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Formal functions and formal embeddings

Dedicated to Professor S. Iyanaga

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Introduction.

This work contains two theorems, among others, which determine the field of formal-rational functions, $\hat{K}$, along a closed algebraic set X in a projective space P and in an abelian variety A, respectively. For obvious reasons, we assume that X is connected and has a positive dimension. In the case of ambient variety P, the answer is that $\hat{K}$ is exactly the field of rational functions on P. If A is the ambient variety, $\hat{K}$ coincides with the field of rational functions on a certain abelian scheme A^* over a certain complete local ring R, which is derived from the given pair (X, A). For instance, if X generates A, then R is nothing but the base field of A and A^* is the maximal one, say $\mathrm{Al}(X, A)$, among those étale and proper (hence abelian) extensions of A which are dominated by the albanese variety of X. In the general case, the origin of A being chosen in X with no loss of generality, let A' be the abelian subvariety of A which is generated by X, and $A''=A/A'$. Then R is the completion of the local ring of A'' at the origin, and A^* is the unique etale extension of $A\times_{A''}\mathrm{Spec}(R)$ that induces the covering $\mathrm{Al}(X, A')$ of the closed fibre A'. (There exists a non-canonical isomorphism of A^* with the product $\mathrm{Al}(X, A')\times\mathrm{Spec}(R)$.)

In [1], a general problem was posed about the existence of a certain universal scheme associated with any given formal scheme. The results of this paper readily imply an affirmative and explicit solution to the problem when the given formal scheme is the completion of P (resp. A) along X as above. Namely, in this case, P itself (resp. A^* described above) is the universal solution, i.e., the most dominant (in the sense of an arbitrary small neighborhood of the image of X) scheme of finite type over the ring of formal-regular functions, which in this case, is k (resp. R). Meanwhile, Hartshorne gave an affirmative answer to the same problem, when X is a smooth (or more generally,

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locally complete intersection) subvariety with ample normal bundle in an arbitrary smooth algebraic variety. Though Hartshorne's result is most general with respect to the ambient variety, it does not cover our results for special ambient varieties for various reasons related to his cohomology-theoretical techniques. Some of the essential points are: (i) in our case, X is arbitrary and need not be locally complete intersection, and (ii) a smooth subvariety X of an abelian variety A can generate A without having ample normal bundle, though the ampleness of normal bundle implies the generation.

If the base field is the complex number field, then the general GAGA-techniques, due to J.-P. Serre, enable us to deduce from our results on formal-rational functions, the corresponding facts on formal-meromorphic functions as we pass from algebraic geometry to complex-analytic geometry. In other words, if P_{an} (resp. A_{an}) denotes the complex-analytic variety derived from P (resp. A), then the field $\hat{K}$ can be identified with the field of formal-meromorphic functions along X on P_{an} (resp. A_{an}), i.e., meromorphic functions on the formal neighborhood of X in P_{an} (resp. A_{an}). Thus, in this complex-analytic setup, one can draw immediate consequences of two kinds. Namely, on one hand, the result on formal-rational functions implies that if $X \to Z$ is any other embedding into an algebraic variety Z, which is formally equivalent to the given one $X \to P$ (resp. $\to A$), then these two are complex-analytically equivalent. (In fact, in the case of P, they are rationally equivalent.) On the other hand, the same result implies that every meromorphic function in a connected neighborhood of X extends to a global meromorphic function on P_{an} (resp. on a finite abelian covering of A_{an}, provided X generates A). The last result has been proven by complex-analytic methods by W. Barth (Münster).

Our proof of the theorems on formal-rational functions are based upon the theory of formal (or, "holomorphic" in the sense of Zariski) functions and formal schemes, which was first introduced by Zariski and then extended by Grothendieck. Using the main theorem of GFGA (formal geometry versus algebraic geometry) on coherent sheaves, due to Zariski and Grothendieck, we prove the following theorem, which plays an important role throughout the paper. Let $f: Z' \to Z$ be a proper morphism of algebraic varieties. Let X be a closed algebraic set in Z, and $X' = f^{-1}(X)$. Let $\hat{K}'$ (resp. $\hat{K}$) be the ring of formal-rational functions along X' (resp. X) on Z' (resp. Z), and let K' (resp. K) the field of rational functions on Z' (resp. Z). Then we have a canonical isomorphism: $\hat{K}' \approx [\hat{K} \otimes_K K']_0$. In particular, when $K' = K$, we get the birational invariance of the ring of formal-rational functions. Besides general techniques and theorems from the theory of formal schemes, we need certain special techniques for the special cases. Some of the special techniques for the case of projective spaces are adopted from [**1**], in which the theorem of formal-rational functions are proven under certain assumptions. We need and prove a special lemma for the case of abelian varieties, which asserts: If $f: V \to A$ is a proper morphism from a normal variety and there exists an embedding of X in V such that f induces an isomorphism of completions along X, then V is also an abelian variety and f is a separable isogeny, provided X generates A.

Notations and conventions. By a ring we shall mean a commutative ring with a unit element. When R is a ring we shall denote its total ring of fractions by $[R]_0$. All the schemes and formal schemes considered in this paper are tacitly assumed to be *locally noetherian.* Let us recall that a formal scheme is said to be locally noetherian (resp. noetherian) if each point has a neighborhood of the form Spf (A), where A is a noetherian ring which is complete with respect to a topology defined by the powers of an ideal (resp. if it is locally noetherian and if its underlying topological space is quasi-compact).

§ 1. The ring of formal-rational functions.

Let Z^* be a (locally noetherian) formal scheme. For each affine open set $U \subset Z^*$, put

$$M^0_{Z^*}(U) = [\mathcal{O}_{Z^*}(U)]_0 .$$

Then we obtain[1] a presheaf of rings $M^0_{Z^*}$. Let M_{Z^*} be the associated sheaf, and put

$$K(Z^*) = H^0(Z^*, M_{Z^*}) .$$

This is called *the ring of formal-rational*[2] *functions.*

Let $\xi \in K(Z^*)$. We define the *pole sheaf* of ξ to be the ideal sheaf $\lambda_\xi^{-1}(\mathcal{O}_{Z^*})$ with the homomorphism $\lambda_\xi : \mathcal{O}_{Z^*} \to M_{Z^*}$ defined by $\lambda_\xi(g) = \xi g$, where $\mathcal{O}_{Z^*}$ is viewed as a subsheaf of M_{Z^*} in a natural manner. The pole sheaf of ξ will be denoted by P_ξ. P_ξ is coherent. In fact, pick up an affine open set U of Z^*, say so small that ξ belongs to $M^0_{Z^*}(U)$. Write $\xi = ab^{-1}$ with $a, b \in H^0(U, \mathcal{O}_{Z^*})$, where b is a non-zero-divisor of $H^0(U, \mathcal{O}_{Z^*})$. We can view λ_a as a homomorphism $\mathcal{O}_U \to \mathcal{O}_U$, and clearly $\lambda_a^{-1}(b\mathcal{O}_U)$ is equal to $P_\xi | U$. Since $b\mathcal{O}_U$ is coherent, so is

1) Let f be a non-zero-divisor of the ring $A = \mathcal{O}_{Z^*}(U)$. Since $\mathcal{O}_{Z^*}$ is a coherent sheaf of rings (EGA I 10.10.3), and since there exists an equivalence between the category of finite A-modules and the category of coherent $\mathcal{O}_{Z^*}$-Modules (cf. EGA I, 10.10), the multiplication by f defines an injective homomorphism $\mathcal{O}_{Z^*} \xrightarrow{f} \mathcal{O}_{Z^*}$, and hence for every affine open $V \subset U$ the restriction $f | V$ is a non-zero-divisor of $\mathcal{O}_{Z^*}(V)$. Therefore there exists a natural restriction map $M^0{}_{Z^*}(U) \to M^0{}_{Z^*}(V)$ for $V \subset U$.

2) If Z is a locally noetherian scheme, then it can be viewed as a formal scheme with (0) as the defining Ideal, and the above construction can be applied to Z; in this case $K(Z)$ is the ring of *rational* functions.

$\lambda_a^{-1}(b\mathcal{O}_U)$.

The sheaf ξP_ξ, which might be called the sheaf of numerators of ξ, is also a coherent sheaf of ideals of $\mathcal{O}_{Z^*}$, because it is the image of $\lambda_\xi : P_\xi \to \mathcal{O}_{Z^*}$. The closed subset $\mathrm{Supp}(\xi P_\xi)$ coincides with the support of the section ξ.

LEMMA (1.1). *Assume Z^* is affine. Then $K(Z^*)$ is the total ring of fractions of $H^0(Z^*, \mathcal{O}_{Z^*})$.*

PROOF. Pick any $\xi \in K(Z^*)$. Then the pole sheaf P_ξ is coherent. Since Z^* is affine, P_ξ is generated by $H^0(P_\xi)$. Let $A = H^0(\mathcal{O}_{Z^*})$ and $P = H^0(P_\xi)$. If P contains a non-zero-divisor of A, then we are done. It suffices that the ideal P is not contained in any associated prime $Q \in \mathrm{Ass}(A)$. For such Q, there exists $a \in A$ such that $0 \to Q \to A \xrightarrow{a} A$ is exact. If $Q \supset P$, then P_ξ lies in the kernel of $\mathcal{O}_{Z^*} \xrightarrow{a} \mathcal{O}_{Z^*}$, which is impossible. Q. E. D.

Let Z^* be a formal scheme. In speaking of inclusion, union and intersection of closed formal subschemes Z_i^*, $i=1$ and 2, we mean:

$$Z_1^* \subset Z_2^* \Leftrightarrow I(Z_1^*) \supset I(Z_2^*),$$

$$I(Z_1^* \cup Z_2^*) = I(Z_1^*) \cap I(Z_2^*),$$

and

$$I(Z_1^* \cap Z_2^*) = I(Z_1^*) + I(Z_2^*),$$

where I denotes the ideal sheaf.

Let Z^* be a noetherian formal scheme, and let X be a closed subset. Let Z^{**} denote the completion of Z^* along X; then there exists a canonical morphism of local-ringed spaces $c : Z^{**} \to Z^*$. c is flat, therefore it sends non-zero-divisors of $\mathcal{O}_{Z^*}$ to non-zero-divisors of $\mathcal{O}_{Z^{**}}$, and hence induces a homomorphism $\nu : K(Z^*) \to K(Z^{**})$.

LEMMA (1.2). *The homomorphism*

$$\nu : K(Z^*) \to K(Z^{**})$$

is injective if the following condition is satisfied;

(1.2.1) *if $Z^* = Z_1^* \cup Z_2^*$ with closed subschemes Z_i^* and if $Z_2^* \cap X = \emptyset$, then $Z_1^* = Z^*$.*

PROOF. Pick any $\xi \in K(Z^*)$ with $\nu(\xi) = 0$. Let P be the pole sheaf of ξ in $\mathcal{O}_{Z^*}$. Then $Q_1 = \xi P_\xi$ is an ideal sheaf in $\mathcal{O}_{Z^*}$ such that $Q_1 \cdot \mathcal{O}_{Z^{**}} = (0)$ by $\nu(\xi) = 0$. Z^* being noetherian, Artin-Rees theorem implies that there exists an integer $q > 0$ such that, if Q_2 denotes the q-th power of the annihilator Ideal of Q_1 in $\mathcal{O}_{Z^*}$, we have $Q_1 \cap Q_2 = (0)$. Let Z_i^* be the formal subscheme of Z^* defined by Q_i, $i = 1, 2$. Then $Z_1^* \cup Z_2^* = Z^*$ and $Z_2^* \cap X = \emptyset$ because, by $Q_1\mathcal{O}_{Z^{**}}$ (0), $\mathrm{Supp}(Q_1) \cap X = \emptyset$ and hence $Q_2 = \mathcal{O}_{Z^*}$ in a neighborhood of X. By (1.2.1) we must have $Z_1^* = Z^*$, i. e. $Q_1 = (0)$. Namely, $\xi P = (0)$, which implies $\xi = 0$.

Q. E. D.

A ring A is called a *primary ring* if it has only one prime ideal, or equivalently, if it is a local ring whose maximal ideal is a nil-ideal. A ring A is said to be *semi-primary* if it is a direct product of a finite number of primary rings. If A is a noetherian ring without embedded primes (i. e. Ass (A) consists of the minimal primes), then the total ring of fractions $[A]_0$ is semi-primary. We shall say that a formal scheme Z^* is *primary* if (1) it is noetherian, (2) the stalks of the structure sheaf have no embedded primes and (3) the only zero-divisors of $K(Z^*)$ are nilpotent elements. (We shall see later that $K(Z^*)$ is actually a primary ring if Z^* is primary.) We shall say that a formal scheme Z^* is *strongly primary* if (1*) it is noetherian, (2*) the zero ideal of $\mathcal{O}_{Z^*,x}$ is a primary ideal for each point $x \in Z^*$ and (3*) the underlying space of Z^* is connected. It is easy to see that a strongly primary formal scheme is primary.

