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Reduction of logics to the primitive logic

By Katuzi ONO

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Introduction

Main conclusion of my work [2] has been the following: *Any logic belonging to* **J**-*series* (*the intuitionistic logic* **LJ**, *the minimal logic* **LM**, *and the positive logic* **LP**, *each without assuming Peirce's rule*) *or to* **K**-*series* (*the classical logic* **LK**, *the minimal logic* **LN**, *and the positive logic* **LQ** *which are stronger than* **LJ**, **LM** *and* **LP** *by Peirce's rule, respectively*) *can be faithfully interpreted in the primitive logic* **LO** (*the sub-logic of the intuitionistic logic* **LJ** *having the logical constants, implication and universal quantification, only*). I call here any logic **L** a *sub-logic* of another logic **L*** if and only if every logical constant of **L** is a logical constant of **L*** and every proposition expressible in terms of the logical constant of **L** is provable in **L** if and only if it is provable in **L***.

Faithful interpretation of the intuitionistic logic **LJ** and the classical logic **LK** in the primitive logic **LO** can be realized by $\mathfrak{R}$-*transform* $\mathfrak{A}^{[\mathfrak{R}]}$ of any proposition $\mathfrak{A}$ with respect to an n-ary relation $\mathfrak{R}$. $\mathfrak{A}^{[\mathfrak{R}]}$ can be defined recursively as follows (ξ stands for a sequence of n distinct variables, none of them is assumed to occur free in $\mathfrak{F}$ and $\mathfrak{G}$):

$\mathfrak{F}^{[\mathfrak{R}]} \rightleftarrows (\xi)((\mathfrak{F} \to \mathfrak{R}(\xi)) \to \mathfrak{R}(\xi))$ for any elementary formula $\mathfrak{F}$,

$$(\mathfrak{F} \to \mathfrak{G})^{[\mathfrak{R}]} \rightleftarrows (\mathfrak{F}^{[\mathfrak{R}]} \to \mathfrak{G}^{[\mathfrak{R}]}),$$

$$((t)\mathfrak{F})^{[\mathfrak{R}]} \rightleftarrows (t)\mathfrak{F}^{[\mathfrak{R}]},$$

$$(\mathfrak{F} \wedge \mathfrak{G})^{[\mathfrak{R}]} \rightleftarrows (\xi)((\mathfrak{F}^{[\mathfrak{R}]} \to (\mathfrak{G}^{[\mathfrak{R}]} \to \mathfrak{R}(\xi))) \to \mathfrak{R}(\xi)),$$

$$(\mathfrak{F} \vee \mathfrak{G})^{[\mathfrak{R}]} \rightleftarrows (\xi)((\mathfrak{F}^{[\mathfrak{R}]} \to \mathfrak{R}(\xi)) \to ((\mathfrak{G}^{[\mathfrak{R}]} \to \mathfrak{R}(\xi)) \to \mathfrak{R}(\xi))),$$

$$((\exists t)\mathfrak{F})^{[\mathfrak{R}]} \rightleftarrows (\xi)((t)(\mathfrak{F}^{[\mathfrak{R}]} \to \mathfrak{R}(\xi)) \to \mathfrak{R}(\xi)),$$

$$(\neg \mathfrak{F})^{[\mathfrak{R}]} \equiv \mathfrak{F}^{[\mathfrak{R}]} \to (\xi)\mathfrak{R}(\xi).$$

Now, we can prove the following theorem: $\mathfrak{A}$ *is provable in* **LJ** *if and only if* $\mathfrak{A}^{[R]}$ *is provable in* **LO**, *assuming that R is an n-ary relation symbol having no occurrence in* $\mathfrak{A}$ *for some n* ($n \geqq 1$). $\mathfrak{A}$ *is provable in* **LK** *if and only if* $\mathfrak{A}^{[R]}$ *is provable in* **LO**, *assuming that R is a 0-ary relation symbol i.e. proposition symbol having no occurrence in* $\mathfrak{A}$.

In my paper [2], I have stated this theorem in weaker form, namely, only for sufficiently large n for **LJ**. Moreover, the proof in [2] has been incomplete for **J**-series logics, because two erroneous formulas (erroneous in case of **J**-series logics) has been employed in the proof[1].

Main purpose of the present paper is to give theorems in stronger form for **J**- and **K**-series logics in general together with their corrected complete proofs.

In my paper [3], I have mentioned that any formal system having just one primitive notion (a single-word vocabulary) and standing on any one of **J**- or **K**-series logics can be interpreted more simply in a formal system having the same vocabulary and standing on the primitive logic **LO**. In showing this, I have employed essentially the same erroneous formulas of [2] (erroneous for **J**-series logics). In my work [4], I have described a method to reduce any vocabulary consisting of a finite number of words (primitive notions) to a single-word vocabulary, expecting that systems having single-word vocabularies are easier to deal with according to the conclusion of my work [3]. In reality, however, the merit of these two works seem to fade away in some cases even when we reformulate them so to avoid fallacious reasonings in them. In the case of **J**-series logics, the device of [3] for introducing new logical constants causes an interesting problem.

In my work [5], I have relied also on the result of my paper [3], so I have to reformulate it anyway. Essentially, however, the conclusion of [5] can be kept true. I will remark only a few words on the subject in the present paper, because a detailed description of taboo theory is expected to appear in the nearest future.

(1) **Preparations.**

The following notations used in my work [2] are also useful in the present paper:

$$\mathfrak{F}^{\mathfrak{R}} \underleftarrow{\equiv} (\xi)((\mathfrak{F} \to \mathfrak{R}(\xi)) \to \mathfrak{R}(\xi)),$$

$$\mathfrak{F} \underset{\mathfrak{R}}{\wedge} \mathfrak{G} \underleftarrow{\equiv} (\xi)((\mathfrak{F} \to (\mathfrak{G} \to \mathfrak{R}(\xi))) \to \mathfrak{R}(\xi)),$$

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1) In (4.11 R) of [2] (p. 350), I have mentioned that

$\mathfrak{A}^R \underset{R}{\wedge} \mathfrak{B}^R \equiv \mathfrak{A}^R \wedge \mathfrak{B}^R,$

$\mathfrak{A}^R \underset{R}{\vee} \mathfrak{B}^R \equiv \mathfrak{A}^R \vee \mathfrak{B}^R,$

$(\underset{R}{\exists} t)\, \mathfrak{A}^R \equiv (\exists t)\, \mathfrak{A}^R$

are all provable in **LP** as well as in **LQ**. This does not hold generally in **LP** except for the first formula if R is an n-ary relation symbol ($n > 1$). I owe to my young colleagues M. Ohta and K. Kawada in finding out this mis-reasoning.

$$\mathfrak{F} \underset{\mathfrak{R}}{\vee} \mathfrak{G} \equiv (\xi)((\mathfrak{F}\to\mathfrak{R}(\xi))\to((\mathfrak{G}\to\mathfrak{R}(\xi))\to\mathfrak{R}(\xi))),$$

$$\underset{\mathfrak{R}}{(\exists t)}\mathfrak{F}(t)\equiv(\xi)((t)(\mathfrak{F}(t)\to\mathfrak{R}(\xi))\to\mathfrak{R}(\xi)),$$

$$\underset{\mathfrak{R}}{\to}\mathfrak{F}\equiv\mathfrak{F}\to(\xi)\mathfrak{R}(\xi) \quad \text{or} \quad \underset{\mathfrak{R}}{\text{人}}\equiv(\xi)\mathfrak{R}(\xi).$$

