $\mathfrak{d}_n = q_n \mathfrak{p}_n^{-p^n}$ . Then

$$T_n(\mathfrak{d}_n \log(1 + \mathfrak{p}_n)) \equiv 0 \pmod{q_n}.$$

PROOF. Since the multiplicative group  $(1 + \mathfrak{p}_n)/(1 + \mathfrak{p}_n^2)$  is generated by the coset of  $\zeta_n$ , we have  $\log(1 + \mathfrak{p}_n) = \log(1 + \mathfrak{p}_n^2)$ . We shall show that  $T_n(\mathfrak{d}_n \log \alpha) \equiv 0 \pmod{q_n}$ , or, equivalently,  $\nu_n(q_n \log \alpha) \geq 2p^n$ , for any  $\alpha$  in  $1 + \mathfrak{p}_n^2$ . Let  $\alpha = 1 - \beta$  with  $\beta$  in  $\mathfrak{p}_n^2$  so that  $\log \alpha = -\sum_{i=1}^{\infty} i^{-1} \beta^i$ . It is sufficient to show  $\nu_n(q_n i^{-1} \beta^i) \geq 2p^n$  for  $i \geq 1$ . Let  $p^e$  ( $e \geq 0$ ) be the exact power of  $p$  dividing  $i$ . Then  $\nu_n(q_n i^{-1} \beta^i) \geq (n+1-e)(p-1)p^n + 2i \geq (n+1-e)(p-1)p^n + 2p^e$ . If  $e \leq n$ , then clearly  $(n+1-e)(p-1)p^n + 2p^e \geq (p-1)p^n \geq 2p^n$ . If  $e > n$ , then  $(n+1-e)(p-1)p^n + 2p^e = (2+2(p^{e-n}-1)-(e-n-1)(p-1))p^n \geq 2p^n$  because  $p^{e-n}-1 = (1+p-1)^{e-n}-1 \geq (e-n)(p-1)$ . Thus  $\nu_n(q_n i^{-1} \beta^i) \geq 2p^n$  in either case.

LEMMA 2.

- i)  $q_m^{-1} T_m(\mathfrak{d}_m \log(1 + \mathfrak{p}_n)) \equiv 0 \pmod{q_n}, \quad m \geq 2n+1,$
- ii)  $q_m^{-1} T_m(\mathfrak{d}_m \log(1 + \mathfrak{p}_n^{p^n})) \equiv 0 \pmod{q_n}, \quad m \geq n+1,$
- iii)  $q_m^{-1} T_m(\mathfrak{d}_m \log(1 + \mathfrak{p}_n^{2p^n})) \equiv 0 \pmod{q_n}, \quad m \geq n.$

PROOF. Clearly  $T_{n,m}(\mathfrak{d}_m) = T_{n,m}(q_m \mathfrak{p}_0^{-1})$  is contained in  $p^{m-n} q_m \mathfrak{p}_0^{-1}$ , and if  $m \geq 2n+1$ , the latter is contained in  $p^{n+1} q_m \mathfrak{p}_0^{-1} = q_m \mathfrak{d}_n$ . Hence, for  $m \geq 2n+1$ ,  $q_m^{-1} T_m(\mathfrak{d}_m \log(1 + \mathfrak{p}_n)) = q_m^{-1} T_n(T_{n,m}(\mathfrak{d}_m) \log(1 + \mathfrak{p}_n))$  is contained in  $T_n(\mathfrak{d}_n \log(1 + \mathfrak{p}_n))$ , and i) follows from Lemma 1.

Since  $(1 + \mathfrak{p}_n^{p^n})/(1 + \mathfrak{p}_n^{p^{n+1}})$  is generated by the coset of  $\zeta_0$ , we have  $\log(1 + \mathfrak{p}_n^{p^n}) = \log(1 + \mathfrak{p}_n^{p^{n+1}}) = \mathfrak{p}_n^{p^{n+1}}$ . Hence  $q_m^{-1} T_m(\mathfrak{d}_m \log(1 + \mathfrak{p}_n^{p^n})) = T_m(\mathfrak{p}_n) \equiv 0 \pmod{p^m}$ , and ii) holds for  $m \geq n+1$ . iii) can be proved similarly by using  $\log(1 + \mathfrak{p}_n^{2p^n}) = \mathfrak{p}_n^{2p^n} = \mathfrak{p}_0^2$  and  $T_m(\mathfrak{p}_0) \equiv 0 \pmod{q_m}$ .

Let  $A$  denote the ring of formal power series in  $T$  with coefficients in  $\mathbf{Z}_p$ :  $A = \mathbf{Z}_p[[T]]$ . For any nonzero element  $\xi$  in  $\mathfrak{o}_n$ , there exists a power series  $f(T)$  in  $A$  such that

$$\xi = f(\pi_n),$$

$$f(T) = \sum_{i=s}^{\infty} a_i T^i, \quad a_i \in \mathbf{Z}_p, \quad s = \nu_n(\xi) \geq 0.$$

Such  $f(T)$  will be simply called a power series for  $\xi$ . Let

2) However,  $(\alpha, \beta)_n$  in the formulas of [6], § 3, is the inverse of the same symbol in the present paper. See the footnote 1).

$$\frac{d\xi}{d\pi_n} = f'(\pi_n),$$

$$\delta_n(\xi) = \frac{\zeta_n}{\xi} \frac{d\xi}{d\pi_n},$$

where  $f'(T)$  denotes the formal derivative of  $f(T)$  with respect to  $T$ . Since a power series for  $\xi$  such as  $f(T)$  is not unique for  $\xi$ ,  $d\xi/d\pi_n$  and  $\delta_n(\xi)$  are not uniquely determined by  $\xi$ . However we can prove the following lemma on the values taken by  $\delta_n(\xi)$ .