REMARK (1.3). If Z is a noetherian scheme, the following conditions are equivalent, as one can easily see:

i) Z is irreducible (as a topological space) and the local rings $\mathcal{O}_{Z,x}$ $(x \in Z)$ have no embedded primes,

ii) Z is strongly primary,

iii) Z is primary.

But they are quite different in the category of noetherian formal schemes.

LEMMA (1.4). *Let Z be an integral noetherian scheme, let X be a connected closed set of Z and let $\hat{Z}$ be the completion of Z along X. Assume that, for every $x \in X$, the completion of the local ring $\mathcal{O}_{Z,x}$ is an integral domain. Then $\hat{Z}$ is a reduced, strongly primary formal scheme, and $K(\hat{Z})$ is a field.*

PROOF. For each $x \in X$, $\mathcal{O}_{\hat{Z},x}$ and $\mathcal{O}_{Z,x}$ have the same completion. Therefore $\mathcal{O}_{\hat{Z},x}$ is an integral domain. Thus $\hat{Z}$ is reduced and strongly primary. The last statement follows from this, as we shall see in Lemma (1.7).

LEMMA (1.5). *Let*

$$A \longrightarrow B \overset{u}{\underset{v}{\rightrightarrows}} C$$

be an exact diagram of rings and ring-homomorphisms (i. e. $A \approx \{b \in B \mid u(b) = v(b)\}$). If B is semi-primary, then so is A.

PROOF. Let $B = B_1 \times \cdots \times B_n$ be the decomposition into primary rings and let e_i be the unit element of B_i. Then $1 = \sum e_i$, $e_i^2 = e_i$ and $e_i e_j = 0$ $(i \neq j)$. Let R be the smallest equivalence relation on the set $\{1, 2, \cdots, n\}$ satisfying

$$u(e_i) \cdot v(e_j) \neq 0 \quad \Longrightarrow \quad (i, j) \in R\,.$$

Let S_α $(1 \leqq \alpha \leqq m)$ be the R-equivalence classes, and put $B'_\alpha = \prod_{i \in S_\alpha} B_i$, $E_\alpha = \sum_{i \in S_\alpha} e_i$. Then $u(E_\alpha) \cdot v(E_\beta) = 0$ for $\alpha \neq \beta$, $1 = \sum_\alpha u(E_\alpha) = \sum_\alpha v(E_\alpha)$, therefore $u(E_\alpha) = v(E_\alpha)$. It follows that $A = \prod_\alpha (A \cap B_\alpha)$. Therefore we can now assume, without loss of generality, that any two indices in $\{1, 2, \cdots, n\}$ are R-equivalent; we will prove, under this assumption, that A is primary. Let P_i be the prime ideal of B corresponding to e_i (i.e. $P_i=$ the radical of the kernel of the projection $B\to B_i$), and put $p_i=P_i\cap A$. We claim $p_1=p_2=\cdots=p_n$. Since the relation $p_i=p_j$ defines an equivalence relation on $\{1, 2, \cdots, n\}$, it suffices to show that

$$u(e_i)\cdot v(e_j)\neq 0 \Longrightarrow p_i=p_j\,.$$

Let $x\in p_i$. Then $e_ix^\nu=0$ for some $\nu>0$, so $u(e_ix^\nu)=u(e_i)\cdot v(x^\nu)=0$. Since $e_jb\to u(e_i)v(e_jb)$ is a ring-homomorphism from the primary ring $B_j=e_jB$ into the ring $e_{ij}C$ (where e_{ij} is the non-zero idempotent $u(e_i)\cdot v(e_j)$ of C), its kernel is a nil-ideal. Since $u(e_i)\cdot v(e_jx^\nu)=0$, we have $(e_jx^\nu)^\mu=e_jx^{\nu\mu}=0$ for some $\mu>0$, which shows $x\in p_j$. By symmetry, we have $p_i=p_j$ as wanted. Put $p=p_1=p_2=\cdots$. Then $p=P_1\cap\cdots\cap P_n\cap A$. Therefore if $x\in A-p$ then x is a unit (in B, hence) in A, and if $x\in p$ then x is nilpotent. Q. E. D.

LEMMA (1.6). *Let Z^* be a formal scheme, and suppose that it has a finite open covering $\{U_i\}$ such that $M_{Z^*}(U_i)$ are semi-primary. Then $K(Z^*)$ is semi-primary. In particular, $K(Z^*)=[K(Z^*)]_0$.*

PROOF. There exists an exact diagram of rings and ring-homomorphisms

$$K(Z^*)=H^0(Z^*, M_{Z^*})\longrightarrow B \rightrightarrows C\,,$$

where $B=\prod_i M_{Z^*}(U_i)$ and $C=\prod_{i,j} M_{Z^*}(U_i\cap U_j)$, to which we can apply the preceding lemma.

COROLLARY (1.7). *Let Z^* be a formal scheme. If Z^* is reduced, then $K(Z^*)$ is reduced. If Z^* is noetherian and the stalks of $\mathcal{O}_{Z^*}$ have no embedded primes, then $K(Z^*)$ is semi-primary. If Z^* is primary, then $K(Z^*)$ is primary.*

PROOF. The first assertion is obvious. Under the hypothesis of the second assertion, Z^* is covered by a finite number of open sets of the form Spf(A), where A is noetherian and complete. Therefore every maximal ideal m of A is open and defines a point $z\in Z^*$. The local ring $\mathcal{O}_{Z^*,z}$ contains A_m and has the same completion as A_m (cf. EGA I. 10.1.5 and 0.7.6.17), hence it is faithfully flat over A_m. Since $\mathcal{O}_{Z^*,z}$ has no embedded primes, A_m has no embedded primes either. This being true for all maximal ideals, A itself has no embedded primes and $[A]_0$ is semi-primary. Therefore $K(Z^*)$ is semi-primary by the preceding lemma. The last assertion is now obvious.

REMARK. When Z^* is the completion of a reduced algebraic scheme (over a field) along a closed subscheme, Z^* is reduced by a theorem of Chevalley (or by Grothendieck's theory of excellent rings, cf. EGA IV. 7.8.3). Thus $K(Z^*)$ is reduced and semi-primary, i.e. a finite direct sum of fields.

The case of a usual scheme is much simpler. Namely, the following lemma is well known.

LEMMA (1.8). *If Z is a primary scheme, then it has a unique generic point,* *say x, and M_Z is a constant sheaf equal to the local ring $\mathcal{O}_{Z,x}$. If Z is a noetherian scheme whose local rings have no embedded primes, and if $x_1, \cdots, x_n$ are the generic points of the irreducible components $Z_1, \cdots, Z_n$ of Z, then $M_Z = M_1 \oplus \cdots \oplus M_n$ and $K(Z) = \mathcal{O}_{Z,x_1} \times \cdots \times \mathcal{O}_{Z,x_n}$, where M_i is the constant sheaf $\mathcal{O}_{Z,x_i}$ on Z_i and is zero outside of Z_i.*

Proof is simple and obvious.

§2. The effect of proper morphisms.

This section is essentially based upon the following theorem of Grothendieck.

THEOREM (2.1). (EGA III, 5.1.4). *Let A be an I-adically complete noetherian ring with an ideal I. Let $Y = \operatorname{Spec}(A)$, $Y' = \operatorname{Spec}(A/I)$ and $f: Z \to Y$ a proper morphism of schemes. Let $X = f^{-1}(Y')$, and let*

$$\hat{Y} = \hat{Y}/Y', \qquad \hat{Z} = \hat{Z}/X$$

denote the completions of Y and Z along Y' and X respectively. Let $c: \hat{Z} \to Z$ be the completion morphism. Then the functor $F \mapsto c^*(F)$ is an equivalence of the category of coherent sheaves on Z with the category of coherent sheaves on $\hat{Z}$.

PROPOSITION (2.2). *Let the notations and the assumptions be the same as in the above theorem. Then we have a canonical isomorphism*

$$\nu : K(Z) \simeq K(\hat{Z})$$

induced by c.

PROOF. Since $f: Z \to Y$ is proper, f maps every closed point of Z to a closed point of Y. Since A is I-adically complete, Y' contains all the closed points of Y. Therefore $X = f^{-1}(Y')$ contains all the closed points of Z and hence meets any non-empty closed subscheme of Z. Thus the condition (1.2.1) is trivially satisfied with Z for Z^*. Hence, by Lemma (1.2), ν is injective. To prove the surjectivity of ν, pick any $\xi \in K(\hat{Z})$. Let $\hat{P}$ be the pole sheaf of ξ in $\mathcal{O}_{\hat{Z}}$, and $\hat{Q}$ be the ideal sheaf $\xi\hat{P}$ in $\mathcal{O}_{\hat{Z}}$. By the above theorem of Grothendieck there exist ideal sheaves P and Q in $\mathcal{O}_Z$ such that $\hat{P} = P\mathcal{O}_{\hat{Z}}$ and $\hat{Q} = Q\mathcal{O}_{\hat{Z}}$. Consider the commutative diagram of sheaves

$$\begin{array}{ccc} \mathcal{O}_{\hat{Z}} & \xrightarrow{\hat{\alpha}} & \underline{\operatorname{Hom}}(\hat{P}, \mathcal{O}_{\hat{Z}}) \\ \uparrow & & \uparrow \\ \mathcal{O}_Z & \xrightarrow[\alpha]{} & \underline{\operatorname{Hom}}(P, \mathcal{O}_Z) \end{array}$$

where α maps $\varphi \in \mathcal{O}_Z(U)$ to the homomorphism $P(U) \to \mathcal{O}_Z(U)$ induced by the multiplication by φ, and $\hat{\alpha}$ is defined similarly. The vertical maps are the completion maps, so that $\operatorname{Ker}(\hat{\alpha})$ is the completion of $\operatorname{Ker}(\alpha)$ by the equivalence of coherent sheaf categories. By definition of $\hat{P}$, $\mathrm{Ker}(\hat{\alpha})=(0)$ and hence $\mathrm{Ker}(\alpha)=(0)$. This means that P is locally everywhere generated by non-zero-divisors.

Using (2.1) again, we have $\mathrm{Hom}(\hat{P}, \hat{Q})\approx \mathrm{Hom}(P, Q)$. Multiplication by ξ defines an element $\lambda_\xi \in \mathrm{Hom}(\hat{P}, \hat{Q})$; let $\mu \in \mathrm{Hom}(P, Q)$ be such that λ_ξ is the completion of μ. Let z be an arbitrary point of Z and let U be an affine neighborhood of z; then $P(U)$ contains non-zero-divisors of $\mathcal{O}_Z(U)$. Let b be such a non-zero-divisor and put $\mu(b)=a\in Q(U)\subset \mathcal{O}_Z(U)$. The quotient $ab^{-1}\in M_Z(U)$ is independent of the choice of b, as one can check immediately. Therefore these local rational functions piece together to a rational function $\xi'\in K(Z)$, and μ is the multiplication by ξ'. It is now clear that $\xi=\nu(\xi')$.

Q. E. D.

We say that a homomorphism of rings $A\to B$ is *admissible* if the non-zero-divisors of A are mapped to non-zero-divisors of B. If B is flat over A then $A\to B$ is admissible.

Let $f: Z'\to Z$ be a morphism of finite type of schemes. We say that f is *quasi-admissible*, if for each pair of points $z'\in Z'$ and $z=f(z')$, there exist affine neighborhoods, $\mathrm{Spec}(B)$ of z and $\mathrm{Spec}(A)$ of z, such that

(2.3.1) f maps $\mathrm{Spec}(B)$ into $\mathrm{Spec}(A)$, and

(2.3.2) $A\to B$ is admissible.

It is easy to see that, if f is quasi-admissible, then (2.3.2) holds for any pair of affine open sets $\mathrm{Spec}(B)\subset Z'$ and $\mathrm{Spec}(A)\subset Z$ which satisfy (2.3.1). If f is quasi-admissible then there is a canonical homomorphism of sheaves of rings $M_Z\to M_{Z'}$, hence a canonical homomorphism $K(Z)\to K(Z')$.

We say that f is *admissible* (resp. *locally birational*) if, in the above definition, we can replace (2.3.2) by the stronger condition

(2.3.3) there exists an element $a\in A$ which is a non-zero-divisor in A as well as in B, scuh that the induced map $A_a\to B_a$ is flat[3], (resp. by the still stronger condition

(2.3.4) there exists a non-zero-divisor a in A for which we have a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r] & A_a \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$

3) A_a and B_a denote the localizations of A and B by the powers of a.

where $A \to B$ and $A \to A_a$ are the canonical maps and $B \to A_a$ is injective[4]. The condition (2.3.4) is equivalent to saying that $A \to B$ is admissible and the induced homomorphism $[A]_0 \to [B]_0$ is an isomorphism (note that B is an A-algebra of finite type).

We say that f is *birational* if it is locally birational and $M_Z \to f_*(M_{Z'})$ is an isomorphism.

LEMMA (2.4). *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Z' & \longleftarrow & W' = Z' \times_Z W \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Z & \underset{g}{\longleftarrow} & W \end{array}$$

where g is flat and f (hence also h) is of finite type. Then h is admissible (resp. locally birational, resp. birational) if f is so.