Clearly, $\mathfrak{F}^{[\mathfrak{R}]}$ *is* $\mathfrak{F}^{\mathfrak{R}}$ *for any elementary proposition* $\mathfrak{F}$, $(\mathfrak{F}\wedge\mathfrak{G})^{[\mathfrak{R}]}$ *is* $\mathfrak{F}^{[\mathfrak{R}]}\underset{\mathfrak{R}}{\wedge}\mathfrak{G}^{[\mathfrak{R}]}$, $(\mathfrak{F}\vee\mathfrak{G})^{[\mathfrak{R}]}$ *is* $\mathfrak{F}^{[\mathfrak{R}]}\underset{\mathfrak{R}}{\vee}\mathfrak{G}^{[\mathfrak{R}]}$, $((\exists t)\mathfrak{F}(t))^{[\mathfrak{R}]}$ *is* $\underset{\mathfrak{R}}{(\exists t)}\mathfrak{F}(t)^{[\mathfrak{R}]}$, *and* $(\to\mathfrak{F})^{[\mathfrak{R}]}$ *is* $\underset{\mathfrak{R}}{\to}\mathfrak{F}^{[\mathfrak{R}]}$.

For any proposition $\mathfrak{F}$, the proposition $\mathfrak{F}^{\mathfrak{R}}$ is called the $\mathfrak{R}$-*closure* of $\mathfrak{F}$. *In any logic equivalent to or stronger than the primitive logic* **LO**, *any proposition* $\mathfrak{F}$ *implies its* $\mathfrak{R}$-*closure* $\mathfrak{F}^{\mathfrak{R}}$ *always.* Any proposition $\mathfrak{F}$ is called $\mathfrak{R}$-*closed* if and only if its $\mathfrak{R}$-closure $\mathfrak{F}^{\mathfrak{R}}$ implies $\mathfrak{F}$ itself. Accordingly, *any proposition* $\mathfrak{F}$ *is equivalent to its* $\mathfrak{R}$-*closure* $\mathfrak{F}^{\mathfrak{R}}$ *if and only if it is* $\mathfrak{R}$-*closed.* Hence, *we can express any* $\mathfrak{R}$-*closed proposition* in the form $\mathfrak{F}^{\mathfrak{R}}$.

We can also easily see that *any elementary formula of the form* $R(\xi)$ *is* R-*closed* for any relation symbol R (including the case where the same variable occurs in different places of ξ), that $\mathfrak{F}\to\mathfrak{G}$ *is* $\mathfrak{R}$-*closed whenever* $\mathfrak{G}$ *is* $\mathfrak{R}$-*closed,* and that $(t)\mathfrak{F}(t)$ *is* $\mathfrak{R}$-*closed if* $\mathfrak{F}(u)$ *is* $\mathfrak{R}$-*closed for any variable* u *whatever.* Accordingly, *propositions of the forms* $\mathfrak{F}^{[\mathfrak{R}]}$, $\mathfrak{F}\underset{\mathfrak{R}}{\wedge}\mathfrak{G}$, $\mathfrak{F}\underset{\mathfrak{R}}{\vee}\mathfrak{G}$, $\underset{\mathfrak{R}}{(\exists t)}\mathfrak{F}$, *and* $\underset{\mathfrak{R}}{\to}\mathfrak{F}$ (*or* $\underset{\mathfrak{R}}{\text{人}}$) *are all* $\mathfrak{R}$-*closed.* Also, we can see easily that *any* $\mathfrak{R}$-*closed proposition is deducible from* $(\xi)\mathfrak{R}(\xi)$.

Taking $\to\mathfrak{A}$ as $\mathfrak{A}\to\text{人}$, the intuitionistic logic **LJ** can be characterized by the following inference rules:

F. $\mathfrak{A}$ *is deducible from* $\mathfrak{A}$ *itself.*

I*. $\mathfrak{A}\to\mathfrak{B}$ *is deducible if* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}$.

I. $\mathfrak{A}$ *is deducible from* $\mathfrak{B}$ *and* $\mathfrak{B}\to\mathfrak{A}$.

C*. $\mathfrak{A}\wedge\mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *and* $\mathfrak{B}$.

C. $\mathfrak{A}$ *as well as* $\mathfrak{B}$ *is deducible from* $\mathfrak{A}\wedge\mathfrak{B}$.

D*. $\mathfrak{A}\vee\mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *as well as from* $\mathfrak{B}$.

D. $\mathfrak{A}$ *is deducible from* $\mathfrak{B}\vee\mathfrak{C}$, $\mathfrak{B}\to\mathfrak{A}$, *and* $\mathfrak{C}\to\mathfrak{A}$.

U*. $(t)\mathfrak{A}(t)$ *is deducible if* $\mathfrak{A}(u)$ *is deducible for any variable* u *whatever.*

U. $\mathfrak{A}(u)$ *is deducible from* $(t)\mathfrak{A}(t)$.

E*. $(\exists t)\mathfrak{A}(t)$ *is deducible from* $\mathfrak{A}(u)$.

E. $\mathfrak{B}$ *is deducible from* $(\exists t)\mathfrak{A}(t)$ *and* $(t)(\mathfrak{A}(t)\to\mathfrak{B})$, *assuming that* t *does not occur in* $\mathfrak{B}$.

N. $\mathfrak{A}$ *is deducible from* 人.

The classical logic **LK** admits further Peirce's rule:

P. $\mathfrak{A}$ *is deducible from* $(\mathfrak{A}\to\mathfrak{B})\to\mathfrak{A}$.

The positive logic **LP** admits the inference rules **F**, **I***, **I**, **C***, **C**, **D***, **D**, **U***, **U**, **E***, and **E** without assuming the propositional constant 人 (nor the logical constant $\rightarrow$). The positive logic **LQ** admits further the inference rule **P**.

The minimal logic **LM** and **LN** admit the same inference rules of **LP** and **LQ**, respectively, but the propositional constant $\curlywedge$ (hence also the negation notion) is assumed in them.

In the primitive logic **LO**, we can prove that *the newly defined logical constants* $\underset{\mathfrak{R}}{\wedge}$, $\underset{\mathfrak{R}}{\vee}$, $(\underset{\mathfrak{R}}{\exists})$, *and* $\underset{\mathfrak{R}}{\rightarrow}$ *(or* $\underset{\mathfrak{R}}{\curlywedge}$*) together with the original logical constants of* **LO** *satisfy all the inference rules of* **LJ** *for* $\mathfrak{R}$*-closed propositions. They satisfy further the inference rule* **P** *for* $\mathfrak{R}$*-closed propositions, if* $\mathfrak{R}$ *is a* 0*-ary relation i.e. a proposition.*

Speaking more precisely, we can prove that *the following inference rules hold in the primitive logic* **LO** (see my work [2].):

C*R. $\mathfrak{A} \underset{\mathfrak{R}}{\wedge} \mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *and* $\mathfrak{B}$.