LEMMA 3. *The values of  $\delta_n(\xi)$  which are obtained from all possible power series for  $\xi$  fulfil a residue class of  $\mathfrak{p}_n^{-1} \bmod \mathfrak{d}_n$ .*

PROOF. It is clear from the definition that  $\delta_n(\xi_n)$  always belongs to  $\mathfrak{p}_n^{-1}$ . In general, let  $f(T)$  and  $g(T)$  be power series in  $A$  such that  $f(\pi_n) = g(\pi_n)$ . Then

$$f(T) = g(T) + u(T)d(T),$$

where  $u(T)$  is a certain power series in  $A$  and  $d(T) = \sum_{i=0}^{p-1} (1-T)^{ip^n}$  is the minimal polynomial of  $\pi_n = 1 - \zeta_n$  over  $\mathbf{Q}_p^{(3)}$ . Differentiating the both sides, we obtain

$$f'(\pi_n) = g'(\pi_n) + u(\pi_n)d'(\pi_n).$$

Since  $\mathfrak{d}_n = (d'(\pi_n))$ , it follows that

$$f'(\pi_n) \equiv g'(\pi_n) \pmod{u(\pi_n)\mathfrak{d}_n}.$$

Now, suppose that both  $f(T)$  and  $g(T)$  are power series for  $\xi$ . Then the coefficient of  $T^i$  vanishes for  $0 \leq i < s = \nu_n(\xi)$  in both  $f(T)$  and  $g(T)$ . Since  $d(0) = p \neq 0$ , the same holds for the coefficient of  $T^i$  in  $u(T)$  for  $0 \leq i < s$ . Hence  $\nu_n(u(\pi_n)) \geq s$  and it follows from the above congruence that

$$\zeta_n \xi^{-1} f'(\pi_n) \equiv \zeta_n \xi^{-1} g'(\pi_n) \pmod{\mathfrak{d}_n}.$$

On the other hand, let  $\eta$  be any element of  $\mathfrak{d}_n = (d'(\pi_n))$ . Then there is a power series  $v(T)$  in  $A$  such that  $\eta = \zeta_n v(\pi_n) d'(\pi_n)$ . Let

$$h(T) = (1 + v(T)d(T))f(T).$$

Then  $h(T)$  is also a power series for  $\xi$  and  $\zeta_n \xi^{-1} h'(\pi_n) = \zeta_n \xi^{-1} f'(\pi_n) + \eta$ . Therefore the lemma is proved.

In the following, we regard  $\delta_n(\xi)$  as a multi-valued function of  $\xi \neq 0$  in  $\mathfrak{o}_n$ , representing any such value as described above.

LEMMA 4.

i) For  $\xi \neq 0$  and  $\eta \neq 0$  in  $\mathfrak{o}_0$ ,

$$\delta_n(\xi\eta) \equiv \delta_n(\xi) + \delta_n(\eta) \pmod{\mathfrak{d}_n}.$$

ii) For any  $\xi \neq 0$  in  $\mathfrak{o}_n$  and for any automorphism  $\sigma$  of  $\Phi$ ,

3) See [2], p. 151.

$$\delta_n(\xi^\sigma) \equiv \kappa(\sigma)\delta_n(\xi)^\sigma \pmod{\mathfrak{d}_n}.$$

iii) For  $m \geq n$  and for any unit  $\xi$  in  $\mathfrak{o}_n$ ,

$$\delta_m(\xi) \equiv p^{m-n}\delta_n(\xi) \pmod{\mathfrak{d}_m}.$$

PROOF. i) follows immediately from the fact that if  $f(T)$  is a power series for  $\xi$  and  $g(T)$  a power series for  $\eta$ , then  $f(T)g(T)$  is a power series for  $\xi\eta$ . To prove ii), let

$$u(T) = 1 - (1-T)^{\kappa(\sigma)} = \kappa(\sigma)T + \dots.$$

Then  $u(T)$  is a power series for  $\pi_n^\sigma$  and

$$u'(\pi_n) = \kappa(\sigma)(1-\pi_n)^{\kappa(\sigma)-1} = \kappa(\sigma)\zeta_n^{\sigma-1}.$$

Let  $f(T)$  be a power series for  $\xi$ . Then  $f(u(T))$  is a power series for  $\xi^\sigma$ , and

$$\begin{aligned} \delta_n(\xi^\sigma) &= \zeta_n(\xi^\sigma)^{-1}f'(u(\pi_n))u'(\pi_n) = \zeta_n(\xi^{-1})^\sigma f'(\pi_n)^\sigma \kappa(\sigma)\zeta_n^{\sigma-1} \\ &= \kappa(\sigma)(\zeta_n\xi^{-1}f'(\pi_n))^\sigma = \kappa(\sigma)\delta_n(\xi)^\sigma. \end{aligned}$$

Finally, let  $\xi$  be a unit in  $\mathfrak{o}_n$ . Then  $\xi$  is also a unit in  $\mathfrak{o}_m$  and every power series  $g(T)$  such that  $g(\pi_m) = \xi$  is a power series for  $\xi$  in  $\mathfrak{o}_m$ . Let  $f(T)$  be a power series for  $\xi$  in  $\mathfrak{o}_n$ . Since  $f(\pi_n) = \xi$ ,  $\pi_n = 1 - (1-\pi_m)^{p^{m-n}}$ ,  $f(1 - (1-T)^{p^{m-n}})$  is a power series for  $\xi$  in  $\mathfrak{o}_m$ . Computing  $d\xi/d\pi_m$  by means of this power series, we obtain immediately the formula in iii). Note that the both sides of the congruence in iii) are well determined mod  $\mathfrak{d}_m$  because  $\mathfrak{d}_m = p^{m-n}\mathfrak{d}_n$ .

Let  $\xi$  now be an arbitrary element of the multiplicative group  $\Phi_n^*$  of  $\Phi_n$ , i. e., an arbitrary nonzero element of  $\Phi_n$ . We write  $\xi$  in the form  $\xi = \xi_1/\xi_2$  with  $\xi_1 \neq 0$ ,  $\xi_2 \neq 0$  in  $\mathfrak{o}_n$ , and define

$$\delta_n(\xi) = \delta_n(\xi_1) - \delta_n(\xi_2).$$

By Lemma 4, i), we see that the values of  $\delta_n(\xi)$  again fulfil a residue class of  $\mathfrak{p}_n^{-1} \pmod{\mathfrak{d}_n}$ . Furthermore, i), ii) of Lemma 4 now hold for any  $\xi$  and  $\eta$  in  $\Phi_n^*$ . Hence  $\delta_n$  defines a so-called  $\kappa$ -homomorphism

$$\delta_n : \Phi_n^* \rightarrow \mathfrak{p}_n^{-1}/\mathfrak{d}_n.$$

$\delta_n$  is continuous in the sense that if  $\xi \equiv 1 \pmod{\mathfrak{p}_n^k}$ ,  $k \gg 0$ , then  $\delta_n(\xi) \equiv 0 \pmod{\mathfrak{d}_n}$ .