PROOF. Since all three properties are local with respect to the bases, we may assume that Z and W are affine: $Z = \operatorname{Spec}(A)$ and $W = \operatorname{Spec}(C)$. Moreover, we may assume that Z' is covered by a finite number of affine open sets $U_i = \operatorname{Spec}(B_i)$ for which (2.3.3) (resp. (2.3.4)) holds. Since C is flat over A, $A \to C$ and $B_i \to B_i \otimes C$ are admissible and (2.3.3) (resp. (2.3.4)) remains valid after tensoring with C. Thus h is admissible (resp. locally birational) if f is so. Now, if f is birational, then we have an exact diagram of rings

$$[A]_0 \longrightarrow \prod_i [B_i]_0 \rightrightarrows \prod_{i,j} [B_{ij}]_0$$

where $B_{ij} = \Gamma(U_i \cap U_j, \mathcal{O}_{Z'})$. Put $K = [A]_0$. Since C is flat over A, $[C]_0$ is flat over K. Thus, tensoring the above diagram with $[C]_0$ over K and noting that $[B_i]_0 \approx K$ and $[B_i \otimes_A C]_0 \approx [C]_0$, we see that

$$[C]_0 \longrightarrow \prod_i [B_i \otimes C]_0 \rightrightarrows \prod_{i,j} [B_{ij}]_0 \otimes_K [C]_0$$

is exact. $C \to B_{ij} \otimes_A C$ is admissible (cf. footnote 4)) and $B_{ij} \to B_{ij} \otimes_A C$ is also admissible (by flatness), therefore $[B_{ij}]_0 \otimes_K [C]_0$ is a subring of $[B_{ij} \otimes_A C]_0$. Thus

$$[C]_0 \longrightarrow \prod_i [B_i \otimes_A C]_0 \rightrightarrows \prod_{i,j} [B_{ij} \otimes_A C]_0$$

is exact, which proves that

$$K(W) \longrightarrow K(W') = \Gamma(W, h_*(M_{W'}))$$

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4) If Spec (B_1) is open in Spec (B) then B_1 is flat over B. So we may replace B by B_1 in (2.3.3). Therefore, the property "$Z' \to Z$ is admissible" is a local property with respect to both Z' and Z. But, despite its name, local birationality is not a local property with respect to Z'. Example: Z = two intersecting lines, Z' = two disjoint lines viewed as the normalization of Z.

is an isomorphism for $W = \operatorname{Spec}(C)$. Since $\operatorname{Spec}(C)$ can be taken arbitrarily small, this means $M_W \approx h_*(M_{W'})$. Q. E. D.

REMARK (2.5). Let $f: Z' \to Z$ be a morphism of finite type of noetherian schemes. According to the theorem of generic flatness (SGA IV. 6.11 or EGA IV. 6.9.1) the following condition is sufficient for f to be admissible:

(2.5.1) Z is *reduced*, the local rings of Z' have no embedded primes, and f maps every irreducible component of Z' generically onto some irreducible component of Z.

We now prove the "birational invariance of the ring of formal-rational functions".

THEOREM (2.6). *Let $f: Z' \to Z$ be a proper birational morphism of schemes. Let X be a closed subset of Z and let $X' = f^{-1}(X)$; put $\hat{Z} = Z_{/X}$, $\hat{Z}' = Z'_{/X'}$. Then the cannonical morphism $\hat{f}: \hat{Z}' \to \hat{Z}$ induces an isomorphism*

$$M_{\hat{Z}} \simeq \hat{f}_*(M_{\hat{Z}'}) \tag{2.6.1}$$

and hence also

$$K(\hat{Z}) \simeq K(\hat{Z}'). \tag{2.6.2}$$

PROOF. Let $U = \operatorname{Spec}(A)$ be an affine open set in Z. It suffices to show (2.6.2) for $Z = U$ and $Z' = f^{-1}(U)$. Therefore we assume that $Z = \operatorname{Spec}(A)$. Let $\hat{A}$ be the completion of A by the powers of the ideal of X, and consider the base change

$$\begin{array}{ccc} Z' & \longleftarrow & W' = Z' \times_Z W \\ \downarrow & & \downarrow \\ Z = \operatorname{Spec}(A) & \longleftarrow & W = \operatorname{Spec}(\hat{A}). \end{array}$$

Now, Z (resp. Z') is canonically isomorphic to the completion $\hat{W}$ of W along the inverse image of X in W (resp. $\hat{W}'$ of W' along the inverse image of X' in W'). Since $\hat{A}$ is flat over A, $W' \to W$ is proper and birational. Therefore, in the commutative diagram

$$\begin{array}{ccc} K(W') & \longrightarrow & K(\hat{W}') = K(\hat{Z}') \\ \uparrow & & \uparrow \\ K(W) & \longrightarrow & K(\hat{W}) = K(\hat{Z}) \end{array}$$

$K(W) \to K(W')$ is an isomorphism by birationality and the horizontal arrows are isomorphisms by (2.2). Hence $K(\hat{Z}) \approx K(\hat{Z}')$, Q. E. D.

Next we consider more general situations.

THEOREM (2.7). *Let $f: Z' \to Z$ be a proper morphism of noetherian schemes; let X be a closed subset of Z and let $X' = f^{-1}(X)$; put $\hat{Z} = Z_{/X}$ and $\hat{Z}' = Z'_{/X'}$. Assume that*

(2.7.1) *the local rings of Z and Z' have no embedded primes, and f maps* *each irreducible component of Z' onto some irreducible component of Z; and*

(2.7.2) *one of the following conditions is satisfied:*

I) *Z is affine,*

II) *f is admissible and $K(Z')$ is a finite $K(Z)$-module,*

III) *Z and $\hat{Z}$ are reduced.*

Then there is a canonical isomorphism

$$[K(Z')\otimes_{K(Z)}K(\hat{Z})]_0 \simeq K(\hat{Z}') . \tag{2.7.3}$$

PROOF. Case I). Let Z be affine: $Z=\mathrm{Spec}(A)$. Let $\hat{A}$ be the I-adic completion of A, where I is the ideal of X. Put $W=\mathrm{Spec}(\hat{A})$ and $W'=Z'\times_Z W$. Let $\mathrm{Spec}(B)\subset Z'$ be an arbitrary affine open set. Then, by (2.7.1), $A\to B$ is admissible and $\mathrm{Ass}_A(B)\subset \mathrm{Ass}(A)$. It follows from this that $\mathrm{Ass}_{\hat{A}}(B\otimes_A\hat{A})\subset \mathrm{Ass}(\hat{A})$ by [Bourbaki, Alg. Comm. Ch. IV, 2.6. Th. 2]. Therefore $\hat{A}\to B\otimes\hat{A}$ is also admissible. So both $Z'\to Z$ and $W'\to W$ are quasi-admissible and there are canonical homomorphisms $K(Z)\to K(Z')$ and $K(W)\to K(W')$. As in the preceding proof, we have commutative diagrams

$$\begin{array}{ccccc} Z' & \longleftarrow & W' & \longleftarrow & \hat{W}'=\hat{Z}' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & W & \longleftarrow & \hat{W}=\hat{Z}, \end{array} \qquad \begin{array}{ccc} K(Z') & \longrightarrow & K(W')\approx K(\hat{Z}') \\ \uparrow & & \uparrow \\ K(Z) & \longrightarrow & K(W)\approx K(\hat{Z}) . \end{array}$$

Therefore we have only to prove

$$[K(Z')\otimes_{K(Z)}K(W)]_0 \simeq K(W') . \tag{2.7.4}$$

W' is covered by affine open sets of the form $\mathrm{Spec}(B\otimes_A\hat{A})$. Both $B\to B\otimes\hat{A}$ and $\hat{A}\to B\otimes\hat{A}$ are admissible, hence we have $[B\otimes_A\hat{A}]_0=[[B]_0\otimes_A[\hat{A}]_0]_0$. Now, by (2.7.1), $K(Z')$ is the direct product of the local rings of the generic points of the irreducible components of Z', and such direct decomposition induces a direct decomposition of $K(W')$. Therefore, to prove (2.7.4), we may assume that Z' is irreducible. Then $[B]_0$ is independent of the open set $\mathrm{Spec}(B)$; more precisely, the restriction map $[B\otimes_A\hat{A}]_0\to[B'\otimes_A\hat{A}]_0$ is an isomorphism for any $\mathrm{Spec}(B')\subset\mathrm{Spec}(B)\subset Z'$. Therefore we get (2.7.4).

Case II). Suppose that f is admissible and $K(Z')$ is a finite $K(Z)$-module. This condition continues to hold when we replace Z by an affine open set U, and Z' by $f^{-1}(U)$, because $K(Z')$ is simply the direct product of the local rings of the generic points of Z'. We may also suppose that Z and Z' are irreducible. Let $(U_i)_{1\leqq i\leqq n}$, $U_i=\mathrm{Spec}(A_i)$, be an affine covering of Z. Then we have an exact diagram of rings

$$K(\hat{Z}) \longrightarrow \prod_i M_{\hat{Z}}(\hat{U}_i) \rightrightarrows \prod_{i,j} M_{\hat{Z}}(\hat{U}_{ij})$$

where $U_{ij}=U_i\cap U_j$. Put $L=K(Z')$ and $K=K(Z)$. Then L and K are primary rings, and L is a flat K-module of finite type. Therefore

$$L\otimes_K K(\hat{Z})\longrightarrow\prod_i L\otimes_K M_{\hat{Z}}(U_i)\rightrightarrows\prod_{i,j} L\otimes_K M_{\hat{Z}}(\hat{U}_{ij})$$

is exact. All rings in this diagram are semi-primary (since L is finite over K). By Case I) proven above, we can rewrite it as

$$L\otimes_K K(\hat{Z})\longrightarrow\prod_i M_{\hat{Z}'}(f^{-1}(U_i))\rightrightarrows\prod_{i,j} M_{\hat{Z}'}(f^{-1}(U_{ij}))\,.$$

This means $L\otimes_K K(\hat{Z})=K(\hat{Z}')$, as was wanted.

Case III). Suppose Z and $\hat{Z}$ are reduced. In order to prove (2.7.3) we can assume that Z and Z' are irreducible. Put $L=K(Z')$, $K=K(Z)$. K is a field, and L is a primary ring which is a localization of a K-algebra of finite type. Let $\mathfrak{m}$ be the maximal ideal of L, let $\bar{L}=L/\mathfrak{m}$ and let $x_1,\cdots,x_n\in L$ be such that their images in $\bar{L}$ constitute a transcendence basis of $\bar{L}$ over K. If $0\neq\varphi(x)\in K[x_1,\cdots,x_n]=K[x]$, then $\varphi(x)\notin\mathfrak{m}$ and so $\varphi(x)$ is a unit in L. Therefore $K\subset K(x)\subset L$ and L is finite over the field $K(x)$.

Let Z be covered by m affine open sets $(U_i)_{1\leqq i\leqq m}$. The theorem being true when $m=1$, we proceed by induction on m. Put $Y=U_1\cap\cdots\cap U_{m-1}$, $W=U_m\cap Y$, $Y'=f^{-1}(Y)$ and $W'=f^{-1}(W)$. Then we can apply the induction hypothesis to Y as well as to W. We have an exact diagram of rings

$$K(\hat{Z})\longrightarrow K(\hat{Y})\times K(\hat{U}_m)\underset{r_2p_2}{\overset{r_1p_1}{\rightrightarrows}}K(\hat{W})$$

where p_i are projections and r_i are restrictions. From this we want to derive the exactness of

$$[L\otimes_K K(\hat{Z})]_0\longrightarrow[L\otimes_K K(\hat{Y})]_0\times[L\otimes_K K(\hat{U}_m)]_0\rightrightarrows[L\otimes_K K(\hat{W})]_0\,.$$

Since $[L\otimes K(\hat{Y})]_0=K(\hat{Y}')$, $[L\otimes K(\hat{U}_m)]_0=K(\hat{U}'_m)$ where $U'_m=f^{-1}(U_m)$, and $[L\otimes K(\hat{W})]_0=K(\hat{W}')$, the above diagram would prove $[L\otimes_K K(\hat{Z})]_0=K(\hat{Z}')$, i.e. (2.7.3).

Since $K(\hat{Y})$, $K(\hat{U}_m)$ and $K(\hat{W})$ are direct products of fields, the following lemma will complete our proof.

LEMMA (2.8). *Let L and K be as abave; let $\hat{K}$, K_1, K_2 and K_3 be finite direct products of fields containing K such that $K_1\subset K_3$, $K_2\subset K_3$ and $\hat{K}=K_1\cap K_2$. Thus there exists an exact diagram*

$$\hat{K}\longrightarrow K_1\times K_2\underset{p_2}{\overset{p_1}{\rightrightarrows}}K_3\,,$$

where p_i $(i=1,2)$ are the projections. Then the induced diagram

$$[L\otimes_K\hat{K}]_0\longrightarrow[L\otimes_K K_1]_0\times[L\otimes_K K_2]_0\rightrightarrows[L\otimes_K K_3]_0$$

is also exact.