CR. *Any* $\mathfrak{R}$*-closed* $\mathfrak{A}$ *is deducible from* $\mathfrak{A} \underset{\mathfrak{R}}{\wedge} \mathfrak{B}$. *Any* $\mathfrak{R}$*-closed* $\mathfrak{B}$ *is deducible from* $\mathfrak{A} \underset{\mathfrak{R}}{\wedge} \mathfrak{B}$.

D*R. $\mathfrak{A} \underset{\mathfrak{R}}{\vee} \mathfrak{B}$ *is deducible from* $\mathfrak{A}$ *as well as from* $\mathfrak{B}$.

DR. *Any* $\mathfrak{R}$*-closed* $\mathfrak{A}$ *is deducible from* $\mathfrak{B} \underset{\mathfrak{R}}{\vee} \mathfrak{C}$, $\mathfrak{B} \rightarrow \mathfrak{A}$, *and* $\mathfrak{C} \rightarrow \mathfrak{A}$.

E*R. $(\underset{\mathfrak{R}}{\exists} t)\mathfrak{A}(t)$ *is deducible from* $\mathfrak{A}(u)$.

ER. *Any* $\mathfrak{R}$*-closed* $\mathfrak{B}$ *is deducible from* $(\underset{\mathfrak{R}}{\exists} t)\mathfrak{A}(t)$ *and* $(t)(\mathfrak{A}(t) \rightarrow \mathfrak{B})$, *assuming that* t *does not occur in* $\mathfrak{B}$.

NR. *Any* $\mathfrak{R}$*-closed* $\mathfrak{A}$ *is deducible from* $\underset{\mathfrak{R}}{\curlywedge}$ *i.e.* $(\xi)\mathfrak{R}(\xi)$.

PR. *For any* $\mathfrak{R}$*-closed* $\mathfrak{A}$ *and* $\mathfrak{B}$, $\mathfrak{A}$ *is deducible from* $(\mathfrak{A} \rightarrow \mathfrak{B}) \rightarrow \mathfrak{A}$ *if* $\mathfrak{R}$ *is a* 0*-ary relation i.e. a proposition.*

For any inference rule **X**, let us denote by **X**$(\wedge, \overset{\circ}{\wedge})$, by **X**$(\vee, \overset{\circ}{\vee})$ by **X**$((\exists\ ), (\overset{\circ}{\exists}\ ))$, and by **X**$(\curlywedge, \overset{\circ}{\curlywedge})$ the inference rule, possibly valid or not valid, obtained on replacing $\wedge$, $\vee$, $(\exists\ )$, and $\curlywedge$ of **X** by $\overset{\circ}{\wedge}$, $\overset{\circ}{\vee}$, $(\overset{\circ}{\exists}\ )$, and $\overset{\circ}{\curlywedge}$, respectively. Then, **C*R**, **CR**, **D*R**, **DR**, **E*R**, **ER**, and **NR** point out that **C***$(\wedge, \underset{\mathfrak{R}}{\wedge})$, **C**$(\wedge, \underset{\mathfrak{R}}{\wedge})$, **D***$(\vee, \underset{\mathfrak{R}}{\vee})$, **D**$(\vee, \underset{\mathfrak{R}}{\vee})$, **E***$((\exists\ ), (\underset{\mathfrak{R}}{\exists}\ ))$, **E**$((\exists\ ), (\underset{\mathfrak{R}}{\exists}\ ))$, *and* **N**$(\curlywedge, \underset{\mathfrak{R}}{\curlywedge})$ *hold for* $\mathfrak{R}$*-closed propositions in any logic stronger than or equivalent to* **LO**, *if* $\mathfrak{R}$ *is an* n*-ary relation* ($n \geqq 0$). *The inference rule* **P** *holds for* $\mathfrak{R}$*-closed propositions in any logic stronger than or equivalent to* **LO**, *if* $\mathfrak{R}$ *is a* 0*-ary relation i.e. a proposition.* Also the inference rules **F**, **I***, **I**, **U***, **U**, **C***, **C**, **D***, **D**, **E***, and **E** hold for $\mathfrak{R}$-closed propositions in any logic stronger than or equivalent to **LP**, and the inference rule **N** holds for $\mathfrak{R}$-closed propositions in any logic stronger than or equivalent to **LJ**, though it is not always true that $\mathfrak{A} \vee \mathfrak{B}$, $\rightarrow \mathfrak{A}$, and $(\exists t)\mathfrak{A}(t)$ are all $\mathfrak{R}$-closed for any $\mathfrak{R}$-closed $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}(u)$ for any u whatever. (It is sure that $\mathfrak{A} \rightarrow \mathfrak{B}$, $\mathfrak{A} \wedge \mathfrak{B}$, and $(t)\mathfrak{A}(t)$ are all $\mathfrak{R}$-closed if $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}(u)$ for any u whatever are $\mathfrak{R}$-closed. It is sure in any logic stronger than or equivalent to **LQ** that $\mathfrak{A} \vee \mathfrak{B}$ and $(\exists t)\mathfrak{A}(t)$ are $\mathfrak{R}$-closed if $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}(u)$ for whatever u are $\mathfrak{R}$-closed. It is not sure even in **LK** that $\rightarrow \mathfrak{A}$ is $\mathfrak{R}$-closed for any $\mathfrak{R}$-closed $\mathfrak{A}$.)

We can now prove the following lemma:

LEMMA 1. *In any logic stronger than or equivalent to* **LP**, $\mathfrak{A} \wedge \mathfrak{B}$ *is equivalent to* $\mathfrak{A} \underset{\mathfrak{R}}{\wedge} \mathfrak{B}$ *if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are both* $\mathfrak{R}$*-closed,* $\mathfrak{A} \vee \mathfrak{B}$ *is equivalent to* $\mathfrak{A} \underset{\mathfrak{R}}{\vee} \mathfrak{B}$ *if* $\mathfrak{A}$, $\mathfrak{B}$, *and* $\mathfrak{A} \vee \mathfrak{B}$ *are all* $\mathfrak{R}$*-closed, and* $(\exists t)\mathfrak{A}(t)$ *is equivalent to* $(\underset{\mathfrak{R}}{\exists} t)\mathfrak{A}(t)$ *if* $(\exists t)\mathfrak{A}(t)$ *is* $\mathfrak{R}$*-closed. In any logic stronger than or equivalent to* **LJ**, $\rightharpoondown\mathfrak{A}$ *is equivalent to* $\underset{\mathfrak{R}}{\rightharpoondown}\mathfrak{A}$ *if* $\rightharpoondown\mathfrak{A}$ *is* $\mathfrak{R}$*-closed (or, if we assume that* 人 *is* $\mathfrak{R}$*-closed). Here we assume that* $\mathfrak{R}$ *is an n-ary relation* $(n \geqq 0)$.

PROOF. Let $\mathfrak{R}$ be any n-ary relation $(n \geqq 0)$. Now, let **L** be any logic stronger than or equivalent to **LP**.