LEMMA 5. For  $m \geq n$  and for  $\xi$  in  $\Phi_m^*$ ,

$$\delta_n(N_{n,m}(\xi)) \equiv p^{-(m-n)}T_{n,m}(\delta_m(\xi)) \pmod{\mathfrak{d}_n}.$$

PROOF. Since  $\mathfrak{d}_m = p^{m-n}\mathfrak{d}_n$ , the both sides of the above are well determined mod  $\mathfrak{d}_n$ . By Lemma 4, i), it is sufficient to prove the lemma for  $\xi \equiv \pi_m$  and for a unit  $\xi$  in  $\mathfrak{o}_m$ . Since

$$\begin{aligned} N_{n,m}(\pi_m) &= \pi_n, & T_{n,m}(\zeta_m\pi_m^{-1}) &= p^{m-n}\zeta_n\pi_n^{-1}, \\ \delta_n(\pi_n) &= \zeta_n\pi_n^{-1}, & \delta_m(\pi_m) &= \zeta_m\pi_m^{-1}, \end{aligned}$$

the lemma holds for  $\xi = \pi_m$ . Let  $\xi$  be a unit in  $\mathfrak{o}_m$ . When  $\sigma$  ranges over all Galois automorphisms of  $\Phi_m/\Phi_n$ , we obtain from Lemma 4, ii), iii) that

$$\begin{aligned} p^{m-n}\delta_n(N_{n,m}(\xi)) &\equiv \delta_m(N_{n,m}(\xi)) \equiv \sum_{\sigma} \delta_m(\xi^{\sigma}) \\ &\equiv \sum_{\sigma} \kappa(\sigma)\delta_m(\xi)^{\sigma} \pmod{\mathfrak{d}_m}. \end{aligned}$$

Apply  $p^{-(m-n)}T_{n,m}$  to the above. Since

$$T_{n,m}(\mathfrak{d}_m) = T_{n,m}(q_m\mathfrak{p}_0^{-1}) = p^{m-n}q_m\mathfrak{p}_0^{-1} = p^{m-n}\mathfrak{d}_m,$$

we have

$$p^{m-n}\delta_n(N_{n,m}(\xi)) \equiv p^{-(m-n)} \sum_{\sigma} \kappa(\sigma)T_{n,m}(\delta_m(\xi)) \pmod{\mathfrak{d}_m}.$$

Here

$$\sum_{\sigma} \kappa(\sigma) \equiv \sum_{i=0}^{p^{m-n}-1} (1+iq_n) \equiv p^{m-n} \pmod{q_n}.$$

Since  $\xi$  is a unit,  $\delta_m(\xi)$  belongs to  $\mathfrak{o}_m$ . Hence  $p^{-(m-n)}T_{n,m}(\delta_m(\xi))$  is in  $\mathfrak{o}_n$ , and we obtain from the above

$$p^{m-n}\delta_n(N_{n,m}(\xi)) \equiv T_{n,m}(\delta_m(\xi)) \pmod{\mathfrak{d}_m},$$

which implies the congruence of the lemma.

2. For  $\alpha$  in  $\Phi_n^*$  and  $\beta$  in  $1+\mathfrak{p}_n$ ,  $n \geq 0$ , let

$$\langle \alpha, \beta \rangle_n = -q_n^{-1}T_n(\delta_n(\alpha) \log \beta)^{4)}.$$

Although the value of  $\langle \alpha, \beta \rangle_n$  depends upon the choice of the value of  $\delta_n(\alpha)$ , it is, in certain cases, well determined modulo a high power of  $p$ .

LEMMA 6. Let  $m \geq n$  and let  $\alpha$  be an arbitrary element of  $\Phi_m^*$ . In each one of the following three cases, the value of  $\langle \alpha, \beta \rangle_m$  is well determined mod  $q_n$ :

- i)  $m \geq 2n+1$  and  $\beta$  in  $1+\mathfrak{p}_n$ ,
- ii)  $m \geq n+1$  and  $\beta$  in  $1+\mathfrak{p}_n^{2^n}$ ,
- iii)  $m \geq n$  and  $\beta$  in  $1+\mathfrak{p}_n^{2^2p^n}$ .

PROOF. Since  $\delta_m(\alpha)$  is well determined mod  $\mathfrak{d}_m$ , the lemma follows immediately from Lemma 2.

In these three cases, we have

$$\begin{aligned} \langle \alpha_1\alpha_2, \beta \rangle_m &\equiv \langle \alpha_1, \beta \rangle_m + \langle \alpha_2, \beta \rangle_m \pmod{q_n}, \\ \langle \alpha, \beta_1\beta_2 \rangle_m &\equiv \langle \alpha, \beta_1 \rangle_m + \langle \alpha, \beta_2 \rangle_m \pmod{q_n}, \end{aligned}$$

for  $\alpha, \alpha_1, \alpha_2$  in  $\Phi_m^*$  and  $\beta, \beta_1, \beta_2$  in  $\Phi_n$  satisfying respective conditions.

LEMMA 7. Let  $l \geq m \geq n$ . Let  $\alpha$  be an element of  $\Phi_l^*$ , and  $\beta$  an element

4) The symbol  $\langle \alpha, \beta \rangle_n$  here is completely different from the same symbol defined in [6], § 1, p. 52.

of  $1+\mathfrak{p}_n$ . Suppose that  $\beta$  and  $m$  satisfy one of the three conditions in Lemma 6. Then

$$\langle N_{m,l}(\alpha), \beta \rangle_m \equiv \langle \alpha, \beta \rangle_l \pmod{q_n}.$$

PROOF. By Lemma 5,

$$\delta_m(N_{m,l}(\alpha)) \equiv p^{-(l-m)} T_{m,l}(\delta_l(\alpha)) \pmod{\mathfrak{d}_m}.$$

Multiply the both sides of the above by  $-\log \beta$  and apply  $q_m^{-1} T_m$ . The congruence of the lemma then follows immediately from Lemma 2.

For  $n \geq 0$ ,  $i \geq 1$ , let

$$\eta_i^{(n)} = 1 - \pi_n^i.$$

For any  $k \geq 0$ , the elements  $\eta_i^{(n)}$ ,  $i \geq k$ , generate the compact multiplicative group  $1+\mathfrak{p}_n^k$  topologically. We shall next prove a key lemma for such  $\eta_i^{(n)}$ .