PROOF. Since $K\subset K(x_1)\subset K(x_1, x_2)\subset \cdots \subset K(x_1, \cdots, x_n)\subset L$, it suffices to consider the following two cases:

(1) L is a simple transcendental extension field of K: $L=K(t)$.

(2) L is a primary ring which is a finite K-module.

In the second case we can omit $[\ ]_0$ in the formula because $L\otimes K$ and $L\otimes K_i$ are semi-primary, and the assertion is immediate (because any K-module is flat over K). Now let us prove the first case. We are to prove $[K_1[t]]_0\cap[K_2[t]]_0 =[\hat{K}[t]]_0$, where t is an indeterminate over K_3. Let $K_3=\Phi_1\times\cdots\times\Phi_r$ be the decomposition of K_3 into fields, and let $1=e_1+\cdots+e_r$ be the corresponding decomposition of 1 into idempotents. Then $[K_3[t]]_0=\Phi_1(t)\times\cdots\times\Phi_r(t)$. When F is a field and $u(t)=g(t)/f(t)$, $g(t)\in F[t]$, $f(t)\in F[t]$ with f and g relatively prime and f monic, we shall call the expression $g(t)/f(t)$ the standard form of the rational function $u(t)$. Then it is unique and is invariant under extension of the coefficient field F. Now pick any element $h\in[K_1[t]]_0\cap[K_2[t]]_0$ and write

$$h=(h_1, \cdots, h_r), \qquad h_i=g_i(t)/f_i(t)\in\Phi_i(t)$$

where g_i/f_i is the standard form of h_i. Put $g(t)=(g_1(t), \cdots, g_r(t))$, $f(t)=(f_1(t), \cdots, f_r(t))$, so that $h=g(t)/f(t)$. We claim that $g(t), f(t)\in K_1[t]$. In fact, if $K_1=\Psi_1\times\cdots\times\Psi_s$ is the decomposition of K_1 into fields and $1=e_1'+\cdots+e_s'$, $e_i'\in\Psi_i$, then we may assume that $e_1'=e_1+\cdots+e_p$, $e_2'=e_{p+1}+\cdots+e_q$, etc., and hence Ψ_1 is a subfield of $\Phi_1\times\cdots\times\Phi_p$. Let $\sigma_j:\Psi_1\to\Phi_j$ $(1\leqq j\leqq p)$ be the projection (which is an isomorphism onto a subfield of Φ_j). Since $h\in[K_1[t]]_0 =\Psi_1(t)\times\cdots\times\Psi_s(t)$, he_1' has a standard form $he_1'=A(t)/B(t)$ with A, B in $\Psi_1[t]$. Then $\sigma_j(A)/\sigma_j(B)$ is the standard form of h_j and hence $\sigma_j(A)=g_j$, $\sigma_j(B)=f_j$ $(1\leqq j\leqq p)$. Therefore $(g_1, \cdots, g_p)=A\in\Psi_1[t]$, $(f_1, \cdots, f_p)=B\in\Psi_1[t]$. The same is true for each of $\Psi_2, \cdots, \Psi_s$, hence $g, f\in K_1[t]$. By the same reason we have $g, f\in K_2[t]$, therefore $g, f\in\hat{K}[t]$ and $h\in[\hat{K}[t]]_0$, as was wanted. Thus we have proven Lemma (2.8) and Theorem (2.7).

DEFINITIONS (2.9). Let Z be a scheme, X a closed subscheme and $\hat{Z}$ the completion of Z along X. We shall say

(2.9.1) that X is $G1$ in Z if the canonical map $H^0(Z, \mathcal{O}_Z)\to H^0(\hat{Z}, \mathcal{O}_{\hat{Z}})$ is an isomorphism,

(2.9.2) that X is $G2$ in Z if $K(\hat{Z})$ is a finite module over $K(Z)$,

(2.9.3) that X is $G3$ in Z if the canonical map $K(Z)\to K(\hat{Z})$ is an isomorphism.

(The letter G is intended to suggest "generate".)

REMARKS (2.10). For simplicity let us consider here the case when Z is an algebraic variety over a field k such that $H^0(Z, \mathcal{O}_Z)=k$. Then $K(Z)$ is a field, and $K(\hat{Z})$ is a finite direct sum of fields. If X is $G1$ or $G3$ in Z then X must be connected. If we assume that $\hat{Z}$ is primary and that k is algebraically closed, then we have $G3\Rightarrow G2\Rightarrow G1$. In fact, $K(\hat{Z})$ is a field, and if $\xi\in H^0(\hat{Z}, \mathcal{O}_{\hat{Z}})$

$\subset K(\hat{Z})$, $\xi \notin k$, then ξ would be analytically independent over k and $H^0(\hat{Z}, \mathcal{O}_{\hat{Z}})$ would contain the formal power series ring $k[[\xi_{-c}]]$, where $c \in k$ is such that $\xi - c = 0$ on X, so $K(\hat{Z})$ would have infinite transcendence degree over k.

If X is $G1$ in Z and if $f: Z \to Z'$ is a dominant morphism to a variety Z' with $\dim Z' > 0$, then $\dim f(X) > 0$. In fact, if $f(X)$ were a point Q, then $H^0(\hat{Z}, \mathcal{O}_{\hat{Z}})$ would contain the completion of the local ring $\mathcal{O}_{Z',Q}$, which contradicts the assumption.

Let X be $G1$ in Z, and let $f: Z \to A$ be a morphism into an abelian variety A such that $f(X)$ passes through the origin O_A (i.e. the zero element) of A. Assume that $f(Z)$ generates A. Then $f(X)$ also generates A. In fact, if $B \subset A$ is the abelian subvariety generated by $f(X)$, then the composite morphism $Z \to A \to A/B$ sends X into a point while the image of Z generates A/B, therefore $A = B$ by what we have just remarked.

Hironaka [1] has proved that, if Z is a smooth variety and X is a smooth and complete subvariety with $\dim X > 1$ and $\operatorname{codim}_Z X = 1$ such that the normal bundle of X in Z is ample, then X is $G3$ in Z. In the next paragraph we shall prove, among other things, that the condition $\dim X > 1$ can be relaxed to $\dim X > 0$. For higher codimensions, Hartshorne recently proved that, if Z and X are smooth, if X is complete and $\dim X > 0$, and if the normal bundle of X in Z is ample in his sense, then X is $G2$ in Z.

We need one more definition. Under the assumption made at the beginning of these remarks (2.10), let R be a normal complete local ring containing k such that the residue field R/m_R (where m_R is the maximal ideal of R) is canonically isomorphic to k (in short, a k-rational normal complete local ring). Put $Z_R = Z \times_k \operatorname{Spec}(R)$, and let $\hat{Z}_R$ denote the completion of Z_R along the closed set $X \times y$, where y is the unique closed point of $\operatorname{Spec}(R)$. We shall say that X is *universally* Gi, $(i = 1, 2, 3)$, if, for any such R, $X \times y$ is Gi in Z_R.

If Z is complete, then Z is universally $G3$ in Z itself by (2,2). Does "X is Gi in Z" always imply "X is universally Gi in Z"? We do not know the answer.

Back to the general case, we can derive the following connectedness theorem from Theorem (2.7). (See (2.7.3).)

PROPOSITION (2.11). *With the notations and the assumptions of* (2.7), *let us assume that* X *is* $G3$ *in* Z. *Then* $f^{-1}(X)$ *is* $G3$ *in* Z'. *In particular,* $f^{-1}(X)$ *is connected if* Z' *is irreducible.*

PROPOSITION (2.12). *Let* Z *be an integral noetherian scheme, and let* X *be a closed subscheme which is* $G2$ *in* Z. *Then the degree* $[K(\hat{Z}) : K(Z)]$ *bounds the index of the canonical homomorphism* $\hat{\pi}_1(X) \to \hat{\pi}_1(Z)$ *induced by the inclusion, where* $\hat{Z}$ *denotes the completion of* Z *along* X. ($\hat{\pi}_1$ *denotes the fundamental group in the sense of Abhyankar-Grothendieck.*)

PROOF. Since the fundamental groups are by definition profinite complete, the image of the homomorphism is closed. But, for every finite Galois (and in particular, étale) connected covering $f: Z' \to Z$, (2.7.3) implies that the number of connected components of $\hat{Z}'$ (i. e., the same of $f^{-1}(X)$) is bounded by the degree $[K(\hat{Z}): K(Z)]$. This means that this degree bounds the index of the homomorphism from the Galois group of the induced connected covering of X to the Galois group of f. (2.12) follows.

The following theorem plays an important role in § 4.

THEOREM (2.13). *Let $Y_1, \cdots, Y_n$ be algebraic varieties over a field k in Weil's sense (i. e. integral schemes of finite type over* Spec(k) *such that $K(Y_i)$ are regular extensions of k). Let $e_i \in Y_i$ be a k-rational, geometrically normal point of Y_i, and let $W_i \subset Y_i$ be a closed set containing e_i and universally G3 in Y_i, for each $i=1, \cdots, n$. Put $Z=Y_1 \times \cdots \times Y_n$, $X_i=e_1 \times \cdots \times e_{i-1} \times W_i \times e_{i+1} \times \cdots \times e_n$, and $X=X_1 \cup \cdots \cup X_n$. Then X is universally G3 in Z.*

PROOF. It follows from our assumptions that $Y_1 \times \cdots \times Y_i$ is a variety over k, and that $e_1 \times \cdots \times e_i$ is a geometrically normal point of $Y_1 \times \cdots \times Y_i$ (cf. EGA IV. 6.14.1). Therefore, by induction, it is enough to consider the case $n=2$.

Let R, m_R, y be the same as in (2.10). We have to prove that $K(Z_R) \simeq K(\hat{Z}_R)$. Here $Z_R=Y_1 \times Y_2 \times \mathrm{Spec}(R)$, $X_R=(W_1 \times e_2 \times y) \cup (e_1 \times W_2 \times y)$ and $\hat{Z}_R$ is the completion of Z_R along X_R. Let $\hat{Z}_1$ (resp. $\hat{Z}_2$) denote the completion of Z_R along $W_1 \times e_2 \times y$ (resp. $e_1 \times W_2 \times y$).

Let o_i $(i=1, 2)$ be the local ring of $e_i \times y$ on $Y_i \times \mathrm{Spec}(R)$, and $\hat{o}_i$ its completion; o_i is normal by EGA IV. 6.14.1, hence $\hat{o}_i$ is also normal as o_i is excellent (cf. EGA IV. 7.8.3). Let F, F_1, F_2 denote the fields of fractions of R, $\hat{o}_1$, $\hat{o}_2$ respectively. Put $K_i=K(Y_i)$, $i=1, 2$. Then, by the definition of "universally $G3$", we have

$$K(\hat{Z}_1)=F_2(K_1), \qquad K(\hat{Z}_2)=F_1(K_2).$$

Therefore we have canonical inclusions

$$K(Z_R)=F(K_1, K_2) \subset K(\hat{Z}_R) \subset K(\hat{Z}_1) \cap K(\hat{Z}_2)=F_2(K_1) \cap F_1(K_2),$$

where all fields under consideration are viewed as subfields of $[\hat{o}_{12}]_0$, o_{12} being the local ring of $e_1 \times e_2 \times y$ on Z_R. We want to show $F_2(K_1) \cap F_1(K_2)=F(K_1, K_2)$; this is a purely algebraic problem and is contained in the following lemma.

LEMMA (2.14). *Notations being as above, the fields F_1 and F_2 are linearly disjoint over F, and we have $F_1(K_2) \cap F_2(K_1)=F(K_1, K_2)$.*

PROOF. Let $\varphi_1, \cdots, \varphi_p \in F_1$ be linearly independent over F. We want to show that they are linearly independent over F_2. Clearing the denominators, we may assume that $\varphi_i \in \hat{o}_1$ for all i. Suppose that there is a relation $\sum f_i \varphi_i =0$ with $f_i \in F_2$ and, say, $f_1 \neq 0$. Again we may assume that $f_i \in \hat{o}_2$ for all i.