At first, let us assume that $\mathfrak{A}$ *and* $\mathfrak{B}$ *are both* $\mathfrak{R}$*-closed.* Then, we can prove easily in **L** that $\mathfrak{A} \wedge \mathfrak{B}$ *is also* $\mathfrak{R}$*-closed.* Because $\wedge$ satisfies **C*** and **C** as well as $\underset{\mathfrak{R}}{\wedge}$ satisfies **C***$(\wedge, \underset{\mathfrak{R}}{\wedge})$ and **C**$(\wedge, \underset{\mathfrak{R}}{\wedge})$ for $\mathfrak{R}$-closed $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \wedge \mathfrak{B}$ the proposition $\mathfrak{A} \wedge \mathfrak{B}$ is equivalent to $\mathfrak{A} \underset{\mathfrak{R}}{\wedge} \mathfrak{B}$ in **L**.

Next, let us assume that $\mathfrak{A}$, $\mathfrak{B}$, *and* $\mathfrak{A} \vee \mathfrak{B}$ *are all* $\mathfrak{R}$*-closed.* Then, $\mathfrak{A} \vee \mathfrak{B}$ *is equivalent to* $\mathfrak{A} \underset{\mathfrak{R}}{\vee} \mathfrak{B}$, because **I** holds in **L**, $\vee$ satisfies **D*** and **D**, and $\underset{\mathfrak{R}}{\vee}$ satisfies **D***$(\vee, \underset{\mathfrak{R}}{\vee})$ and **D**$(\vee, \underset{\mathfrak{R}}{\vee})$ for $\mathfrak{R}$-closed propositions $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \vee \mathfrak{B}$.

Lastly, let us assume that $(\exists t)\mathfrak{A}(t)$ *and* $\mathfrak{A}(u)$ *for whatever variable u are* $\mathfrak{R}$*-closed.* Then, $(\exists t)\mathfrak{A}(t)$ *is equivalent to* $(\underset{\mathfrak{R}}{\exists} t)\mathfrak{A}(t)$, because **I** holds in **L**, $(\exists\ )$ satisfies **E*** and **E**, and $(\underset{\mathfrak{R}}{\exists}\ )$ satisfies **E***$((\exists\ ), (\underset{\mathfrak{R}}{\exists}\ ))$ and **E**$((\exists\ ), (\underset{\mathfrak{R}}{\exists}\ ))$ for any $\mathfrak{A}(u)$ and $\mathfrak{R}$-closed $\mathfrak{B}$.

Now, let **L** be any logic stronger than or equivalent to **LJ** and let us assume that $\mathfrak{A}$ *and* $\rightharpoondown\mathfrak{A}$ *are* $\mathfrak{R}$*-closed (or,* $\mathfrak{A}$ *and* 人 *are* $\mathfrak{R}$*-closed).* Then, $\rightharpoondown\mathfrak{A}$ *is equivalent to* $\underset{\mathfrak{R}}{\rightharpoondown}\mathfrak{A}$, because **I*** and **I** hold in **L**, 人 satisfies **N**, $\underset{\mathfrak{R}}{人}$ satisfies **N**(人, $\underset{\mathfrak{R}}{人}$) for $\mathfrak{R}$-closed proposition $\mathfrak{A}$.

In any logic stronger than or equivalent to **LQ**, $\mathfrak{A} \vee \mathfrak{B}$ *is* $\mathfrak{R}$*-closed if* $\mathfrak{A}$ *(or* $\mathfrak{B}$*) is so.* This can be shown in my way of practical description (see [**6**]) as follows:

A) $\mathfrak{A}^{\mathfrak{R}} \rightarrow \mathfrak{A}$.

$\in$)) $(\mathfrak{A} \vee \mathfrak{B})^{\mathfrak{R}} \rightarrow \mathfrak{A} \vee \mathfrak{B}$ /**I***.

$\in A$) $(\mathfrak{A} \vee \mathfrak{B})^{\mathfrak{R}}$ *i.e.* $(\xi)((\mathfrak{A} \vee \mathfrak{B} \rightarrow \mathfrak{R}(\xi)) \rightarrow \mathfrak{R}(\xi))$.

$\in b$)) $(\mathfrak{A} \vee \mathfrak{B} \rightarrow \mathfrak{A}) \rightarrow \mathfrak{A} \vee \mathfrak{B}$ /**I***.

$\in bA$) $\mathfrak{A} \vee \mathfrak{B} \rightarrow \mathfrak{A}$.

$\in bb$)) $\mathfrak{A}^{\mathfrak{R}}$ *i.e.* $(\xi)((\mathfrak{A} \rightarrow \mathfrak{R}(\xi)) \rightarrow \mathfrak{R}(\xi))$ /**I***, **U***.

$\in bbA$) $\forall\xi : \mathfrak{A} \rightarrow \mathfrak{R}(\xi)$. ('$\forall$:' stands for '$\forall$!' in [**6**].)

$\in bbb$)) $\mathfrak{A} \vee \mathfrak{B} \rightarrow \mathfrak{R}(\xi)$ /**I***.

$\in bbbA$) $\mathfrak{A} \vee \mathfrak{B}$.

$\in bbbb$) $\mathfrak{A}$ /$\in bA$, $\in bbbA$; **I**.

$\in bbb\in$) $\mathfrak{R}(\xi)$ /$\in bbA$, $\in bbbb$; **I**.

$\in bbc$) $(\mathfrak{A}\vee\mathfrak{B}\to\mathfrak{R}(\xi))\to\mathfrak{R}(\xi)$ $/\in A$; **U**.
$\in bb\in$) $\mathfrak{R}(\xi)$ $/\in bbb$, $\in bbc$; **I**.
$\in bc$) $\mathfrak{A}$ $/A$, $\in bb$; **I**.
$\in b\in$) $\mathfrak{A}\vee\mathfrak{B}$ $/\in bc$; **D***.
$\in\in$) $\mathfrak{A}\vee\mathfrak{B}$ $/\in b$; **P**.