LEMMA 8. For  $i \geq 1$ ,  $j \geq ((n+2)(p-1)+1)p^{n-1}$ ,

$$[\eta_i^{(n)}, \eta_j^{(n)}]_n = \langle \eta_i^{(n)}, \eta_j^{(n)} \rangle_n.$$

PROOF. Since  $n$  is fixed throughout the following proof, we shall write  $\eta_i$  for  $\eta_i^{(n)}$ ,  $i \geq 1$ . By Lemma 6, iii)  $\langle \eta_i, \eta_j \rangle_n$  is well determined mod  $q_n$ . Hence it is sufficient to prove the lemma for the following particular value of  $\delta_n(\eta_i)$ :

$$\delta_n(\eta_i) = -\zeta_n \sum_{r=1}^{\infty} i \pi_n^{ir-1}.$$

Since

$$\log \eta_j = -\sum_{s=1}^{\infty} s^{-1} \pi_n^{js},$$

we have

$$\begin{aligned} \langle \eta_i, \eta_j \rangle_n &= -q_n^{-1} \sum_{r,s \geq 1} i s^{-1} T_n(\zeta_n \pi_n^{ir+js-1}) \\ &= -q_n^{-1} \sum_{\substack{r,s \geq 1 \\ (r,s)=1}} \sum_{u=1}^{\infty} i (us)^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}). \end{aligned}$$

On the other hand, it is known that

$$[\eta_i, \eta_j]_n = \sum_{\substack{r',s' \geq 1 \\ (r',s')=1}} (ir'+js') [\pi_n, \eta_{ir'+js'}]_n,$$

where for each pair  $r, s$ , the integers  $r', s'$  are chosen so that  $rs' - sr' = 1$ <sup>5)</sup>. Using Artin-Hasse's formula

$$[\pi_n, \beta]_n = -q_n^{-1} T_n(\zeta_n \pi_n^{-1} \log \beta), \quad \beta \in 1+\mathfrak{p}_n,$$

we obtain

5) See [3].

$$\begin{aligned} [\eta_i, \eta_j]_n &= -q_n^{-1} \sum_{\substack{r, s \geq 1 \\ (r, s) = 1}} (ir' + js') T_n(\zeta_n \pi_n^{-1} \log \eta_{ir+js}) \\ &= q_n^{-1} \sum_{\substack{r, s \geq 1 \\ (r, s) = 1}} \sum_{u=1}^{\infty} (ir' + js') u^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}). \end{aligned}$$

Since  $rs' - sr' = 1$ , we see that

$$[\eta_i, \eta_j]_n - \langle \eta_i, \eta_j \rangle_n = q_n^{-1} \sum_{\substack{r, s \geq 1 \\ (r, s) = 1}} \sum_{u=1}^{\infty} s'(ir + js)(us)^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}).$$

However

$$\begin{aligned} q_n^{-1} \sum_{u=1}^{\infty} (ir + js) u^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}) &= -q_n^{-1} (ir + js) T_n(\zeta_n \pi_n^{-1} \log \eta_{ir+js}) \\ &= (ir + js) [\pi_n, \eta_{ir+js}]_n \\ &= [\pi_n^{ir+js}, 1 - \pi_n^{ir+js}]_n \\ &\equiv 0 \pmod{q_n^6}. \end{aligned}$$

Hence

$$\begin{aligned} [\eta_i, \eta_j]_n - \langle \eta_i, \eta_j \rangle_n \\ \equiv q_n^{-1} \sum_{r, s} \sum_{u=1}^{\infty} s'(ir + js)(us)^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}) \pmod{q_n}, \end{aligned}$$

where the outer sum on the right is taken over all pairs of integers  $r, s \geq 1$  with  $(r, s) = 1$  and with  $s$  divisible by  $p$ . We shall next show that each term on the right hand side of the above is divisible by  $q_n$ .

With  $r, s$ , and  $u$  fixed, let  $p^e$  be the exact power of  $p$  dividing  $us$ . Then  $e \geq 1$  by the assumption on  $s$ . Since  $j \geq ((n+2)(p-1)+1)p^{n-1}$ , we have

$$\begin{aligned} \nu_n(q_n^{-1}(us)^{-1} \zeta_n \pi_n^{u(ir+js)-1}) &= u(ir+js) - 1 - (n+1+e)(p-1)p^n \\ &\geq ((n+2)(p-1)+1)p^{n-1+e} - (n+1+e)(p-1)p^n. \end{aligned}$$

For real  $x$ , let  $f(x) = ((n+2)(p-1)+1)p^{n-1+x} - (n+1+x)(p-1)p^n$ . Then  $f(1) = p^n$  and  $f'(x) \geq 0$  for  $x \geq 1$ . Hence  $f(x) \geq p^n$  for  $x \geq 1$ . Since  $e \geq 1$ , it follows that

$$\nu_n(q_n^{-1}(us)^{-1} \zeta_n \pi_n^{u(ir+js)-1}) \geq p^n$$

so that

$$q_n^{-1} s'(ir + js)(us)^{-1} T_n(\zeta_n \pi_n^{u(ir+js)-1}) \equiv 0 \pmod{q_n}.$$

Thus we obtain

$$[\eta_i, \eta_j]_n \equiv \langle \eta_i, \eta_j \rangle_n \pmod{q_n},$$

and the lemma is proved. Note that  $[\eta_i, \eta_j]_n$  is a  $p$ -adic integer, determined only mod  $q_n$ . Note also that if  $j > ((n+1)(p-1)+1)p^n$ , then  $\eta_j$  is a  $q_n$ -th power of an element in  $1 + p_n^{p^{n+1}}$  and the equality of the lemma can be verified immedi-

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6) In general,  $(\alpha, 1-\alpha)_n = 1$ .

ately because both sides are simply zero.