Let m_2 be the maximal ideal of O_{Y_2,e_2}. Since $\hat{o}_2$ is a noetherian integral domain, by Krull's Durchschnittssatz there exists an integer $\nu > 0$ such that $f_1 \notin m_2^\nu \hat{o}_2$. Now, since $\hat{o}_2$ is the complete tensor product[5] of O_{Y_2,e_2} and R over k, we have

$$\hat{o}_2/m_2^\nu \hat{o}_2 = R \otimes_k (O_{Y_2,e_2}/m_2^\nu)\,.$$

Similarly,

$$\hat{o}_{12}/m_2^\nu \hat{o}_{12} = \hat{o}_1 \otimes_k (O_{Y_2,e_2}/m_2^\nu)\,.$$

Moreover, $\hat{o}_{12}$ is flat over $\hat{o}_2$ since $Y_1 \times Y_2 \times \operatorname{Spec}(R)$ is flat over $Y_2 \times \operatorname{Spec}(R)$ (cf. Bourbaki, Alg. Comm. Chap. III. 5. Prop. 4), hence we have $m_2^\nu \hat{o}_{12} \cap \hat{o}_2 = m_2^\nu \hat{o}_2$. Let (ω_j) be a k-basis of $O_{Y_2,e_2}/m_2^\nu$, and write

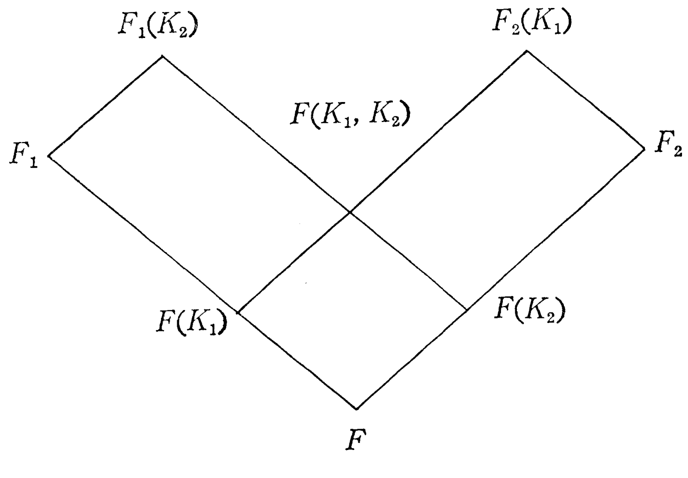
$$f_i \bmod m_2^\nu \hat{o}_2 = \sum_j a_{ij} \otimes \omega_j\,, \qquad a_{ij} \in R\,.$$

Then, taking the relation $\sum f_i \varphi_i = 0$ modulo $m_2^\nu \hat{o}_{12}$, we get

$$\sum_j \Big(\sum_i a_{ij}\varphi_i\Big) \otimes \omega_j = 0\,,$$

therefore all $a_{ij} = 0$. But this is absurd since $f_1 \bmod m_2^\nu \hat{o}_2 \neq 0$.

The second assertion of the lemma follows from the first, by considering the following diagram:



In fact, by a well-known property of linear disjointness, $F_1(K_2)$ and $F_2(K_1)$ are linearly disjoint over $F(K_1, K_2)$, hence the assertion. Q. E. D.

§ 3. Embeddings into projective spaces.

Let P^N be the projective space of dimension N over a field k, where $N \geqq 2$. We are interested in the function field of the completion of P^N along a con-

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5) Complete tensor product=produit tensoriel complété. Cf. EGA 0_I. 7.7, Bourbaki: Alg. Comm. Chap. III. § 2. ex. 38, and Nagata [**5**], § 42.

nected closed subscheme X. Let us first consider the most special case in which X is a linear subspace $P^1 \subset P^N$.

LEMMA (3.1). *P^1 is universally G3 in P^N.*

PROOF. Take a k-rational normal complete local ring R as in (2.10). We have to prove that $K(\hat{P}_R) = K(P_R)$, where $P_R = P^N \times \mathrm{Spec}(R)$ and $\hat{P}_R$ is the completion of P_R along $P^1 \times y$. Choose a homogeneous coordinate system $(z_0, z_1, \cdots, z_N)$ of P^N such that P^1 is defined by $z_2 = \cdots = z_N = 0$. Put $v_i = z_i/z_0$ $(1 \leqq i \leqq N)$. Let Q be the point on P^1 defined by $z_1 = \cdots = z_N = 0$, and let $f: V \to P^N$ be the quadratic transformation of P^N with center Q. Put $E = f^{-1}(Q)$ and let l be the strict transform of P^1 in V. Thus $f^{-1}(P^1) = E \cup l$; let $\hat{V}_R$ (resp. $\hat{V}_1$, $\hat{V}_2$) be the completion of V_R along $(E \cup l) \times y$ (resp. along $E \times y$, along $l \times y$). By the birational invariance (2.6), it suffices to show $K(V_R) \simeq K(\hat{V}_R)$.

Put $t_i = v_i/v_1$ $(2 \leqq i \leqq n)$. Then $l \times y$ is covered by two affines, $\mathrm{Spec}(A_0)$ and $\mathrm{Spec}(A_1)$, where $A_0 = R[v_1, t_2, \cdots, t_N]$, $A_1 = R[1/v_1, t_2, \cdots, t_N]$, and in each ring the ideal of $l \times y$ is generated by m_R and the t's. Thus $\mathrm{Spec}(A_0) \cup \mathrm{Spec}(A_1) \approx P^1 \times S \times \mathrm{Spec}(R)$, where $S = S^{N-1}$ is the affine space $\mathrm{Spec}(k[t_2, \cdots, t_N])$, and in this isomorphism $l \times y$ is mapped to $P^1 \times e \times y$, where e is the point $t_2 = \cdots = t_N = 0$ on S. Therefore, by (2.2) or by (2.7), we have

$$K(\hat{V}_2) = [R[[t_2, \cdots, t_N]]]_0(v_1).$$

On the other hand, again by the birational invariance, we have

$$K(\hat{V}_1) = [R[[v_1, \cdots, v_N]]]_0.$$

But $R[[v_1, \cdots, v_N]] \subset R[t_2, \cdots, t_N]\, [[v_1]]$. When Φ is a field, the field of fractions of a power series ring $\Phi[[u]]$ is denoted by $\Phi((u))$. In this notation we have $K(\hat{V}_1) \subset F(t)((v_1))$, where $F = [R]_0$. Therefore

$$K(V_R) = F(v_1, t) \subset K(\hat{V}_R) \subset K(\hat{V}_1) \cap K(\hat{V}_2) \subset F(t)((v_1)) \cap L(v_1), \tag{3.1.1}$$

where $L = [R[[t]]]_0$. Here all the fields under consideration are embedded in $L((v_1))$. (If x is the unique intersection point of E and l and if $\hat{V}_3$ is the completion of V_R along $x \times y$, then $K(\hat{V}_R)$, $K(\hat{V}_1)$ and $K(\hat{V}_2)$ are canonically embedded in $K(\hat{V}_3)$, and $K(\hat{V}_3) = [R[[v_1, t]]]_0 \subset L((v_1))$, therefore the inclusions (3.1.1) are justified.) Since $F(t) \subset L$, Lemma (3.1) is now reduced to the following algebraic lemma.

LEMMA (3.2). *Let Φ, L be fields, $\Phi \subset L$, and let v be an indeterminate over L. Then*

$$\Phi((v)) \cap L(v) = \Phi(v).$$

PROOF. Let $\xi \in \Phi((v)) \cap L(v)$, $\xi \neq 0$. As an element of $\Phi((v))$, ξ can be uniquely written in the form $\xi = \sum_{\nu=p}^{\infty} c_\nu c^\nu$, $c_\nu \in \Phi$, $c_p \neq 0$, where p may be negative.

On the other hand, $\xi=(\sum_0^m a_i v^i)/(\sum_0^n b_i v^i)$ for some $a_i \in L$, $b_i \in L$. Then

$$(\sum_0^n b_i v^i)(\sum_\nu^\infty c_\nu v^\nu)=\sum_0^m a_i v^i\,,$$

therefore $\sum_{i+\nu=j} b_i c_\nu=0$ for all $j>m$. Let (w_λ) be a linear basis of L over Φ, and write $b_i=\sum_\lambda f_{i\lambda} w_\lambda$, $f_{i\lambda}\in\Phi$. Then, for each λ, we have

$$\sum_{i+\nu=j} f_{i\lambda} c_\nu=0 \qquad \text{for all } j>m\,.$$

This means $(\sum_{i=0}^n f_{i\lambda} v^i)\cdot\xi\in\Phi(v)$. Since not all $f_{i\lambda}$ are zero, we get $\xi\in\Phi(v)$,

Q. E. D.

From now on we assume that k is an infinite field.

THEOREM (3.3). *Let X be any conneceted closed subscheme of positive dimension of $P=P^N$, $N\geqq 2$. Then X is universally G3 in P.*

PROOF. Let R and y be as in (2.10), and let $\hat{P}_R$ be the completion of P_R along $X\times y$. Take an irreducible reduced curve Y contained in X, and let $\hat{P}'_R$ be the completion of P_R along $Y\times y$. Since the local rings of P_R are normal and excellent, and since $X\times y$ is connected, $\hat{P}_R$ is primary and $K(\hat{P}_R)$ is a field by (1.4). Therefore we have a monomorphism $K(\hat{P}_R)\to K(\hat{P}'_R)$. Thus it is enough to prove $K(\hat{P}'_R)=K(P_R)$; namely, we may assume that $X=Y$. Take a linear subspace L of dimension $N-2$ and another linear subspace P^1 in P, such that $L\frown(P^1\cup X)=\emptyset$. Let $V=P-L$, $\pi: V\to P^1$ the projection with center L, and $\lambda: X\to P^1$ the morphism induced by π; λ is finite. Let W be the fibre product of V and X over P^1, and $\pi': W\to X$ the projection. We know (cf. [1]) that π can be given a structure of vector bundle whose zero section is the inclusion $P^1\subset V$. Hence π' inherits a structure of vector bundle. Let X_1 be the zero section of π', which is equal to $\gamma^{-1}(P^1)$ where $\gamma: W\to V$ is the natural finite morphism. We have another section of π', say X_2, which induces the inclusion $X\subset V$. Then we have an automorphism σ of W such that $\sigma(X_1)=X_2$. Let $\hat{W}_i$ $(i=1, 2)$ be the completion of W_R along $X_i\times y$. Then σ extends to an isomorphism σ_R: $\hat{W}_1\simeq\hat{W}_2$, which induces an isomorphism $K(\hat{W}_2)\simeq K(\hat{W}_1)$, and the subfield $K(W_R)$ is mapped onto itself by this isomorphism. Now, by (3.1) and by (2.7), we have $K(\hat{W}_1)=K(W_R)$. Therefore $K(\hat{W}_2)=K(W_R)$. Since $\gamma(X_2)=X$, we have a natural morphism $\varphi: \hat{W}_2\to\hat{P}_R$, which induces a monomorphism $K(\hat{P}_R)\to K(\hat{W}_2)$ $=K(W_R)$. Let F be the field of fractions of R, and let $P_F=P\times\mathrm{Spec}(F)$, $W_F=W\times\mathrm{Spec}(F)$. Then $K(W_R)=K(W_F)$ is a finite algebraic extension of $K(P_R)=K(P_F)$. Therefore $K(\hat{P}_R)$ is finite algebraic over $K(P_F)$ and its branch locus in P_F is contained in that of $K(W_F)$ over $K(P_F)$. As we can choose L and P^i in various positions, the purity of branch locus implies that $K(\hat{P}_R)$ is unramified over $K(P_F)$. P_F is simply connected, hence $K(\hat{P}_R)=K(P_F)=K(P_R)$, D. E. D.

COROLLARY (3.4). *Let k be an infinite field, and let Z be an algebraic scheme over k; suppose that Z is proper over* Spec(k) *and that the local rings of Z have no embedded primes. Let X be a closed subset of Z, and suppose that there exists a morphism $f: Z \rightarrow P^N$ from Z onto a projective space P^N (over k) such that* (i) *every irreducible component of Z is mapped onto P^N, and* (ii) *$f(X)$ is connected and is not a point, and $X=f^{-1}(f(X))$. Then X is universally G3 in Z. Hence X is connected if Z is irreducible.*

PROOF. Immediate from (3.3) and (2.7).

REMARK (3.5). This corollary can be applied, for example, to the following case: Z is a normal variety, X is a closed subset of codimension 1, and there exists an effective divisor D with Supp$(D)=X$ such that the complete linear system $|D|$ has no base point and is neither a pencil nor a composite of a pencil[6]. Indeed, $|D|$ defines then a morphism $g: Z \rightarrow P^N$ such that $\dim g(Z) = m > 1$, and $D=g^{-1}(H)$ for some hyperplane H of P^N. Take a linear subspace P^{N-m-1} in H which does not meet $g(Z)$. Then, the projection of P^N onto a suitable linear subspace P^m with center P^{N-m-1} induces a finite morphism $h: g(Z) \rightarrow P^m$, and $H \cap g(Z)=h^{-1}(H')$ with $H'=H \cap P^m$. Therefore $f=h \circ g$ satisfies the requirements of (3.4). We note a particularly simple case in the following

PROPOSITION (3.6). *Let F be a non-singular projective surface over an algebraically closed field k, and let D be an effective divisor on F with $(D^2)>0$ and such that every prime component of D has a non-negative self-intersection number. Then $K(\hat{F})=K(F)$, where $\hat{F}$ is the completion of F along* Supp(D).

PROOF. According to Riemann-Roch inequality for surfaces, $\dim|nD|$ grows with n in the order of n^2 if $(D^2)>0$. Therefore $|nD|$ is neither a pencil nor a composite of a pencil for large n. On the other hand, Zariski has shown ([**7**] p. 586-588) that $|nD|$ has no base points for large n under the assumption of the proposition. Therefore we can apply the preceding remark. Q. E. D.

In the case when D is an irreducible non-singular curve on F, $(D^2)>0$ means that the normal bundle of D in F is ample. Thus the proposition (3.6) supplements § 1 of Hironaka [**1**].