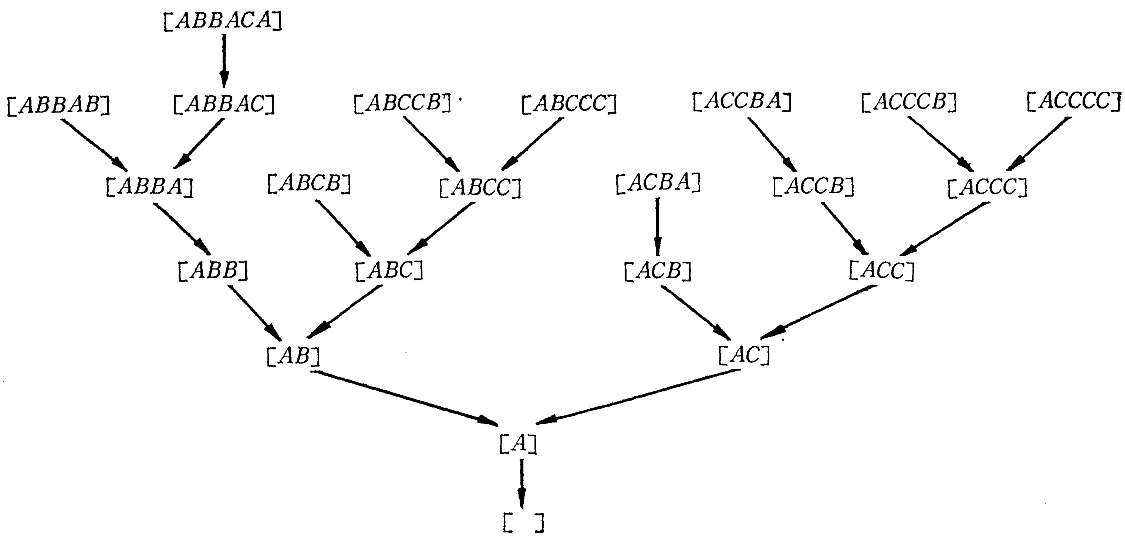
Quite similarly, we can prove that, *in any logic stronger than or equivalent to* **LQ**, $(\exists t)\mathfrak{A}(t)$ *is* $\mathfrak{R}$*-closed if* $\mathfrak{A}(u)$ *is* $\mathfrak{R}$*-closed for any variable u.* (We prove $((\exists t)\mathfrak{A}(t)\to\mathfrak{A}(u))\to(\exists t)\mathfrak{A}(t)$ in place of $(A\vee B\to A)\to A\vee B$ in the above proof.)

Naturally, in any logic stronger than or equivalent to **LM**, $\to\mathfrak{A}$ *i.e.* $\mathfrak{A}\to$人 is $\mathfrak{R}$-closed if we assume 人$^{\mathfrak{R}}\to$人.

Accordingly, we can prove the following lemma:

LEMMA 2. *In any logic stronger than or equivalent to* **LQ**, $\mathfrak{A}\wedge\mathfrak{B}$ *and* $\mathfrak{A}\vee\mathfrak{B}$ *are equivalent to* $\mathfrak{A}\underset{\mathfrak{R}}{\wedge}\mathfrak{B}$ *and* $\mathfrak{A}\underset{\mathfrak{R}}{\vee}\mathfrak{B}$, *respectively, if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are both* $\mathfrak{R}$*-closed, and* $(\exists t)\mathfrak{A}(t)$ *is equivalent to* $\underset{\mathfrak{R}}{(\exists t)}\mathfrak{A}(t)$ *if* $\mathfrak{A}(u)$ *is* $\mathfrak{R}$*-closed for whatever variable u. In any logic stronger than or equivalent to* **LJ**, $\to\mathfrak{A}$ *is equivalent to* $\underset{\mathfrak{R}}{\to}\mathfrak{A}$ *if we assume* 人$^{\mathfrak{R}}\to$人. *Here we assume that* $\mathfrak{R}$ *is an n-ary relation* $(n\geqq 0)$.

Before stating theorems on reductions of the logics **LJ** and **LK** to the primitive logic **LO**, I would like to remark here that **LJ** and **LK** can be regarded as Gentzen's **LJ** and **LK**, respectively. *In Gentzen's formalism, any proof is described in a tree-form proof-figure having its end-sequent at its bottom. Tree-form proof-figure* can be described more exactly by supplying every sequent an *index* of it. The following example would give a global image of our index system.



Namely, *any index is a finite (possibly null) sequence of three kinds of symbols A, B, and C,* which is occasionally denoted by a small German letters.

If $\mathfrak{b}$ is of the form $\mathfrak{a}\mathfrak{c}$, $\mathfrak{a}$ is called *equal to or less than* $\mathfrak{b}$ according as $\mathfrak{c}$ is a null sequence or not. The ordering '*less than*' of indices is evidently a partial order.

I will call any non-empty set **T** of indices a *tree* if and only if **T** satisfies the following conditions:

T1. *For any index of the forms $\mathfrak{c}A$, $\mathfrak{c}B$, or $\mathfrak{c}C$ in* **T**, $\mathfrak{c}$ *is a member of T unless $\mathfrak{c}A$, $\mathfrak{c}B$, or $\mathfrak{c}C$ is the minimum index of* **T**.

T2. *For any $\mathfrak{c}$ in* **T**, *either $\mathfrak{c}$ is a maximal index of* **T**, *or* **T** *contains $\mathfrak{c}A$ but no $\mathfrak{c}B$ nor $\mathfrak{c}C$, or* **T** *contains $\mathfrak{c}B$ and $\mathfrak{c}C$ but no $\mathfrak{c}A$.*

It follows easily from **T1** that there is a *minimum index* (usually a null sequence) in every tree **T**.

Any set $\Pi[\mathbf{T}]$ of sequents, each supplied by an index of a tree **T**, is called a (*Gentzen-type*) *proof-figure* if and only if $\Pi[\mathbf{T}]$ satisfy the following conditions ($[\mathfrak{c}]$ denotes the sequent of $\Pi[\mathbf{T}]$ which is supplied by the index $\mathfrak{c}$):

Π T1. *If $\mathfrak{c}$ is an maximal index of* **T**, *the sequent $[\mathfrak{c}]$ is fundamental (sequent of the form $\mathfrak{M} \vdash \mathfrak{M}$).*

Π T2. *$[\mathfrak{c}]$ is deducible from $[\mathfrak{c}A]$ if $\mathfrak{c}$ and $\mathfrak{c}A$ belong to* **T**. *$[\mathfrak{c}]$ is dedubible from $[\mathfrak{c}B]$ and $[\mathfrak{c}C]$ if $\mathfrak{c}$, $\mathfrak{c}B$, and $\mathfrak{c}C$ belong to* **T**.

If $\mathfrak{a}$ is the minimum index of **T** and $\Pi[\mathbf{T}]$ is a proof-figure, $\Pi[\mathbf{T}]$ is called a (*Gentze-type*) *proof* of the sequent $[\mathfrak{a}]$. If $\mathfrak{c}$ is an index of **T** of a proof-figure $\Pi[\mathbf{T}]$, the set of all the sequents $[\mathfrak{d}]$ supplied by indices $\mathfrak{d}$ equal to or greater than $\mathfrak{c}$ can be regarded as a proof of $[\mathfrak{c}]$. The set of all the indices $\mathfrak{d}$ of **T** equal to or greater than $\mathfrak{c}$ is denoted by $\mathbf{T}[\mathfrak{c}]$ and is occasionally called a *sub-tree* of the tree **T**. Naturally, $\mathbf{T}(\mathfrak{c})$ *is a tree, and the proof of $[\mathfrak{c}]$ is denoted by $\Pi[\mathbf{T}(\mathfrak{c})]$.*

Now, I will prove the following lemma:

LEMMA 3. *Let* **T** *be any tree containing two indices $\mathfrak{b}$ and $\mathfrak{c}$. Then, one of $\mathbf{T}(\mathfrak{b})$ and $\mathbf{T}(\mathfrak{c})$ is included in the other if they have a common index.*

PROOF. Let **T** be a tree, and let $\mathfrak{b}$ and $\mathfrak{c}$ be two indices in it. Let us further assume that $\mathbf{T}(\mathfrak{b})$ and $\mathbf{T}(\mathfrak{c})$ have a common index. Then, we can find out at least one minimal common index $\mathfrak{d}$ of these two sub-trees of **T**. If $\mathfrak{d}$ were neither equal to $\mathfrak{b}$ nor equal to $\mathfrak{c}$, it would be expressed in the forms $\mathfrak{d}'A$, $\mathfrak{d}'B$, or $\mathfrak{d}'C$. $\mathfrak{d}'$ would be less than $\mathfrak{d}$ and $\mathfrak{d}'$ would be a common index of $\mathbf{T}(\mathfrak{b})$ and $\mathbf{T}(\mathfrak{c})$ according to **T1**. Hence, $\mathfrak{d}$ must be either equal to $\mathfrak{b}$ or equal to $\mathfrak{c}$. Consequently, either $\mathbf{T}(\mathfrak{b})$ is included in $\mathbf{T}(\mathfrak{c})$ or $\mathbf{T}(\mathfrak{c})$ is included in $\mathbf{T}(\mathfrak{b})$.