LEMMA 9. Let  $m \geq 3n+1$ . Then

$$[N_{n,m}(\alpha), \beta]_n = \langle \alpha, \beta \rangle_m, \quad \alpha \in \Phi_m^*, \beta \in 1 + \mathfrak{p}_n.$$

PROOF. We first note that  $\langle \alpha, \beta \rangle_m$  is well determined mod  $q_n$  by Lemma 6, i). Let  $k = ((m-2n)(p-1)+1)p^m$ . It follows from  $m \geq 3n+1$ ,  $p \geq 3$  that  $k \geq ((m+2)(p-1)+1)p^{m-1}$ . Since  $\beta$  is in  $1 + \mathfrak{p}_n$ ,  $\beta^{p^n}$  is contained in  $1 + \mathfrak{p}_n^{p^n}$ , and  $\beta^{p^{m-n}}$  in  $1 + p^{m-2n}\mathfrak{p}_n^{p^n} = 1 + \mathfrak{p}_m^k$ . Hence  $\beta^{p^{m-n}}$  is the limit of a sequence of certain products of  $\eta_j^{(n)}$ ,  $j \geq ((m+2)(p-1)+1)p^{m-1}$ . Therefore it follows from Lemma 8 that

$$[\eta_i^{(m)}, \beta^{p^{m-n}}]_m = \langle \eta_i^{(m)}, \beta^{p^{m-n}} \rangle_m, \quad i \geq 1.$$

Using the well known properties of the norm residue symbol, we then have

$$\begin{aligned} (N_{n,m}(\eta_i^{(n)}), \beta)_n &= (\eta_i^{(m)}, \beta)_m^{p^{m-n}} = (\eta_i^{(m)}, \beta^{p^{m-n}})_m \\ &= \zeta_m^{[\eta_i^{(m)}, \beta^{p^{m-n}}]_m} = \zeta_m^{\langle \eta_i^{(m)}, \beta^{p^{m-n}} \rangle_m} \\ &= \zeta_m^{p^{m-n} \langle \eta_i^{(m)}, \beta \rangle_m}. \end{aligned}$$

Since the first term of the above is a  $q_n$ -th root of unity,

$$q_n p^{m-n} \langle \eta_i^{(m)}, \beta \rangle_m \equiv 0 \pmod{q_m}.$$

Therefore  $\langle \eta_i^{(m)}, \beta \rangle_m$  is a  $p$ -adic integer, and the last term of the above equalities can be written as  $\zeta_n^{\langle \eta_i^{(m)}, \beta \rangle_m}$ . Hence

$$[N_{n,m}(\eta_i^{(n)}), \beta]_n = \langle \eta_i^{(m)}, \beta \rangle_m, \quad i \geq 1.$$

Since  $1 + \mathfrak{p}_m$  is topologically generated by  $\eta_i^{(m)}$ ,  $i \geq 1$ , it follows that the lemma holds for any  $\alpha$  in  $1 + \mathfrak{p}_m$ .

If  $\alpha$  is an element of the group  $V$  of  $(p-1)$ -st roots of unity in  $\Phi_m$ , then  $[N_{n,m}(\alpha^{p^{-1}}), \beta]_n = \langle \alpha^{p^{-1}}, \beta \rangle_m = 0$  so that  $[N_{n,m}(\alpha), \beta]_n = \langle \alpha, \beta \rangle_m = 0$ . Hence the lemma again holds for such  $\alpha$ . Let  $\alpha = \pi_m$ . Then  $N_{n,m}(\alpha) = \pi_n$  and  $q_m^{-1} T_{n,m}(\delta_m(\alpha)) = q_m^{-1} T_{n,m}(\zeta_m \pi_m^{-1}) = q_n^{-1} \zeta_n \pi_n^{-1}$ . Hence the equality of the lemma in this case is nothing but the formula of Artin-Hasse. Since the group  $\Phi_m^*$  is generated by  $\pi_m$ ,  $V$ , and  $1 + \mathfrak{p}_m$ , the lemma is completely proved.

We are now able to prove the main result of the paper. For each  $n \geq 0$ , let  $\Phi'_n$  denote the intersection of all  $N_{n,m}(\Phi_m^*)$ ,  $m \geq n$ . Then

$$\Phi_n^* = U_0 \times \Phi'_n, \quad \Phi'_n = N_{n,m}(\Phi'_m), \quad m \geq n,$$

with  $U_0 = 1 + p\mathbf{Z}_p^{\times}$ . It follows in particular that for each  $\alpha$  in  $\Phi'_n$  and for any  $m \geq n$ , there exists an element  $\alpha'$  in  $\Phi'_m$  such that  $\alpha = N_{n,m}(\alpha')$ . Any such  $\alpha'$

7) See [6], §2, Proposition 9.



will be denoted by  $N_{n,m}^{-1}(\alpha)$ .

THEOREM 1. For  $\alpha$  in  $\Phi'_n$ ,  $\beta$  in  $1+\mathfrak{p}_n$ ,

$$[\alpha, \beta]_n = \langle N_{n,m}^{-1}(\alpha), \beta \rangle_m$$

for any  $m \geq 2n+1$ . More explicitly,

$$(\alpha, \beta)_n = \zeta_n^{-q_n^{-1} T_m \langle \zeta_m \alpha'^{-1} (d\alpha'/d\pi_m) \log \beta \rangle}$$

with  $\alpha' \in \Phi'_m$ ,  $N_{n,m}(\alpha') = \alpha$ ,  $m \geq 2n+1$ .

PROOF. Let  $l \geq m$ ,  $3n+1$  and  $\alpha'' = N_{m,l}^{-1}(\alpha')$  so that  $\alpha' = N_{m,l}(\alpha'')$ ,  $\alpha = N_{n,l}(\alpha'')$ . By Lemma 9,

$$[\alpha, \beta]_n = \langle \alpha'', \beta \rangle_l,$$

and by Lemma 7,

$$\langle \alpha', \beta \rangle_m \equiv \langle \alpha'', \beta \rangle_l \pmod{q_n}.$$

Hence

$$[\alpha, \beta]_n = \langle \alpha', \beta \rangle_m.$$

Let  $\alpha = \zeta_n$ . Then  $N_{n,m}^{-1}(\alpha) = \zeta_m$ . Since  $\delta_m(\zeta_m) = \zeta_m \zeta_m^{-1} (d\zeta_m/d\pi_m) = -1$ , the theorem gives us the first formula of Artin-Hasse:

$$[\zeta_n, \beta]_n = q_n^{-1} T_n(\log \beta), \quad \beta \in 1+\mathfrak{p}_n.$$

Next, let  $\alpha = \pi_n$ . Using the equalities for  $\pi_n$  and  $\pi_m$  given in the proof of Lemma 5, we see immediately that Theorem 1 in this case gives us the second formula of Artin-Hasse:

$$[\pi_n, \beta]_n = -q_n^{-1} T_n(\zeta_n \pi_n^{-1} \log \beta), \quad \beta \in 1+\mathfrak{p}_n.$$

3. In Theorem 1, if  $\beta-1$  is divisible by a higher power of  $\mathfrak{p}_n$ , then the formula  $[\alpha, \beta]_n = \langle N_{n,m}^{-1}(\alpha), \beta \rangle_m$  holds with a smaller  $m$ . The proof of the theorem shows that

$$[\alpha, \beta]_n = \langle N_{n,n+1}^{-1}(\alpha), \beta \rangle_{n+1} \quad \alpha \in \Phi'_n, \beta \in 1+\mathfrak{p}_n^{p^n},$$

$$[\alpha, \beta]_n = \langle \alpha, \beta \rangle_n, \quad \alpha \in \Phi'_n, \beta \in 1+\mathfrak{p}_n^{2p^n},$$

We shall next prove that the second equality holds not only for  $\alpha$  in  $\Phi'_n$  but also for arbitrary  $\alpha$  in  $\Phi_n^*$ . Since  $\Phi_n^* = U_0 \times \Phi'_n$ , it is sufficient to prove it for  $\alpha$  in  $U_0 = 1+p\mathbb{Z}_p$ .