§ 4. Embeddings into abelian varieties.

(4.1). Let k be an algebraically closed field. Let A be an abelian variety over k, and X a connected closed subset of positive dimension of A. Let $\hat{A}$

6) By definition, a linear system $|D|$, free from fixed components, is a pencil or a composite of a pencil if the associated rational map $Z \rightarrow P^n$ ($n=\dim|D|$) has a curve as image. Cf. Weil [**6**].

be the completion of A along X. We assume that X goes through the origin (or the unit element) O_A of A. Let A' be the abelian subvariety of A generated by X, i. e., the image of $X\times\cdots\times X\to A$ by addition with sufficiently many copies of X. (Since X is connected and $O_A\in X$, the images of such products are independent of the number of copies. This stationary image is exactly the smallest abelian subvariety of A that contains X.) One can prove that there exists a solution (g, B) with $g: X\to B$ for the universal mapping property with respect to the morphisms from X into abelian varieties. The existence of (g, B) can be shown as follows. First of all, the pairs (g_α, B_α) of morphisms $g_\alpha: X\to B_\alpha$ with abelian varieties B_α constitute a directed system, as one defines: $(g_\alpha, B_\alpha)>(g_\beta, B_\beta)$ if $g_\beta=hg_\alpha$ with a morphism $h: B_\alpha\to B_\beta$. To find (g, B) as above, it is enough to consider only those (g_α, B_α) such that $g_\alpha(X)$ generates B_α. Then $\dim B_\alpha$ has a universal bound. In fact, let X_i be the irreducible components of X, and $A_i=\mathrm{Alb}(X_i)$, the albanese variety of X_i in the sense of rational maps. Then we can find a rational map (morphism in an open dense subset of X) $f: X\to\prod_i A_i$ and $h:\prod_i A_i\to B_\alpha$ such that $hf=g_\alpha$, with reference to any given (g_α, B_α); h is then a homomorphism up to a translation and hence surjective. Thus $\sum_i \dim A_i$ bounds $\dim B_\alpha$. It then follows that there exists a universal integer m such that the morphism from the m-fold product of X to B_α induced by g_α is surjective for all α. The existence of (g, B), which is called the strict albauese variety of X, follows from this by the same argument as in the case of ordinary albanese variety.

By translation we may assume that $g(O_A)=O_B$, so that we have an epimorphism $p: B\to A'$ with $p\circ g=i$. The morphism p factors into a connected morphism[7] $p_1: B\to B^*$ and a finite separable homomorphism $p_2: B^*\to A'$; indeed it suffices to take $B^*=B/$(the connected component of O_B in $\mathrm{Ker}\, p$). Since $p_2\circ p_1\circ g=i: X\to A'$ is a closed immersion, $p_1\circ g: X\to B^*$ is also a closed immersion, and p_2 is etale, so that the completion $\hat{B}^*$ of B^* along $p_1(g(X))$ and the completion $\hat{A}'$ of A' along X are isomorphic. We shall see later that $K(\hat{A}')=K(\hat{B}^*)=K(B^*)$.

Let $A''=A/A'$, the quotient abelian variety, and let $q: A\to A''$ be the natural morphism. Let R be the completion of the local ring of A'' at $O_{A''}$. Since q is smooth with fibre $q^{-1}(O_{A''})=A'$, after the base extension $\mathrm{Spec}(R)\to A''$, the morphism $\hat{q}: A\times_{A''}\mathrm{Spec}(R)\to\mathrm{Spec}(R)$ has sections; take one such section, say σ. Since q (and hence $\hat{q}$) is a principal fibre space with geometric fibre A', the existence of a section σ induces an isomorphism

$$A\times_{A''}\mathrm{Spec}(R)\approx A'\times_k\mathrm{Spec}(R). \tag{4.1.1}$$

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7) Cf. EGA. Ch. IV, 4.5.5.

Let us define

(4.1.2) $$A^* = B^* \times_k \mathrm{Spec}\,(R).$$

Note that the completion morphism $\hat{A} \to A$ is factored by a morphism $\hat{A} \to A^*$; call this λ.

THEOREM (4.2). *Under the assumptions stated above, X is universally $G2$ in A', the embedding $p_1 \circ g : X \to B^*$ is universally $G3$, and the morphism $\lambda : \hat{A} \to A^*$ induces an isomorphism*

$$[R]_0(K(B^*)) = K(A^*) \simeq K(\hat{A}).$$

REMARK. The last isomorphism depends on the choice of the section σ, hence it is not canonical unless $A = A'$.

PROOF. Let $X_1, \cdots, X_m$ be the irreducible components of X, arranged in such a way that $(X_1 \cup \cdots \cup X_{i-1}) \cap X_i \neq \emptyset$ for $1 < i \leqq m$. We proceed by induction on m. Put $Y = X_1 \cup \cdots \cup X_{m-1}$ and assume, as we may, that the origin O_A is chosen in $Y \cap X_m$. (When $m=1$, we take O_A in X_m and put $Y = O_A$.) Let A'_Y be the abelian subvariety of A generated by Y, and let

$$p_{2Y} : B_Y^* \to A'_Y$$

be defined in the same way as $B^* \to A'$. Then, by induction hypothesis, Y is universally $G2$ in A'_Y and universally $G3$ in B_Y^*.

LEMMA (4.3). *Let X be an irreducible subvariety of a projective space P^N over k, and let X' be the intersection of X and a linear subspace P^{N-s} such that* $\dim X' = \dim X - s > 0$. *Then X' is universally $G3$ in X. Let $f : X \to A$ be a morphism of X into an abelian variety and suppose that $O_A \in f(X)$ and that $f(X)$ generates A. Then $f(X')$ also generates A.*

PROOF. The first assertion can be proven as in (3.5), and hence X' is $G1$ in X also. Therefore the second assertion follows from (2.10).

Returning to the proof of (4.2), we first prove that $X = Y \cup X_m$ is universally $G2$ in A'. Choose a projective embedding of A; if $\dim X_m > 1$ then almost all hyperplanes through O_A cut irreducible subvarieties on X_m by Bertini's theorem. This and (4.3) show that there exists an irreducible curve $\Gamma \subset X_m$ passing through O_A and such that its image in A'/A'_Y generates A'/A'_Y. Let $\tilde{\Gamma}$ be the normalization of Γ, $\varphi : \tilde{\Gamma} \to \Gamma$ the canonical finite morphism and e a point of $\tilde{\Gamma}$ lying over O_A. Let $n = \dim A'/A'_Y$ and let W be the product of B_Y^* and n copies of $\tilde{\Gamma}$:

$$W = B_Y^* \times \tilde{\Gamma} \times \cdots \times \tilde{\Gamma}.$$

Let $S : W \to A'$ be the morphism defined by

$$S(b \times P_1 \times \cdots \times P_n) = p_{2Y}(b) + \sum \varphi(P_i).$$

S is surjective. Let R be any k-rational normal complete local ring as in (2.10).

Let $\hat{A}'_R$ (resp. $\hat{A}'_{\Gamma R}$) be the completion of $A'_R = A' \times \mathrm{Spec}(R)$ along $(Y \cup X_m) \times y$ (resp. along $(Y \cup \Gamma) \times y$), where y is the closed point of Spec(R). Since $K(A'_R) \subset K(\hat{A}'_R) \subset K(\hat{A}'_{\Gamma R})$, we have only to show that $K(\hat{A}'_{\Gamma R})$ is finite algebraic over $K(A'_R)$. Let $s_R : W_R \to A'_R$ be the morphism $s \times 1_{\mathrm{Spec}(R)}$. Put

$$W' = (Y \times e \times \cdots \times e) \cup (0 \times \tilde{\Gamma} \times e \times \cdots \times e) \cup \cdots \cup (0 \times e \times \cdots \times e \times \tilde{\Gamma})$$

and let $\hat{W}_R$ denote the completion of W_R along $W' \times y$. Since $s_R^{-1}((Y \cup \Gamma) \times y)$ contains W', we have a canonical injection $K(\hat{A}'_{\Gamma R}) \subset K(\hat{W}_R)$. Y is universally $G3$ in B^*_Y, therefore by (2.12) we have $K(\hat{W}_R) = K(W_R)$, which is finite algebraic over $K(A'_R)$. Thus X is universally $G2$ in A'.

We need one more lemma to complete the proof of (4.2).

LEMMA (4.4). *In the notations of* (4.2), *let $f : V \to A'$ be a finite surjective morphism from a normal variety V, and suppose that there exists a morphism $j : X \to V$ with $i = f \circ j$, and that f induces an isomorphism $\hat{f}$ from the completion $\hat{V}$ of V along $j(X)$ to $\hat{A}'$. Then V is an abelian variety and f is étale.*

PROOF. First we prove that $K(V)$ is separable over $K(A)$. For that purpose, take a k-rational point x of X and put $z = j(x)$. Let S be the completion of the local ring of V at z. Then $[S]_0 \supset K(V)$. By assumption S is isomorphic to the completion of $\mathcal{O}_{A',x}$. Since $\mathcal{O}_{A',x}$ is an excellent ring, S is separable over it (cf. EGA IV. 7.8.3. (V)). In view of the canonical inclusions $[S]_0 \supset K(V) \supset K(A')$, $K(V)$ is separable over $K(A')$. Then, by the purity of branch locus, the branch locus D in A' of $f : V \to A'$ is purely of codimension 1 in A'. By applying (4.3), we can find a curve $\Gamma_i \subset X_i$ which is $G3$ in X_i; adding a few more curves in X we can make $\cup_i \Gamma_i$ connected. Then A' is the smallest abelian subvariety of A' containing $\Gamma_1 \cup \cdots \cup \Gamma_m$. Suppose D is not empty. Then the intersection number $(\Gamma \cdot D)$ on A' must be positive, where Γ is the 1-cycle $\Gamma_1 + \cdots + \Gamma_m$. In fact, if $(\Gamma \cdot D) = 0$, then for any $d \in D$ we have $\Gamma_d \subset D$, where Γ_d denotes the translation of Γ by d. This implies immediately that D contains a translation of the abelian subvariety of A' generated by Γ, which is absurd. Take a generic point a of A' over k (in Weil's sense). Then $\Gamma_a \cap D \neq \emptyset$. Let Z be an irreducible component of Γ such that $Z_a \cap D \neq \emptyset$. Then $f^{-1}(Z_a)$ is a prime rational cycle over the field $k(a)$. Thus every $\overline{k(a)}$-irreducible component Z^* of $f^{-1}(Z_a)$ must meet every k-irreducible component of $f^{-1}(D)$. Hence Z^* must meet the ramification locus E of f in V. Clearly, $j(Z)$ is a specialization of such Z^* over k and hence $j(Z) \cap E \neq \emptyset$. This contradicts the assumption that $\hat{f}$ is an isomorphism (so that f induces an étale morphism in some neighborhood of $j(X)$ in V). We now have that f is étale. But this implies that V is an abelian variety ([4]). Q. E. D.

We can now complete the proof of (4.2). In the notations of (4.1), $\hat{A}$ is also the completion of $A \times_{A''} \mathrm{Spec}(R)$ along $X \times_{A''} y$, where y is the closed point of Spec(R). By (4.1.1) $\hat{A}$ is isomorphic to the completion $\hat{A}'_R$ of $A'_R = A' \times \mathrm{Spec}(R)$ along $X \times y$. We can now forget the definition of R in (4.1) and let R be an arbitrary k-rational normal complete local ring. We propose to prove that $K(B^*_R) \approx K(\hat{A}'_R)$. Then it will follow, as in the special case of $A = A' = B^*$, that X is universally $G3$ in B^*, and the proof of (4.2) will be completed.