I will call any inference of the following two kinds

1) $\mathfrak{a}: \Gamma, (\exists t)\mathfrak{F}(t) \vdash \Delta, \qquad \mathfrak{a}A: \Gamma, \mathfrak{F}(u) \vdash \Delta,$

2) $\mathfrak{a}: \Gamma \vdash \Delta, (t)\mathfrak{F}(t), \qquad \mathfrak{a}A: \Gamma \vdash \Delta, \mathfrak{F}(u).$

a *u-inference* depending on its *proper variable u*.

The following lemma can be checked without difficulty.

LEMMA 4. *Let* $\mathfrak{F} \to \mathfrak{F}^*$ *be a transformation of propositions satisfying the following conditions:*

$$(\mathfrak{F} \to \mathfrak{G})^* \equiv \mathfrak{F}^* \to \mathfrak{G}^*, \qquad (\neg \mathfrak{F})^* \equiv \neg \mathfrak{F}^*,$$

$$(\mathfrak{F} \wedge \mathfrak{G})^* \equiv \mathfrak{F}^* \wedge \mathfrak{G}^*, \qquad ((t)\mathfrak{F})^* \equiv (t)\mathfrak{F}^*,$$

$$(\mathfrak{F} \vee \mathfrak{G})^* \equiv \mathfrak{F}^* \vee \mathfrak{G}^*, \qquad ((\exists t)\mathfrak{F})^* \equiv (\exists t)\mathfrak{F}^*.$$

Then, any inference is transformed into a right inference except for u-inferences with respect to any variable u.

I will call $\mathbf{T}(\mathfrak{c})$ a *u-tree* if and only if the last inference of $\Pi[\mathbf{T}(\mathfrak{c})]$ is a u-inference of the type from $[\mathfrak{c}A]$ to $[\mathfrak{c}]$. Any proof-figure $\Pi[\mathbf{T}]$ (or any proof $\Pi[\mathbf{T}]$ of a sequent) is called *normal* if and only if every free variable u having a u-tree does not occur outside of $\Pi[\mathbf{T}(\mathfrak{c})]$ for any u-tree $\mathbf{T}(\mathfrak{c})$ of it. Accordingly, in any normal proof-figure, any free variable u in it has at most one u-tree.

LEMMA 5. *If any sequent is provable in Gentzen's* **LJ** *or* **LK**, *it has a normal cut-free proof of it.*

PROOF. Let $\Gamma \vdash \Delta$ be any provable sequent in Gentzen's **LJ** or **LK**. Then, according to Gentzen's cut-elimination theorem (see [1]), $\Gamma \vdash \Delta$ is provable by a cut-free proof $\Pi[\mathbf{T}]$ having $\Gamma \vdash \Delta$ as $[\mathfrak{a}]$ for the minimum index $\mathfrak{a}$ of $\mathbf{T}$. If there is a free variable u which occurs outside of $\Pi[\mathbf{T}(\mathfrak{c})]$ for a u-tree $\mathbf{T}(\mathfrak{c})$ of $\Pi[\mathbf{T}]$, we transform the figure as follows.

According to Lemma 3, we can find out for any free occurrence of u in a sequent $[\mathfrak{b}]$ of $\Pi[\mathbf{T}]$ the smallest u-tree $\mathbf{T}[\mathfrak{c}]$ in so far as there is a u-tree containing $\mathfrak{b}$. Let us denote the function $\mathfrak{b}$ to $\mathfrak{c}$ by $\mathfrak{c} = |\mathfrak{b}|$. Then, $|\mathfrak{b}|$ is surely less than $\mathfrak{b}$, because u does not occur in the sequent $[|\mathfrak{b}|]$.

Now, *we replace every free occurrence of u in the sequent* $[\mathfrak{b}]$ *by the variable* $u_{|\mathfrak{b}|}$ in so far as $|\mathfrak{b}|$ is defined, where variables of the form $u_{|\mathfrak{b}|}$ denote new variables mutually distinguished by their indices $|\mathfrak{b}|$. The variable u in $[\mathfrak{b}]$ remains unchanged if $|\mathfrak{b}|$ is not defined.

Let us denote by $\Pi^*[\mathbf{T}]$ the transformed figure of $\Pi[\mathbf{T}]$. Then, we assert that $\Pi^*[\mathbf{T}]$ *is a cut-free proof of* $[\mathfrak{a}]$ in which the number of variables w occcurring outside of $\Pi^*[\mathbf{T}(\mathfrak{c})]$ for some w-tree $\mathbf{T}(\mathfrak{c})$ of $\Pi^*[\mathbf{T}]$ is less than the number of such kind of free variables in $\Pi[\mathbf{T}]$.

To show this, we remark at first that *any fundamental sequent is transformed into a fundamental sequent.* Next, $[\mathfrak{a}]$ *is invariant by the transformation.* For, $\mathfrak{a}$ is the minimum index of $\mathbf{T}$, so $|\mathfrak{a}|$ is not defined. Thirdly, *any u-inference from* $[\mathfrak{c}A]$ *to* $[\mathfrak{c}]$ *is transformed into a right u-inference,* because u does not occur in $[\mathfrak{c}]$. Lastly, *any inference from* $[\mathfrak{c}A]$ *to* $[\mathfrak{c}]$ *or from* $[\mathfrak{c}B]$ *and* $[\mathfrak{c}C]$ *to* $[\mathfrak{c}]$ *other than u-inferences is transformed into a right inference of the same type* according to Lemma 4, because $[\mathfrak{c}]$ and $[\mathfrak{c}A]$ (or, $[\mathfrak{c}]$, $[\mathfrak{c}B]$, and $[\mathfrak{c}C]$) undergo the same transformation satisfying the conditions of Lemma 4. For, $|\mathfrak{c}|=|\mathfrak{c}A|$ (or, $|\mathfrak{c}|=|\mathfrak{c}B|=|\mathfrak{c}C|$) holds in these cases.