For integers  $a \geq 1$ ,  $n \geq 0$ , let

$$s_n(a) = p^{-n} \sum_{\substack{0 \leq i \leq a \\ p^n | i}} (-1)^i i \binom{a}{i} = \sum_{0 \leq k p^n \leq a} (-1)^k k \binom{a}{k p^n}.$$

Using

$$\binom{a}{kp^n} \equiv \binom{ap}{kp^{n+1}} \pmod{p^8},$$

one sees immediately that

$$s_n(a) \equiv s_{n+1}(ap) \pmod{p}.$$

Let  $a > 0$ ,  $(a, p) = 1$ ,  $n > 0$ . Then  $kp^n \leq a$  implies  $kp^n \leq a-1$ . Hence

$$\begin{aligned} s_n(a) &= \sum_{0 \leq kp^n \leq a-1} (-1)^k \frac{a}{a-kp^n} k \binom{a-1}{kp^n} \\ &= \sum_{0 \leq kp^n \leq a-1} (-1)^k k \binom{a-1}{kp^n} \pmod{p}, \end{aligned}$$

namely,

$$s_n(a) \equiv s_n(a-1) \pmod{p}.$$

LEMMA 10. Let  $a \geq 2p^n$ ,  $n \geq 0$ . Then

$$s_n(a) \equiv 0 \pmod{p}.$$

PROOF. We use induction on  $n$ . For  $n = 0$ ,

$$\begin{aligned} s_0(a) &= \sum_{k=0}^a (-1)^k k \binom{a}{k} = \sum_{k=1}^a (-1)^k a \binom{a-1}{k-1} \\ &= -a(1-1)^{a-1} = 0. \end{aligned}$$

Let  $n > 0$  and  $a = bp + c$ ,  $b \geq 0$ ,  $p > c \geq 0$ . Then we have  $bp \geq 2p^n$ , and we see from the above remarks and from the induction assumption on  $s_{n-1}(b)$ ,  $b \geq 2p^{n-1}$  that

$$s_n(a) \equiv s_n(bp) \equiv s_{n-1}(b) \equiv 0 \pmod{p}.$$

LEMMA 11. Let  $a \geq 2p^n$ ,  $n \geq 0$ . Then

$$T_n(a\zeta_n\pi_n^{a-1}) \equiv 0 \pmod{p^{2n+1}}.$$

PROOF. Since

$$T_n(\zeta_n^i) = \begin{cases} 0, & p^n \nmid i, \\ -p^n, & p^n \mid i, \quad p^{n+1} \nmid i, \\ p^{n+1} - p^n, & p^{n+1} \mid i, \end{cases}$$

and since

$$a\zeta_n\pi_n^{a-1} = \sum_{i=0}^{a-1} (-1)^i a \binom{a-1}{i} \zeta_n^{i+1} = - \sum_{i=0}^a (-1)^i i \binom{a}{i} \zeta_n^i,$$

we obtain by Lemma 10

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8) See [1], Hilfssatz 2.

$$\begin{aligned} T_n(a\zeta_n\pi_n^{a-1}) &= p^n \sum_{\substack{0 \leq i \leq a \\ p^n | i}} (-1)^i i \binom{a}{i} - p^{n+1} \sum_{\substack{0 \leq i \leq a \\ p^{n+1} | i}} (-1)^i i \binom{a}{i} \\ &\equiv p^{2n} s_n(a) \pmod{p^{2n+1}} \\ &\equiv 0 \pmod{p^{2n+1}}. \end{aligned}$$

LEMMA 12. For any  $\xi$  in  $1 + \mathfrak{p}_n^{2p^n}$ ,

$$T_n(\delta_n(\xi)) \equiv 0 \pmod{p^{2n+1}}.$$

PROOF. Note first that since  $T_n(\mathfrak{d}_n) \equiv 0 \pmod{p^{2n+1}}$ ,  $T_n(\delta_n(\xi))$  is well determined mod  $p^{2n+1}$  (for any  $\xi$  in  $\mathcal{O}_n^*$ ). For the proof of the lemma, it suffices to show that for each  $a \geq 2p^n$ , there exists an element  $\xi_a$  such that  $\nu_n(\xi_a - 1) = a$  and  $T_n(\delta_n(\xi_a)) \equiv 0 \pmod{p^{2n+1}}$ . Let  $b = ap - (p-1)p^n$  and let

$$\alpha = 1 - \pi_n^{ap}, \quad \beta = \exp(p^{-1} \log \alpha).$$

We see from  $ap > p^{n+1}$  that  $\beta$  is defined,  $\alpha = \beta^p$ , and  $\nu_n(\beta - 1) \geq b$ . Let  $f(T)$  be a power series for  $\beta - 1$  and let

$$(1 + f(T))^p = 1 + g(T).$$

Then the coefficient of  $T^i$  vanishes for  $0 \leq i < b$  in both  $f(T)$  and  $g(T)$ . Since  $g(\pi_n) = -\pi_n^{ap}$ ,  $ap \geq b$ , we obtain from the proof of Lemma 3 that

$$g'(\pi_n) = p(1 + f(\pi_n))^{p-1} f'(\pi_n) \equiv -ap\pi_n^{ap-1} \pmod{\mathfrak{p}_n^b \mathfrak{d}_n}.$$

It then follows from  $b \geq 2p^{n+1} - (p-1)p^n > (p-1)p^n = \nu_n(p)$  that

$$\frac{f'(\pi_n)}{1 + f(\pi_n)} \equiv -\frac{ap\pi_n^{ap-1}}{1 - \pi_n^{ap}} \pmod{\mathfrak{d}_n}$$

so that

$$\delta_n(\beta) \equiv -\sum_{i=1}^{\infty} a\zeta_n\pi_n^{api-1} \pmod{\mathfrak{d}_n}.$$

