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Closed images of countable-dimensional spaces

Dedicated to Professor K. Noshiro on his sixtieth birthday

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A metric space X is called countable-dimensional or σ_0 if X is the sum of subsets X_i, $i=1, 2, \cdots$, with $\dim X_i \leqq 0$, where $\dim X_i$ denotes the covering dimension of X_i defined by means of finite open coverings. A metric space is called uncountable-dimensional if it is not σ_0. The purpose of this paper is to prove the following:

THEOREM. *Let X be a metric σ_0-space, Y a metric uncountable-dimensional space and f a closed mapping (continuous transformation) of X onto Y. Then Y contains an uncountable-dimensional subset Y_0 such that for every point y in Y_0, $\mathrm{B}(f^{-1}(y))$, the boundary of $f^{-1}(y)$, is dense-in-itself and non-empty.*

This was proved firstly by E. Sklyarenko [3] for the case when X is compact metric and generalized by A. Arhangelskii [1] to some class of spaces which contains all separable metric spaces. But Arhangelskii's generalization is not effective for general metric spaces yet. We need three lemmas.

LEMMA 1. *If f is a closed mapping of a metric space X onto a metric space Y, then $\mathrm{B}(f^{-1}(y))$ is compact for each point y in Y.*

This was proved by K. Morita—S. Hanai [2] and by A. H. Stone [4].

LEMMA 2. *Let X be a metric space which is locally σ_0. Then X is σ_0.*

PROOF. Let $\mathfrak{U}$ be a σ-discrete base of X and $\mathfrak{U}'$ be the set of all elements U of $\mathfrak{U}$ such that U is σ_0. Since $\mathfrak{U}'$ is a σ-discrete open covering of X, we can set $\mathfrak{U}'=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$ where each $\mathfrak{U}_i$ is a discrete collection of open sets. Set $U_i=\cup\{U : U\in\mathfrak{U}_i\}$. Then U_i is evidently σ_0. Since $X=\cup U_i$, X is σ_0.

LEMMA 3. *Let X and Y be metric spaces and f a closed mapping of X onto Y such that $f^{-1}(y)$ is compact and is not dense-in-itself for any point y in Y. If X is σ_0, then Y is.*

PROOF. Let $\cup\mathfrak{U}_i$ be a σ-discrete base of X, where $\mathfrak{U}_i=\{U_\alpha : \alpha\in A_i\}$ and each $\mathfrak{U}_i$ is discrete. By the condition every $f^{-1}(y)$ has an isolated point $x(y)$. There exists an $\alpha(y)\in\cup A_i$ such that

$$x(y)=\bar{U}_{\alpha(y)}\cap f^{-1}(y).$$

For each $\alpha\in\cup A_i$ let

$$Y_\alpha = \{y \in Y : \alpha(y) = \alpha\}\,,$$

$$Y_i = \cup \{Y_\alpha : \alpha \in A_i\}\,.$$

Then $Y = \cup Y_i$. For each $\alpha \in \cup A_i$ let

$$X_\alpha = \{x(y) : y \in Y_\alpha\}\,,$$

$$X_i = \{x(y) : y \in Y_i\}\,.$$

Then $f(X_\alpha) = Y_\alpha$ and $f(X_i) = Y_i$. Since $f | \bar{U}_\alpha$ is closed and $\bar{U}_\alpha \cap f^{-1}(Y_\alpha) = X_\alpha$, f maps X_α onto Y_α homeomorphically. Hence each Y_α is σ_0.

Let us prove that $f(\bar{\mathfrak{U}}_i) = \{f(\bar{U}_\alpha) : \alpha \in A_i\}$ is locally finite in Y. For every point y in Y, $f^{-1}(y)$ meets at most finite elements of $\bar{\mathfrak{U}}_i$ by the compactness of $f^{-1}(y)$ and the discreteness of $\bar{\mathfrak{U}}_i$. Set

$$V = Y - \cup \{f(\bar{U}_\alpha) : \alpha \in A_i,\ \bar{U}_\alpha \cap f^{-1}(y) = \emptyset\}\,.$$

Then V is an open neighborhood of y by the closedness of f and the discreteness of $\bar{\mathfrak{U}}_i$. Since V meets at most finite elements of $f(\bar{\mathfrak{U}}_i)$, $f(\bar{\mathfrak{U}}_i)$ is locally finite in Y. Therefore $\{f(X_\alpha) = Y_\alpha : \alpha \in A_i\}$ is locally finite in Y. Hence Y_i is locally σ_0. By Lemma 2 Y_i is σ_0 and hence Y itself is σ_0.

PROOF OF THE THEOREM. Let Y_1 be the aggregate of all points y in Y with $\mathrm{B}(f^{-1}(y)) = \emptyset$. For each y in Y_1 select a point $x(y)$ from $f^{-1}(y)$. Let

$$X_1 = \{x(y) : y \in Y_1\}\,.$$

Then X_1 is closed in $f^{-1}(Y_1)$ since $\{x(y)\}$ is a discrete collection in $f^{-1}(Y_1)$. Therefore $f | X_1$ is a one-one closed mapping and hence a homeomorphism of X_1 onto Y_1. Since $\dim X_1 \leqq 0$, $\dim Y_1 \leqq 0$.

Let Y_2 be the aggregate of all points y in $Y - Y_1$ such that $\mathrm{B}(f^{-1}(y))(\neq \emptyset)$ is not dense-in-itself. Set

$$X_2 = \cup \{\mathrm{B}(f^{-1}(y)) : y \in Y_2\}\,.$$

Since $X_2 = f^{-1}(Y_2) - \cup \{\mathit{Interior\ of}\ f^{-1}(y) : y \in Y_2\}$, X_2 is closed in $f^{-1}(Y_2)$. Hence $f_1 = f | X_2$ is a closed mapping of X_2 onto Y_2. Since $f_1^{-1}(y) = \mathrm{B}(f^{-1}(y))$ for every $y \in Y_2$, $f_1^{-1}(y)$ is compact by Lemma 1. Hence by Lemma 3 Y_2 is σ_0. If we set

$$Y_0 = Y - Y_1 \cup Y_2\,,$$

Y_0 satisfies the desired condition.

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