Now, I will show that *neither u nor variables of the form $u_{|\mathfrak{b}|}$ occur outside of $\Pi^*[\mathbf{T}(\mathfrak{c})]$ for any u- or $u_{|\mathfrak{b}|}$-tree* $\mathbf{T}(\mathfrak{c})$. For, there is no u-tree in $\Pi^*[\mathbf{T}(\mathfrak{c})]$, and $u_{|\mathfrak{b}|}$ occurs only in $\Pi^*[\mathbf{T}(|\mathfrak{b}|)]$ for the $u_{|\mathfrak{b}|}$-tree $\mathbf{T}(|\mathfrak{b}|)$ which does not include any other $u_{|\mathfrak{b}|}$-trees.

Since $\Pi[\mathbf{T}]$ is assumed to be cut-free and any inference of the proof-figure $\Pi^*[\mathbf{T}]$ is of the same type as the corresponding inference of $\Pi[\mathbf{T}]$, *the proof-figure $\Pi^*[\mathbf{T}]$ is also cut-free.*

Thus, we have a cut-free proof $\Pi^*[\mathbf{T}]$ of $[\mathfrak{a}]$ in which the number of free variables w occurring outside of $\Pi^*[(\mathfrak{c})]$ for a w-tree $\mathbf{T}(\mathfrak{c})$ of $\Pi^*[\mathbf{T}]$ is less than the number of such kind of free variables in $\Pi[\mathbf{T}]$. Hence, after a finite number of steps of the same kind, we can attain a normal cut-free proof of $[\mathfrak{a}]$.

(2) Reductions.

THEOREM 1. *Any proposition $\mathfrak{A}$ is provable in* **LJ** *if and only if its R-transform $\mathfrak{A}^{[R]}$ is provable in* **LO** *for any n-ary relation symbol R which does not occur in $\mathfrak{A}$ ($n \geqq 1$).*

PROOF. Let $\mathfrak{A}$ be any proposition and R be any n-ary relation symbol which does not occur in $\mathfrak{A}$ ($n \geqq 1$). Then, any elementary formula of the form $R(\xi)$ can be expressed in the form $R(\xi', x)$ (including the case where ξ' is a null sequence).

I will show at first that $\mathfrak{A}$ *is provable in* **LJ** *if* $\mathfrak{A}^{[R]}$ *is provable in* **LO**. Namely, let us assume that $\mathfrak{A}^{[R]}$ is provable in **LO**. Then, $\mathfrak{A}^{[R]}$ must be also provable in **LJ**, because **LO** is a sub-logic of **LJ**.

Now, I will call any proposition of the form $(\zeta)(\exists x)(R(\xi', x) \equiv \mathfrak{F}(\zeta))$ an $\mathfrak{A}$-*formula* if and only if $\mathfrak{F}(\zeta)$ is a sub-formula of $\mathfrak{A}$ and x does not occur in $\mathfrak{F}(\zeta)$. (ζ denotes a sequence of distinct variables. We call here any formula a *sub-formula for itself* and any formula of the form $\mathfrak{H}(t)$ a *sub-formula of* $(x)\mathfrak{H}(x)$ *as well as of* $(\exists x)\mathfrak{H}(x)$ for any variable t.)

It should be remarked here that *any proposition of the form* $(\zeta)(\mathfrak{F}(\zeta)^R \rightarrow \mathfrak{F}(\zeta))$ *is deducible from* $(\zeta)(\exists x)(R(\xi', x) \equiv \mathfrak{F}(\zeta))$ *in* **LJ**. We describe the proof in my way of practical description (see [6]) in the following:

A) $(\zeta)(\exists x)(R(\xi', x) \equiv \mathfrak{F}(\zeta))$.

$\in$)) $(\zeta)(\mathfrak{F}(\zeta)^R \rightarrow \mathfrak{F}(\zeta))/\mathbf{I}^*, \mathbf{U}^*$.

$\in A$) $\forall\zeta : \mathfrak{F}(\zeta)^R$ *i.e.* $(\mu)((\mathfrak{F}(\zeta) \rightarrow R(\mu)) \rightarrow R(\mu))$.

$\in b$) $(\exists x)(R(\xi', x) \equiv \mathfrak{F}(\zeta))/A$; $\mathbf{U}$.

$\in c$)) $(x)((R(\xi', x) \equiv \mathfrak{F}(\zeta)) \rightarrow \mathfrak{F}(\zeta))/\mathbf{I}^*, \mathbf{U}^*$.

$\in cA$) $\forall x : R(\xi', x) \equiv \mathfrak{F}(\zeta)$.

$\in cb)$ $(\mathfrak{F}(\zeta)\to R(\xi', x))\to R(\xi', x)/\in A$; **U.**

$\in cc)$ $\mathfrak{F}(\zeta)\to R(\xi', x)/\in cA$; **C**, Def. of $\equiv$.

$\in cd)$ $R(\xi', x)/\in cb, \in cc$; **I.**

$\in ce)$ $R(\xi', x)\to\mathfrak{F}(\zeta)/\in cA$; **C**, Def. of $\equiv$.

$\in c\in)$ $\mathfrak{F}(\zeta)/\in cd, \in ce$; **I.**

$\in\in)$ $\mathfrak{F}(\zeta)/\in b, \in c$; **E**, Assumption that x does not occur in $\mathfrak{F}(\zeta)$.

For any sub-formula $\mathfrak{B}$ of $\mathfrak{A}$, we can deduce $\mathfrak{B}^{[R]}\equiv\mathfrak{B}$ from a certain number of $\mathfrak{A}$-formulas. Lemma 1 makes it possible to prove this by structural induction.

Namely, let us assume at first that $\mathfrak{B}$ *is an elementary formula of the form* $\mathfrak{F}(\eta)$. Then, $\mathfrak{F}(\eta)^{[R]}$ is $\mathfrak{F}(\eta)^{R}$, which is surely deducible from $\mathfrak{F}(\eta)$. Accordingly, $\mathfrak{B}^{[R]}\equiv\mathfrak{B}$ *i.e.* $\mathfrak{F}(\eta)^{[R]}\equiv\mathfrak{F}(\eta)$ is deducible from $(\zeta)(\mathfrak{F}(\zeta)^{R}\to\mathfrak{F}(\zeta))$, so also from the $\mathfrak{A}$-formula $(\zeta)(\exists x)(R(\xi', x)\equiv\mathfrak{F}(\zeta))$.

Next, let us assume that $\mathfrak{B}$ *is a composite proposition of the forms* $\mathfrak{C}\to\mathfrak{D}$, $\mathfrak{C}\wedge\mathfrak{D}$, *or* $\mathfrak{C}\vee\mathfrak{D}$ and that $\mathfrak{C}^{[R]}\equiv\mathfrak{C}$ *and* $\mathfrak{D}^{[R]}\equiv\mathfrak{D}$ *are deducible from the sets* Γ *and* Δ *of* $\mathfrak{A}$*-formulas, respectively.* Then, $(\mathfrak{C}\to\mathfrak{D})^{[R]}\equiv(\mathfrak{C}\to\mathfrak{D})$ *is deducible from the set* $\Gamma\cup\Delta$ *of* $\mathfrak{A}$*-formulas,* because $(\mathfrak{C}\to\mathfrak{D})^{[R]}$ is $\mathfrak{C}^{[R]}\to\mathfrak{D}^{[R]}$. $(\mathfrak{C}\wedge\mathfrak{D})^{[R]}\equiv(\mathfrak{C}\wedge\mathfrak{D})$ *is also deducible from the set* $\Gamma\cup\Delta$ *of* $\mathfrak{A}$*-formulas,* because $(\mathfrak{C}\wedge\mathfrak{D})^{[R]}$ is $\mathfrak{C}^{[R]}\underset{R}{\wedge}\mathfrak{D}^{[R]}$ which is equivalent to $\mathfrak{C}^{[R]}\wedge\mathfrak{D}^{[R]}$ for the R-closed propositions $\mathfrak{C}^{[R]}$ and $\mathfrak{D}^{[R]}$ according to Lemma 1. $(\mathfrak{C}\vee\mathfrak{D})^{[R]}\equiv(\mathfrak{C}\vee\mathfrak{D})$ *is deducible from the set of* $\mathfrak{A}$*-formulas*

$$\Gamma\cup\Delta\cup\{(\exists x)(R(\xi', x)\equiv\mathfrak{C}\vee\mathfrak{D})\}\,.$$