Let

$$\xi_a = (1 - \pi_n^a)\beta^{-1}.$$

Then  $\nu_n(\beta - 1) \geq b > a$  implies  $\nu_n(\xi_a - 1) = a$ . We also see from the above that

$$\begin{aligned} \delta_n(\xi_a) &\equiv \delta_n(1 - \pi_n^a) - \delta_n(\beta) \\ &\equiv -\sum_{i=1}^{\infty} a\zeta_n\pi_n^{ai-1} + \sum_{i=1}^{\infty} a\zeta_n\pi_n^{api-1} \\ &\equiv -\sum_{\substack{i=1 \\ (i,p)=1}}^{\infty} a\zeta_n\pi_n^{ai-1} \pmod{\mathfrak{d}_n} \end{aligned}$$

so that

$$T_n(\delta_n(\xi_a)) \equiv -\sum_{\substack{i=1 \\ (i,p)=1}}^{\infty} T_n(a\zeta_n\pi_n^{ai-1}) \pmod{p^{2n+1}}.$$

However, it follows from Lemma 11 that if  $(i, p) = 1$ , then

$$T_n(a\zeta_n\pi_n^{ai-1}) = i^{-1}T_n(ai\zeta_n\pi_n^{ai-1}) \equiv 0 \pmod{p^{2n+1}}.$$

Hence  $T_n(\delta_n(\xi_a)) \equiv 0 \pmod{p^{2n+1}}$ , and the lemma is proved.

THEOREM 2. For any  $\alpha$  in  $\Phi_n^*$  and any  $\beta$  in  $1 + \mathfrak{p}_n^{2p^n}$ ,  $n \geq 0$ ,

$$[\alpha, \beta]_n = \langle \alpha, \beta \rangle_n,$$

namely

$$(\alpha, \beta)_n = \zeta_n^{-q_n T_n(\zeta_n \alpha^{-1} (d\alpha/d\pi_n)) \log \beta}.$$

PROOF. As noted earlier, we need only to prove the theorem for  $\alpha$  in  $U_0$ . Since  $\delta_n(\alpha) = 0$  in this case, we have to show

$$(\alpha, \beta)_n = 1, \quad \alpha \in U_0, \beta \in 1 + \mathfrak{p}_n^{2p^n}.$$

Let

$$\beta = \xi\eta, \quad \xi \in U_0, \eta \in \Phi'_n.$$

Then  $(\alpha, \xi)_n = 1$  because both  $\alpha$  and  $\xi$  are powers of  $1 + p$  with exponents in  $\mathbb{Z}_p$  and  $(1+p, 1+p)_n = 1$ . Since  $\eta$  is in  $\Phi'_n$  and  $\alpha$  in  $1 + \mathfrak{p}_n^{2p^n}$ , we know that

$$\begin{aligned} [\eta, \alpha]_n &= \langle \eta, \alpha \rangle_n = -q_n^{-1} T_n(\delta_n(\eta) \log \alpha) \\ &= -q_n^{-1} \log \alpha T_n(\delta_n(\eta)) \end{aligned}$$

with  $\log \alpha$  in  $p\mathbb{Z}_p$ . On the other hand, it follows from  $\beta \equiv 1 \pmod{\mathfrak{p}_n^{2p^n}}$ ,  $\xi \equiv 1 \pmod{p}$  that  $\eta \equiv 1 \pmod{\mathfrak{p}_n^{2p^n}}$ . Hence  $T_n(\delta_n(\eta)) \equiv 0 \pmod{p^{2n+1}}$  by Lemma 12. Therefore  $[\eta, \alpha]_n \equiv 0 \pmod{q_n}$ ,  $(\eta, \alpha)_n = 1$ , and

$$(\alpha, \beta)_n = (\alpha, \xi)_n (\alpha, \eta)_n = (\alpha, \xi)_n (\eta, \alpha)_n^{-1} = 1.$$

Hence the theorem is proved.

Obviously Theorem 2 is a generalization of the third formula stated in the introduction. We also note that as the proof of the theorem indicates, Lemma 12 is equivalent with the fact that the conductor of the abelian extension  $\Phi_n((1+p)^{q_n^{-1}})/\Phi_n$  is a divisor of  $\mathfrak{p}_n^{2p^n}$ .

4. For  $n \geq 0$ , let  $X_n$  denote the set of all elements  $\xi$  in  $\Phi_n$  such that

$$T_n(\xi \log(1 + \mathfrak{p}_n)) \equiv 0 \pmod{\mathbb{Z}_p}.$$

$X_n$  is a compact subgroup of the additive group of  $\Phi_n$ . As we saw in [6], the compact modules  $X_n$ ,  $n \geq 0$ , play important roles in the theory of cyclotomic fields. In Proposition 14 of that paper, we proved that for each  $n \geq 0$ , there exists a unique map

$$\phi_n : \Phi'_n \rightarrow X_n / q_n X_n$$

such that

$$(\alpha, \beta)_n = \zeta_n^{T_n(\phi_n(\alpha) \log \beta)},$$

for any  $\alpha$  in  $\Phi'_n$  and  $\beta$  in  $1+\mathfrak{p}_n$ . It was also proved that  $\phi_n$  is a surjective  $\kappa$ -homomorphism.

THEOREM 3. For  $m \geq 2n+1$ ,

$$\phi_n(\alpha) = -q_m^{-1}T_{n,m}(\delta_m(N_{n,m}^{-1}(\alpha))), \quad \alpha \in \Phi'_n.$$

PROOF. Let  $\phi'_n(\alpha)$  denote the right hand side of the above. By Theorem 1,

$$[\alpha, \beta]_n = T_n(\phi'_n(\alpha) \log \beta), \quad \beta \in 1+\mathfrak{p}_n.$$

This shows that  $\phi'_n(\alpha)$  is contained in  $\mathfrak{X}_n$  and is well determined mod  $q_n\mathfrak{X}_n$  for any choice of the value of  $\delta_m(N_{n,m}^{-1}(\alpha))$ . Hence we obtain a map

$$\psi' : \Phi'_n \rightarrow \mathfrak{X}_n/q_n\mathfrak{X}_n$$

such that

$$(\alpha, \beta)_n = \zeta_n^{T_n(\psi'_n(\alpha) \log \beta)}, \quad \alpha \in \Phi'_n, \beta \in 1+\mathfrak{p}_n.$$

The uniqueness of  $\phi_n$  then implies  $\phi'_n = \phi_n$ .

