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Supplement to: Holomorphic imbeddings of symmetric domains

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There are some incomplete explanations in p. 274 (§ 2) of my previous paper [1], which I will supplement here. I wish to express my sincere thanks to Professor. I. Satake who has kindly indicated me the incompleteness together with valuable advices.

1. We used a property without proof, which can be formulated as follows:

PROPOSITION. *Let* $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ *and* $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ *be Cartan decompositions of Lie algebras of hermitian type without compact factors,* $\rho:\mathfrak{g}\to\mathfrak{g}'$ *a homomorphism satisfying the analytic condition* (H_2) *w.r.t. complex structures* H_0 *and* H_0' *of* $(\mathfrak{g}, \mathfrak{k})$ *and* $(\mathfrak{g}', \mathfrak{k}')$ *respectively. Let further* $\mathfrak{g}=\sum_{i=1}^{e}\mathfrak{g}_i$ $(e\geqq 2)$ *be a decomposition of* $\mathfrak{g}$ *into the direct sum of simple ideals. Then there are regular subalgebras* $\mathfrak{g}_j'$ $(1\leqq j\leqq e)$ *of* $\mathfrak{g}'$ *such shat*

i) $\rho(\mathfrak{g}_i)\subset\mathfrak{g}_i'$, *and the restriction of* ρ *to* $\mathfrak{g}_i$ *satisfies the condition* (H_2) *w.r.t. the complex structures of* $(\mathfrak{g}_i, \mathfrak{k}_i)$ *and* $(\mathfrak{g}_i', \mathfrak{k}_i')$ *compatible to those of* $(\mathfrak{g}, \mathfrak{k})$ *and* $(\mathfrak{g}', \mathfrak{k}')$ *respectively;*

ii) $[\mathfrak{g}_i', \mathfrak{g}_j']=0$ *if* $i\neq j$.

PROOF. We may assume that $\mathfrak{g}'$ is simple ([1], p. 273). Let H_{0i} be the projection of H_0 to $\mathfrak{g}_i$, and put $H_{0i}'=\rho(H_{0i})$. Since ρ satisfies (H_2), we have $H_0=\sum_{i=1}^{e}H_{0i}$, $H_0'=\sum_{i=1}^{e}H_{0i}'$. Put

$$\mathfrak{v}_{i+}'=\{\alpha'\in\mathfrak{v}_+'\mid\alpha'(H_{0i}')=\sqrt{-1}\}\,.$$

$$\mathfrak{p}_i'=\mathfrak{g}'\cap\sum_{\alpha'\in\mathfrak{v}_{i+}'}(\mathfrak{g}_{\alpha'}'+\mathfrak{g}_{-\alpha'}')\,,$$

$$\mathfrak{k}_i'=[\mathfrak{p}_i', \mathfrak{p}_i'] \quad\text{and}\quad \mathfrak{g}_i'=\mathfrak{k}_i'+\mathfrak{p}_i'\,,$$

where $\mathfrak{v}_+'$ denotes the set of all positive non-compact roots, and $\mathfrak{g}_{\alpha'}'$ the root space belonging to α'. Then $\mathfrak{g}_i'$ is a regular subalgebra such that the restriction of ρ to $\mathfrak{g}_i$ defines a homomorphism of $\mathfrak{g}_i$ into $\mathfrak{g}_i'$ satisfying (H_2) w.r.t. H_{0i} and H_{0i}' (cf. [1], Theorem 2). On the other hand, one knows ([2], p. 96) that

$$H_{0i}'=2\sqrt{-1}\sum_{\alpha'\in\mathfrak{v}_{i+}'}\tilde{\alpha}'\,,$$

where $\tilde{\alpha}'$ denotes the restriction of α' to the Cartan subalgebra $\mathfrak{g}_i'\cap\mathfrak{h}'$ of $\mathfrak{g}_i'$. Hence, if we denote by $c_{\alpha'}$ (>0) for each $\alpha'\in\mathfrak{v}_{i+}'$ the ratio of the Killing form of $\mathfrak{g}'$ and that of the simple component of $\mathfrak{g}_i'$ whose root system contains the restriction of α', we have

$$H_{0i}'=2\sqrt{-1}\sum_{\alpha'\in\mathfrak{v}_{i+}'}c_{\alpha'}\frac{2H_{\alpha'}'}{\langle H_{\alpha'}', H_{\alpha'}'\rangle'}=2\sqrt{-1}\sum_{\alpha'\in\mathfrak{v}_{i+}'}c_{\alpha'},\alpha'.$$

It can be easily seen that

$$0=\langle H_{0i}', H_{0j}'\rangle'=-2\sum_{\substack{\alpha'\in\mathfrak{v}_{i+}'\\ \alpha'\in\mathfrak{v}_{i+}'}}c_{\alpha'}, c_{\beta'}\langle\alpha', \beta'\rangle' \tag{1}$$

if $i\neq j$. If α' and β' are positive non-compact roots of $\mathfrak{g}_c'$, then $\alpha'+\beta'$ cannot be a root, and so $\langle\alpha', \beta'\rangle'\geqq 0$. Hence it follows from (1) that $\langle\alpha', \beta'\rangle'=0$, and $\alpha'-\beta'$ cannot be a root. Therefore we see $[\mathfrak{p}_i', \mathfrak{p}_j']=0$, and hence $[\mathfrak{k}_i', \mathfrak{k}_j']=0$. Then we can see at once that $[\mathfrak{g}_i', \mathfrak{g}_j']=0$, q. e. d.

2. Thus, if $\mathfrak{g}$ has no compact factors, it is sufficient for finding all homomorphism $\rho:\mathfrak{g}\mapsto\mathfrak{g}'$ under the condition (H_2) to determine for each simple factor $\mathfrak{g}_i$ a regular subalgebra $\mathfrak{g}_i'$ such that $[\mathfrak{g}_i', \mathfrak{g}_j']=0$ if $i\neq j$ and that there is a homomorphism $\rho_i:\mathfrak{g}_i\to\mathfrak{g}_i'$ satisfying (H_2); in fact, the homomorphism $\rho=\rho\oplus\cdots\oplus\rho_e$ of $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_e$ into a regular subalgebra $\mathfrak{g}''=\mathfrak{g}_1'\oplus\cdots\oplus\mathfrak{g}_e'$ of $\mathfrak{g}'$ satisfies (H_2). This procedure can be carried out by the results in §4 and §5 of [**1**]. If $\mathfrak{g}$ contains a compact ideal, the reductions given in p. 274 of [**1**] are valid only modulo compact factors. They are, however, sufficient for our main purpose of determining holomorphic imbeddings between symmetric domains.

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References

[1] S. Ihara, Holomorphic imbeddings of symmetric domains, J. Math. Soc. Japan, 19 (1967), 261-302.

[2] S. Murakami, Cohomology groups of vector valued forms on symmetric spaces, Lecture notes, Univ. of Chicago, 1966.