$K(\hat{A}'_R)$ is finite algebraic over $K(A'_R) = F(K(A'))$, where $F = [R]_0$, as we have already seen. Moreover, we know that $K(\hat{A}'_R) \subset K(W_R) = F(K(W))$, where W is a variety over k such that $K(W)$ is finite algebraic over $K(A')$. $K(\hat{A}'_R)$ is separable over $F(K(A'))$ by the same reason as in the proof of (4.4). Therefore, by Galois theory, there exists a unique field L such that $K(A') \subset L \subset K(W)$ and $F(L) = K(\hat{A}'_R)$. Let V be the normalization of A' in L, and let $f: V \to A'$ be the canonical finite morphism. Then we have a commutative diagram of R-morphisms

$$\begin{tikzcd}
& V_R \arrow[d, "f_R"] \\
\hat{A}'_R \arrow[ur, "g"] \arrow[r, "h"'] & A'_R
\end{tikzcd}$$

where h is the completion morphism. Since $f_R(g(\hat{A}'_R)) = X \times y$, $g(\hat{A}'_R)$ is contained in $V \times y$. We have $\hat{A}'_R \otimes_R (R/m_R) = \hat{A}'$ and $V_R \otimes_R (R/m_R) = V$, therefore g induces a k-morphism $\hat{A}' \to V$, and combining it with the canonical morphism $X \to \hat{A}'$ we have a morphism of k-schemes $j: X \to V$. $j(X)$ is closed in V since X is proper over Spec(k). We have $g(\hat{A}'_R) = j(X) \times y$. Let $\hat{V}$ (resp. $\hat{V}_R$) be the completion of V along $j(X)$ (resp. of V_R along $j(X) \times y$). Then f_R induces a canonical morphism $\hat{f}_R: \hat{V}_R \to \hat{A}'_R$, which factors the identity automorphism of $\hat{A}'_R$ by means of the completion of g. Since f_R is finite and since $\hat{V}_R$ is normal (because the local rings of V_R are excellent and normal), it follows that $\hat{f}_R$ is an isomorphism. Therefore $\hat{f}_R \otimes_R (R/m_R)$, which is equal to the completion $\hat{f}: \hat{V} \to \hat{A}'$ of f, must be an isomorphism. By (4.4), V is then an abelian variety, and after a suitable choice of the origin in V, f is a separable isogeny. By the definition of the strict albanese variety B of X we get a canonical morphism $B \to V$, which is surjective as $V \to A'$ is finite. Since $f: V \to A'$ is étale, $B \to V$ factors through B^*. Thus we have a commutative diagram

$$\begin{tikzcd}
& B^* \arrow[dr] \arrow[dd] & \\
X \arrow[ur] \arrow[dr] & & V \arrow[dl] \\
& A' &
\end{tikzcd}$$

whence $K(\hat{A}'_R)=K(V_R)\subset K(B^*_R)$. On the other hand $\hat{B}^*_R=\hat{A}'_K$ because $\hat{B}^*_R$ is the product (over k) of the formal schemes $\hat{B}^*=\hat{A}'$ and $\mathrm{Spf}\,(R)$. Therefore $K(B^*_R)\subset K(\hat{B}^*_R)\subset K(\hat{A}'_R)$. Thus we obtain $K(\hat{A}'_R)=K(B^*_R)$, Q. E. D.

THEOREM (4.5). *Let A and X be the same as in* (4.2). *Let us assume that X goes through O_A and the morphism $X\times X\to A$ by addition is surjective. Then X is universally $G3$ in A.*

PROOF. By (4.2), it is enough to prove that $B^*=A$. First of all, the assumption implies $A'=A$. Thus we have a separable isogeny $p_2: B^*\to A$ and an imbedding $j: X\to B^*$. ($j=p_1\circ g$) Since $p_2\circ j=$ the inclusion, the assumption on $X\times X\to A$ implies the same on $X\times X\to B^*$, induced by j. If p_2 is not an isomorphism, then there exists $t\in p_2^{-1}(O_A)$, $\neq O_{B^*}$, and $j(X)_t\cap j(X)\neq\emptyset$. But this is impossible, because p_2 induces an étale morphism in some neighborhood of $j(X)$. Q. E. D.

REMARK (4.5.1). The surjectivity assumption of (4.5) is satisfied if A is a simple abelian variety and $2\dim X\geqq\dim A$.

§ 5. Examples (the case of curves in surfaces).

(5.1) Let k be an algebraically closed field, F a smooth projective surface over k, and C a smooth irreducible curve on F. We have shown that C is $G3$ in F if $C^2>0$. On the other hand, if $C^2<0$, a result of Hironaka [2] shows that C is contractible to a point not only formally but also in some neighborhood of C in the étale topology. Anyway C is not $G1$ in F if $C^2<0$, cf. the proof of (5.11). In the remaining case in which $C^2=0$, we have the following result.

PROPOSITION (5.1.1). *Let F and C be as above and assume that $C^2=0$. Then C cannot be $G2$ in F. But, if the divisor class $C\cdot C$ on C is not a torsion element in* $\mathrm{Pic}\,(C)$, *then C is $G1$ in F but not $G2$.*

PROOF. For any divisor or divisor class (modulo linear equivalence) D on F or on C, we denote by $[D]$ the corresponding invertible sheaf. Let J be the ideal sheaf of C in $\mathcal{O}_F$. Then $J\approx[C]^{-1}$. Put $L=[C\cdot C]^{-1}\cong J/J^2$. Consider the following exact sequences

$$0\to J^n/J^{n+1}\to\mathcal{O}_F/J^{n+1}\to\mathcal{O}_F/J^n\to 0$$

for $n=1, 2, \cdots$, and note that $J^n/J^{n+1}\approx L^{\otimes n}$. Take a divisor D on F such that $(D\cdot C)=d>\mathrm{Max}\,((2g-2, g-1)$, where $g=$ genus of C. Tensoring the above sequences with $[D]$, we have exact sequences

$$0\to[D]\otimes L^{\otimes n}\to[D]/J^{n+1}[D]\to[D]/J^n[D]\to 0\,,$$

where $[D]\otimes L^{\otimes n}\approx[(D-nC)\cdot C]$, and $H^1(C, [D]\otimes L^{\otimes n})=0$ since $\deg((D-nC)\cdot C)=d>2g-2$. Therefore

$$0 \to H^0(C, [D] \otimes L^{\otimes n}) \to H^0(C, [D]/J^{n+1}[D]) \to H^0(C, [D]/J^n[D]) \to 0$$

is exact, and $\dim H^0(C, [D] \otimes L^{\otimes n}) = d-g+1 > 0$. Hence we must have

$$\dim (\varprojlim H^0(C, [D]/J^{n+1}[D])) = \infty .$$

If $\hat{F}$ is the completion of F along C, the projective limit in the above formula is equal to $H^0(\hat{F}, \mathcal{O}_{\hat{F}} \otimes [D])$, which would be of finite dimension if $K(\hat{F})$ were finite algebraic over $K(F)$. Thus C is not $G2$ in F.

(Remark. A similar but simpler argument shows that if Z is a smooth variety and X is a smooth subvariety such that the locally free sheaf J/J^2 (wtere J is the ideal sheaf of X in $\mathcal{O}_Z$) is ample, in other words such that the normal bundle of X in Z is negative, then $\dim H^0(\hat{Z}, \mathcal{O}_{\hat{Z}}) = \infty$.)

Suppose C is not $G1$ in F, i.e. that $H^0(\hat{F}, \mathcal{O}_{\hat{F}}) \neq k$, and take a non-constant formal function $\xi \in H^0 (\hat{F}, \mathcal{O}_{\hat{F}})$. Then, since $H^0(\hat{F}, \mathcal{O}_{\hat{F}}/J\mathcal{O}_{\hat{F}}) = H^0(F, \mathcal{O}_F/J) = H^0(C, \mathcal{O}_C) = k$, there exists a constant $c \in k$ such that $\xi - c \in J^n, \notin J^{n+1}$. Then $H^0(F, J^n/J^{n+1}) = H^0(C, L^{\otimes n}) \neq 0$, but as $L^{\otimes n}$ represents the divisor class $nC \cdot C$ which is of degree zero, this means $nC \cdot C \sim 0$.

REMARK (5.1.2). Let X be a smooth projective variety of dimension $n \geqq 2$, and Y a smooth subvariety of codimension 1 of X. Suppose that the self-intersection Y^2, viewed as a divisor class on Y, is numerically equivalent to a sum of distinct smooth subvarieties $D_1 + \cdots + D_r$ on Y, and that the difference $Y^2 - (D_1 + \cdots + D_r)$ is a non-torsion element of $\mathrm{Pic}^\tau(Y)$. For instance,

(a) $\dim X = 2$, $Y^2 > 0$ and $p_a(Y) > 0$,

or (b) Y is the zero section of a line bundle X associated with a non-torsion element of $\mathrm{Pic}^\tau(Y_0)$, where Y_0 is a smooth projective variety with $h^{01}(Y_0) > 0$.

Now, Y and X being as above, we apply the monoidal transformations to X whose centers are first D_1 and then the successive strict transform of D_i, $i = 2, \dots, r$. Let $f: X' \to X$ be their composition. Let Y' be the strict transform of Y by f. Then one can show that $H^0(\hat{X}', \mathcal{O}_{\hat{X}'}) = k$ but $K(\hat{X}')$ has an infinite transcendence degree over k, where $\hat{X}'$ is the completion of X' along Y'. The proof is the same as above, except that we need a theorem of "simultaneous amplification" due to Matsusaka-Mumford-Kleiman, which asserts: with respect to a fixed projective embedding of Y', there exists a positive integer ν such that $L(\nu)$ satisfies the theorems A and B of Cartan-Serre for all $L \in \mathrm{Pic}^\tau(Y')$ (Kleiman [**11**] § 2, Th. 2).

(5.2). If a smooth *rational* curve C is $G1$ in a smooth surface F, then F must be a rational surface. In fact,

(i) $C^2 \geqq 0$ by (5.1) (or by [**2**]);

(ii) $-2 = 2p_a(C) - 2 = C^2 + C \cdot K$, where K is the canonical class of F, hence $C \cdot K < 0$ by (i);

(iii) $l(K)=h^{02}(F)=0$ by (i) and (ii);

(iv) the Picard scheme of F is reduced by (iii)[8], therefore $h^{01}(F)=$ the dimension of $\mathrm{Alb}(F)$, which is zero by (2.10);

(v) we have an exact sequence

$$0\to\mathcal{O}_F\to[C]\to[C\cdot C]\to 0$$

and $H^i(F,\mathcal{O}_F)=(0)$ by (iv). Hence

$$(*)\qquad 0\to H^0(F,\mathcal{O}_F)\to H^0(F,[C])\to H^0(C,[C\cdot C])\to 0$$

is exact. Since $C\approx P^1$, $l(C)=\dim_k H^0([C])=(C^2)+2\geqq 2$. Therefore F is rational by a theorem of Max Noether. But we get more. In fact, since $T_{rC}|C|$ is complete by (*), the complete linear system $|C|$ has no base point and defines a morphism $f:F\to P^{n+1}$, where $n=(C^2)$, such that $C=f^{-1}$ (hyperplane). If $(C^2)=0$, then $f:F\to P^1$ defines a structure of $P^1\times P^1$ on F, at least in a neighborhood of C in F. (This is a consequence of $(C^2)=0$ and $h^{01}(F)=0$, but in this case C is not $G1$ in F.) When $(C^2)>0$, the image $f(F)$ has dimension 2, for if otherwise $|C|$ would be a composite of pencil, contradicting the fact that C is irreducible. Moreover, since $T_{rC}|C|$ is complete, for any point Q of C one can find a divisor $D\in|C|$ such that Q has coefficent 1 in $D\cdot C$, and this implies that f is a birational correspondence between F and $f(F)$ which is an isomorphism in a neighborhood of C. $f(F)$ is a surface of degree n in P^{n+1}, not contained in any hyperplane, and such a surface is well known to be rational. This proves again the rationality of F (cf. Nagata [8] § 10).

(vi) In particular, if $(C^2)=1$, then $f:F\to P^2$ is an isomorphism in a neighborhood of C and maps C onto a line. In other words, a neighborhood of C in F is isomorphic to a neighborhood of the zero section of $V(\mathcal{O}(1))$ over a projective line P^1.

(5.3) Let $F=P^2$ and $C=P^1$, a line in P^2. Then (5.2) shows that this $C\to F$ is the unique (within a neighborhood of C) embedding of C with the normal bundle of degree 1. We shall show that there are infinitely many distinct formal imbeddings $C\to F^*$ with regular formal surfaces F^*, such that the normal bundles are of degree 1. There F^* cannot be obtained from algebraic surfaces by completion, except the unique one $\hat{F}$ which is the completion of F along C. Let J be the ideal sheaf of C on F, and F_m the subscheme of F defined by J^{m+1} for every integer $m\geqq 0$. Let $\mathrm{Aut}(X)$ denote the group of automorphisms of a (formal) scheme X over k. If $X=\hat{F}$ or F_m then $\underline{\mathrm{Aut}}_q(X)$ (resp. $\mathrm{Aut}_q(X)$) denote the sheaf (resp. the group) of those automorphisms of X which induce the identity of F_q, where $0\leqq q<m$. Let $\underline{\mathrm{Aut}}_0^*(X)$ denote the

8) This result is due to Severi-Nakai (cf. Nakai [**9**]), and is also a consequence of the obstruction theory of Picard functor (cf. e. g. Mumford [**10**]).

sheaf of those automorphisms of X which induce the identity in $F_0=C$ and also the identity in the normal bundle of C in F. Let S_m be the sheaf J^m/J^{m+1} restricted to C, so that S_m is canonically isomorphic to the m-th symmetric tensor power of S_1. By the smoothness of F, we get

(a) an exact sequence of group sheaves

$$1 \to \underline{\mathrm{Aut}}_{m-1}(F_m) \to \underline{\mathrm{Aut}}_p(F_m) \to \underline{\mathrm{Aut}}_p(F_{m-1}) \to 1$$

for each $m-1 \geqq p \geqq 0$.

(b) An isomorphism (from "multiplicative" to "additive")

$$\underline{\mathrm{Aut}}_{q-1}(F_q) \underset{\rightarrow}{\approx} \bar{T} \otimes S_q$$

for each $q>1$, where $\bar{T}$ is the sheaf on C induced by T_F (= the tangent sheaf of F).