For  $m \geq n \geq 0$ , let  $\mathfrak{X}_{n,m}$  denote the set of all elements of the form

$$q_m^{-1}T_{n,m}(\delta_m(\xi))$$

with  $\xi$  ranging over  $\Phi_m^*$  and with  $\delta_m(\xi)$  taking all possible values of the multi-valued  $\delta_m(\xi)$ .  $\mathfrak{X}_{n,m}$  is a subgroup of the additive group of  $\Phi_n$ , containing the open subgroup  $q_m^{-1}T_{n,m}(\mathfrak{d}_m) = q_n^{-1}\mathfrak{d}_n$ . Since  $\Phi_m^* = U_0 \times \Phi'_m$  and  $\delta_m(U_0) \equiv 0 \pmod{\mathfrak{d}_m}$ , we have

$$\mathfrak{X}_{n,m} = q_m^{-1}T_{n,m}(\delta_m(\Phi'_m)) = q_m^{-1}T_{n,m}(\delta_m(N_{n,m}^{-1}(\Phi'_n))).$$

Let  $m \geq 2n+1$ . Since  $\phi_n : \Phi'_n \rightarrow \mathfrak{X}_n/q_n\mathfrak{X}_n$  is surjective, it follows from Theorem 3 that

$$\mathfrak{X}_n = \mathfrak{X}_{n,m} + q_n\mathfrak{X}_n.$$

However,  $\mathfrak{X}_n$  is a profinite  $p$ -group and  $\mathfrak{X}_{n,m}$  is an open subgroup of  $\mathfrak{X}_n$ . Hence we obtain from the above that  $\mathfrak{X}_n = \mathfrak{X}_{n,m}$  for  $m \geq 2n+1$ .

THEOREM 4. For any  $m \geq n$ ,

$$\mathfrak{X}_n = \mathfrak{X}_{n,m}.$$

In particular,  $\mathfrak{X}_n = \mathfrak{X}_{n,n}$ , and  $\mathfrak{X}_n$  consists of all elements of the form

$$q_n^{-1}\delta_n(\alpha) = q_n^{-1}\zeta_n\alpha^{-1}(d\alpha/d\pi_n), \quad \alpha \in \Phi_n^*.$$

PROOF. It is sufficient to show that  $\mathfrak{X}_{n,m} = \mathfrak{X}_{n,n}$  for  $m \geq n$ . By Lemma 5,

$$q_m^{-1}T_{n,m}(\delta_m(\xi)) \equiv q_n^{-1}\delta_n(N_{n,m}(\xi)) \pmod{q_n^{-1}\mathfrak{d}_n}$$

for any  $\xi$  in  $\Phi_m^*$ . Since  $q_n^{-1}\mathfrak{d}_n$  is contained in  $\mathfrak{X}_{n,n}$ , we see that  $\mathfrak{X}_{n,m}$  is a subgroup of  $\mathfrak{X}_{n,n}$ .

Let  $\xi$  be a unit of  $\mathfrak{o}_n$  and let  $\delta_n(\xi)$  be fixed. By Lemma 4, iii), one value of  $\delta_m(\xi)$  is given by  $p^{m-n}\delta_n(\xi)$ . Hence  $q_n^{-1}\delta_n(\xi) = q_m^{-1}T_{n,m}(\delta_m(\xi))$  is contained in

$\mathfrak{X}_{n,m}$ . On the other hand, since

$$q_n^{-1}\delta_n(\pi_n) = q_n^{-1}\zeta_n\pi_n^{-1} = q_m^{-1}T_{n,m}(\delta_m(\pi_m)),$$

$q_n^{-1}\delta_n(\pi_n)$  is also contained in  $\mathfrak{X}_{n,m}$ . Therefore  $\mathfrak{X}_{n,n}$  is a subgroup of  $\mathfrak{X}_{n,m}$ .

In [6], the compact module  $\mathfrak{X}_n$  was explicitly described as follows<sup>9)</sup>: Let  $G_n$  denote the Galois group of  $\Phi_n/\mathbb{Q}_p$  and let

$$\begin{aligned}\mu_n &= q_n^{-1}\zeta_n\pi_n^{-1} \\ \theta_n &= q_n^{-1}\sum_{i=0}^n \zeta_i^{-1}.\end{aligned}$$

Then the elements  $\theta_n^\sigma$ ,  $\sigma \in G_n$ , form a normal basis of  $\Phi_n$  over  $\mathbb{Q}_p$ , and  $\mathfrak{X}_n$  consists of all elements of the form

$$\sum_{\sigma} c_{\sigma}\mu_n^{\sigma} + \sum_{\sigma} d_{\sigma}\theta_n^{\sigma}$$

with arbitrary  $p$ -adic integers  $c_{\sigma}$  and  $d_{\sigma}$  satisfying  $\sum_{\sigma} d_{\sigma} = 0$ .

We like to note here that a similar description of the compact module  $\log(1+\mathfrak{p}_n)$  can be given as follows: Let

$$\lambda_n = p^{-n}\sum_{i=0}^n p^i\pi_i.$$

Then the elements  $\lambda_n^\sigma$ ,  $\sigma \in G_n$ , also form a normal basis of  $\Phi_n$  over  $\mathbb{Q}_p$ , and  $\log(1+\mathfrak{p}_n)$  consists of all elements of the form

$$ap + \sum_{\sigma} a_{\sigma}\lambda_n^{\sigma}$$

with arbitrary  $p$ -adic integers  $a$  and  $a_{\sigma}$  satisfying

$$\sum_{\sigma} \kappa(\sigma)a_{\sigma} \equiv 0 \pmod{q_n^{10)}.$$

The proof is obtained easily from the result on  $\mathfrak{X}_n$  stated above and from the fact that if we put

$$\theta'_n = p^{-n}(-1 + \sum_{i=0}^n p^i\zeta_i),$$

then we have

$$T_n(\theta_n^{\sigma}\theta_n^{\tau}) = \begin{cases} 1, & \sigma = \tau, \\ 0, & \sigma \neq \tau, \sigma, \tau \in G_n. \end{cases}$$

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9) See [6], §1, Theorem 1.

10) For  $\sigma$  in  $G_n$ ,  $\kappa(\sigma)$  is defined in the obvious manner. It is a  $p$ -adic integer well determined mod  $q_n$ .

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