(c) an exact sequence

$$1 \to \underline{\mathrm{Aut}}_0^*(F_1) \to \underline{\mathrm{Aut}}_0(F_1) \to \underline{\mathrm{Hom}}_{\mathcal{O}}^*(S_1, S_1) \to 1$$

where $\mathcal{O}=\mathcal{O}_C$ and $\underline{\mathrm{Hom}}^*$ denotes the sheaf of invertible elements in $\underline{\mathrm{Hom}}$.

(d) isomorphisms

$$\underline{\mathrm{Hom}}^*(S_1, S_1) \cong \mathcal{O}^*$$

and

$$\underline{\mathrm{Aut}}_0^*(F_1) \underset{\rightarrow}{\approx} T_C \otimes S_1$$

where T_C = the tangent sheaf of C.

LEMMA (5.3.1). *The natural homomorpihsm $\lambda_q : \mathrm{Aut}(\hat{F}) \to \mathrm{Aut}(F_q)$ is surjective for all $q \geqq 3$.*

PROOF. $\sigma \in \mathrm{Aut}(F)$ induces an element of $\mathrm{Aut}(\hat{F})$ if and only if $\sigma(C)=C$. (In fact, $K(\hat{F})=K(F)$ implies that every element of $\mathrm{Aut}(\hat{F})$ is obtained in this manner.) Let (x_0, x_1, x_2) be a homogeneous coordinate system of F such that C is defined by $x_2=0$. Then every σ as above can be represented by a matrix:

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}$$

Those σ with $a_{31}=a_{32}=0$ induce all the automorphisms of C, so that λ_0 is surjective. Hence it is enough to prove

$$\lambda'_q : \mathrm{Aut}_0(\hat{F}) \to \mathrm{Aut}_0(F_q)$$

is surjective for all $q \geqq 3$. Those σ which induce the identity in $F_0=C$ are represented by matrices of the form

$$\begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ a' & a'' & 1 \end{pmatrix}$$

From now on, we consider only those σ of this type. Let $t=x_1/x_0$ and $u=x_2/x_0$. Pick any σ with $b=1$. Then σ induces a section of $\underline{\mathrm{Aut}}_0^*(F_1)$. We have

$$\left.\begin{aligned} \sigma(t) &= (t+a''u)/(1+a'u) \equiv t+D(a', a'')t \\ \sigma(u) &= u/(1+a'u) \equiv u+D(a', a'')u \end{aligned}\right\} \bmod J^2$$

where $D(a', a'')=(a''+a't)u\dfrac{\partial}{\partial t}$.

The section of $\underline{\mathrm{Aut}}_0^*(F_1)$, induced by σ, corresponds to the section of $T_C\otimes S_1$, induced by the above $D(a', a'')$, under the isomorphism of (d). But $H^0(T_C\otimes S_1)=H^0(\mathcal{O}(1))=k\oplus k$. Hence those σ induces all the sections of $\underline{\mathrm{Aut}}_0^*(F_1)$. Now, pick any σ with an arbitrary $b\in k^*$. Then, by a homomorphism of (c), σ induces an element of $H^0(\mathrm{Hom}_{\mathcal{O}}^*(S_1, S_1))$, which is isomorphic to $H^0(\mathcal{O}^*)=k^*$ by (d). This element induced by σ is nothing but b. It is now clear that λ_1' is surjective. Therefore, it suffices to show that

$$\mu_q : \mathrm{Aut}_1(F) \to \mathrm{Aut}_1(F_q)$$

is surjective for all $q\geqq 3$. We have an exact sequence

$$0 \to \mathcal{O}_F \to \mathcal{O}_F(1)\oplus\mathcal{O}_F(1)\oplus\mathcal{O}_F(1) \to T_F \to 0$$

which induces

$$\text{(e)} \qquad 0 \to \mathcal{O} \to \mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(1) \to \overline{T} \to 0\,.$$

By this exact sequence, we can easily compute

$$H^0(\overline{T}\otimes S_2)\cong H^0(\overline{T}(-2))\cong H^1(\mathcal{O}(-2))\cong k\,.$$

It follows that $H^0(\overline{T}\otimes S_q)=(0)$ for all $q\geqq 3$. By (a) and (b), this implies that

$$\kappa_q : \mathrm{Aut}_1(F_q) \to \mathrm{Aut}_1(F_{q-1})$$

is injective for every $q\geqq 3$. For $q=3$, we have $\mathrm{Aut}_1(F_2)\cong H^0(\overline{T}\otimes S_2)\cong k$. We shall prove that $\kappa_3=0$ and hence $\mathrm{Aut}_1(F_q)=1$ for all $q\geqq 3$. Take any element $\xi\in k$, say $\xi\neq 0$. Let $D_0(\xi)$ (resp. $D_1(\xi)$) be the derivation in $k[t, u]$ (resp. $k\,[1/t,\ u/t]$) defined by

$$D_0(\xi)u=0\,, \qquad D_0(\xi)t=\xi u^2$$
$$D_1(\xi)u'=0\,, \qquad D_1(\xi)t'=-\xi u'^2$$

where $t'=1/t$ and $u'=u/t$. Then $D_0\equiv D_1 \bmod J^3T_F$, and hence they define an element $\overline{D}(\xi)\in H^0(\overline{T}\otimes S_2)$. This is clearly the derivation which corresponds to ξ by $H^0(\bar{T}\otimes S_2)\cong k$. Let σ_0 (resp. σ_1) be the automorphism of $k[t, u]$ (resp. $k[t', u']$) defined by

$$\sigma_0(u)=u,\ \sigma_0(t)=t+\xi u^2$$

$$\sigma_1(u')=u',\ \sigma_1(t')=t'-\xi u'^2$$

then $\sigma_0\equiv\sigma_1 \bmod J^3$, and they define an element $\bar{\sigma}\in \mathrm{Aut}_1(F_2)$, which corresponds to $\bar{D}(\xi)$ by $\mathrm{Aut}_1(F_2)\cong H^0(\bar{T}\otimes S_2)$. Let δ be the connecting homomorphism $\mathrm{Aut}_1(F_2)\to H^1(\underline{\mathrm{Aut}}_2(F_3))$, associated with

$$1\to\underline{\mathrm{Aut}}_2(F_3)\to\underline{\mathrm{Aut}}_1(F_3)\to\underline{\mathrm{Aut}}_1(F_2)\to 1\,.$$

Let us identify $\underline{\mathrm{Aut}}_2(F_3)$ with $\bar{T}\otimes S_3$ by (b). Let N be the normal sheaf to C in F, i.e., the dual of S_1. We have an exact sequence

$$0\to T_C\to\bar{T}\to N\to 0$$

which induces $\delta': H^1(\bar{T}\otimes S_3)\to H^1(N\otimes S_3)$. We have $H^1(N\otimes S_3)\cong H^1(\mathcal{O}(-2))\cong k$. $\delta(\bar{\sigma})$ is obtained by taking $\sigma_1^{-1}\sigma_0 \bmod J^4$, which is an automorphism of F_3 in $\mathrm{Spec}(k[t, u])\cap\mathrm{Spec}(k[t', u'])$. By a direct calculation, we get

$$\left.\begin{aligned}\sigma_1^{-1}\sigma_0(u)&=u-\frac{\xi}{t}u^3\\ \sigma_1^{-1}\sigma_0(t)&=t\end{aligned}\right\}\bmod J^4$$

or

$$\left.\begin{aligned}\sigma_1^{-1}\sigma_0(u')&=u'-\frac{\xi}{t'}u'^3\\ \sigma_1^{-1}\sigma_0(t')&=t'\end{aligned}\right\}\bmod J^4\,.$$

Let $E=-\dfrac{\xi}{t}u^3\dfrac{\partial}{\partial u}=-\dfrac{\xi}{t'}u'^3\dfrac{\partial}{\partial u'}$, which induces an element $\bar{E}$ of $H^1(\mathcal{O}(-2))$, such that

$$\bar{E}=-\frac{\xi}{t}e_0=-\frac{\xi}{t'}e_1(=\delta'\delta(\bar{\sigma}))$$

where e_0 (resp. e_1) is a suitable generator of $\mathcal{O}(-2)$ in $\mathrm{Spec}(k[t, u])$ (resp. in $\mathrm{Spec}(k[t', u'])$). We can get an isomorphism $H^1(\mathcal{O}(-2))\cong k$ by

$$0\to\mathcal{O}(-2)\to\mathcal{O}(-1)\to k_P\to 0$$

where P is for instance, the point $t=0$. Here the connecting homomorphism $H^0(k_P)\ (=k)\underset{\to}{\approx} H^1(\mathcal{O}(-2))$ is an isomorphism and maps $-\xi\in k$ to the above $\bar{E}$. It is now clear that $\mathrm{Aut}_1(F_2)\to H^1(\mathrm{Aut}_2(F_3))$ is injective. (In fact, one can prove that this is an isomorphism.) Hence $\kappa_3=0$, and $\mathrm{Aut}_1(F_q)=(1)$ for all $q\geqq 3$. This together with the surjectivity of $\mathrm{Aut}(F)\to\mathrm{Aut}(F_1)$, proves Lemma (5.3.1).

Let $\hat{U}=\mathrm{Spf}(k[t][[u]])$, $\hat{U}'=\mathrm{Spf}(k[t'][[u']])$ with $t'=1/t$ and $u'=u/t$, and $\hat{W}=\mathrm{Spf}(k[t, t'][[u]])$. Then $\hat{F}=\hat{U}\cup\hat{U}'$ and $\hat{W}=\hat{U}\cap\hat{U}'$. Take any automorphism γ of $\hat{W}$ which induces the identity in $\hat{W}\cap F_{q-1}$, where $q\geqq 3$. Then there exists a derivation $w\in H^0(J^qT_{\hat{W}})$, i.e., $w=g\frac{\partial}{\partial t}+h\frac{\partial}{\partial u}$ with $g, h \in (u^q)k[t, t'][[u]]$, such that

$$\gamma(f)\equiv f+w(f) \bmod J^{q+1}$$

for all $f\in H^0(\mathcal{O}_{\hat{W}})$. We have a canonical isomorphism $J^qT_{\hat{W}}/J^{q+1}T_{\hat{W}}\approx \bar{T}\otimes S_q|\hat{W}$ (= the restriction to $|\hat{W}|=|W\cap C|$). By means of the covering $\hat{F}=\hat{U}\cap\hat{U}'$, the Cěch method gives a surjective homomorphism $H^0(\bar{T}\otimes S_q|\hat{W})\to H^1(\bar{T}\otimes S_q)$. Thus w induces an element $\bar{w}\in H^1(\bar{T}\otimes S_q)$. We shall write this $\bar{w}(\gamma)$.

On the other hand, any γ as above defines a formal scheme

$$F_\gamma^*=\hat{U}\cup_\gamma\hat{U}'$$

where the piecing together is done by

$$\hat{U}\supset\hat{W}\underset{\gamma}{\rightarrow}\hat{W}\subset\hat{U}'.$$

LEMMA (5.3.2). *Assume $q\geqq 4$. For every $\bar{w}\in H^1(\bar{T}\otimes S_q)$, we can find $\gamma\in \mathrm{Aut}_{q-1}(\hat{W})$ such that $\bar{w}=\bar{w}(\gamma)$. Moreover, if $\bar{w}(\gamma)\neq 0$, then F_γ^* is not isomorphic to F as a formal scheme over k.*

PROOF. The first assertion is already proven. To prove the second, assume that there exists an isomorphism $\sigma:\hat{F}\to F_\gamma^*$. By (5.3.1), we can modify σ by an automorphism of $\hat{F}$ so that σ induces the identity modulo J^q. Here we refer to the definition of F_γ^* which gives a canonical way of identifying F_{q-1} as a subscheme of F_γ^*. Moreover, $\sigma|\hat{U}$ and $\sigma|\hat{U}'$ can be viewed as automorphisms of $\hat{U}$ and $\hat{U}'$ in a natural way. Viewing the equality

$$(\sigma|\hat{U}')\circ(\sigma|\hat{U})^{-1}=\gamma$$

from this point of view, we get $\bar{v}\in H^0(\bar{T}\otimes S_q|\hat{U})$ and $\bar{v}'\in H^0(\bar{T}\otimes S_q)|\hat{U}')$, associated with $(\sigma|\hat{U})$ and $(\sigma|\hat{U}')$ respectively, such that $\bar{v}'-\bar{v}|\hat{W}$ is the element of $H^0(\bar{T}\otimes S_q|\hat{W})$ associated with γ. In view of the definition of $\bar{w}(\gamma)$, we get $\bar{w}(\gamma)=0$.

THEOREM (5.3.3). *F_γ^* is not algebraizable, i.e., it cannot be obtained as the completion of an algebraic scheme, for every $\gamma\in\mathrm{Aut}^{q-1}(\hat{W})$ such that $\bar{w}(\gamma)\neq 0$, provided $q\geqq 4$.*

PROOF. Immediate from (5.3.2), because we have the uniqueness of embedding of P^1 into a non-singular surface as was seen in (5.3.1).

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