For, $(\mathfrak{C}\vee\mathfrak{D})^{[R]}$ is $\mathfrak{C}^{[R]}\underset{R}{\vee}\mathfrak{D}^{[R]}$ which is equivalent to $\mathfrak{C}^{[R]}\vee\mathfrak{D}^{[R]}$ for the R-closed propositions $\mathfrak{C}^{[R]}$ and $\mathfrak{D}^{[R]}$ according to Lemma 1, since it can be deduced from the above set of $\mathfrak{A}$-formulas that $\mathfrak{C}^{[R]}\vee\mathfrak{D}^{[R]}$ *i.e.* $\mathfrak{C}\vee\mathfrak{D}$ is also R-closed. Anyway, in all of these cases, $\mathfrak{B}^{[R]}\equiv\mathfrak{B}$ is deducible from a certain numder of $\mathfrak{A}$-formulas.

If we assume that $\mathfrak{B}$ *is* $\neg\mathfrak{C}$ *and that* $\mathfrak{C}^{[R]}\equiv\mathfrak{C}$ *is deducible from the set* Γ *of* $\mathfrak{A}$*-formulas,* then $\mathfrak{B}^{[R]}\equiv\mathfrak{B}$ *i.e.* $(\neg\mathfrak{C})^{[R]}\equiv\neg\mathfrak{C}$ *is also deducible from the set of* $\mathfrak{A}$*-formulas*

$$\Gamma\cup\{(\exists x)(R(\xi', x)\equiv\neg\mathfrak{C})\}\,.$$

For, $(\neg\mathfrak{C})^{[R]}$ is $\underset{R}{\neg}\mathfrak{C}^{[R]}$ which is equivalent to $\neg\mathfrak{C}^{[R]}$, hence also to $\neg\mathfrak{C}$, under the assumption Γ according to Lemma 1, since it can be deduced from the above set of $\mathfrak{A}$-formulas that $\neg\mathfrak{C}$ is also R-closed.

Now, let us assume that $\mathfrak{B}$ *is a formula of the forms* $(t)\mathfrak{C}(t)$ *or* $(\exists t)\mathfrak{C}(t)$ and that $\mathfrak{C}(u)^{[R]}\equiv\mathfrak{C}(u)$ *is deducible from the set* $\Gamma(u)$ *of* $\mathfrak{A}$*-formulas of the form* $(\zeta)(\exists x)(R(\xi', x)\equiv\mathfrak{F}(\zeta, u))$. Then, $((t)\mathfrak{C}(t))^{[R]}\equiv(t)\mathfrak{C}(t)$ *is deducible from* Γ, bacause $((t)\mathfrak{C}(t))^{[R]}$ is $(t)(\mathfrak{C}(t))^{[R]}$. Here, Γ denotes the set of $\mathfrak{A}$-formulas $(\zeta)(t)(\exists x)(R(\xi', x)\equiv\mathfrak{F}(\zeta, t))$ for every proposition of the form $(\zeta)(\exists x)(R(\xi', x)\equiv\mathfrak{F}(\zeta, u))$ in $\Gamma(u)$.

$((\exists t)\mathfrak{G}(t))^{[R]}\equiv(\exists t)\mathfrak{G}(t)$ *is deducible from the set of* $\mathfrak{A}$*-formulas*

$$\Gamma\cup\{(\exists x)(R(\xi', x)\equiv(\exists t)\mathfrak{G}(t))\}$$

according to Lemma 1, since $((\exists t)\mathfrak{G}(t))^{[R]}$ *i.e.* $(\underset{R}{\exists} t)\mathfrak{G}(t)^{[R]}$ is equivalent to $(\underset{R}{\exists} t)\mathfrak{G}(t)$ under the assumption Γ and $(\underset{R}{\exists} t)\mathfrak{G}(t)$ is proved to be equivalent to $(\exists t)\mathfrak{G}(t)$ by assuming that $(\exists t)\mathfrak{G}(t)$ is R-closed, so by assuming $(\exists x)(R(\xi', x)\equiv(\exists t)\mathfrak{G}(t))$. Anyway, also in these cases, $\mathfrak{B}^{[R]}\equiv\mathfrak{B}$ is deducible from a certain number of $\mathfrak{A}$-formulas.

Thus, we have shown that $\mathfrak{A}^{[R]}\equiv\mathfrak{A}$ is deducible from a set $\sum$ of $\mathfrak{A}$-formulas in **LJ**. Because $\mathfrak{A}^{[R]}$ is assumed to be provable in **LJ**, $\mathfrak{A}$ must be deducible from $\sum$ in **LJ**. Hence, according to Lemma 5, there must be a normal cut-free proof $\Pi[\mathbf{T}]$ of the sequent $\sum\vdash\mathfrak{A}$.

Let us now list up all the free variables w occurring at least once in a part of the form $R(\xi', w)$ of propositons in $\Pi[\mathbf{T}]$. Then, *for any listed variable* w there is one and only one w-tree $\mathbf{T}(\mathfrak{c})$, becasue $\Pi[\mathbf{T}]$ is a normal cut-free proof of $\sum\vdash\mathfrak{A}$ in which R occurs only in propositions of the form $(\zeta)(\exists x)R(\xi', x)\equiv\mathfrak{F}(\zeta))$ for $\mathfrak{F}(\zeta)$ containing no R in $\sum$. *Let us denote the function* w *to* $\mathfrak{c}$ *by* $\mathfrak{c}=|w|$. Then, *for any listed variable* w, $\mathbf{T}(|w|)$ contains the index $|w|A$ and $\Pi[\mathbf{T}]$ contains an inference of the form

$$|w|:\Gamma, (\exists x)(R(\xi', x)\equiv\mathfrak{F}(\eta))\vdash\Delta\,,$$
$$|w|A:\Gamma, R(\xi', w)\equiv\mathfrak{F}(\eta)\vdash\Delta\,.$$

Thus, we can associate to every listed free variable w one and only one formula $\mathfrak{F}(\eta)$ in the sequent $[|w|]$. *I will denote the function* w *to* $\mathfrak{F}(\eta)$ *by* $\mathfrak{F}(\eta)=R[w]$.

Now, we transform $\Pi[\mathbf{T}]$ by the following rule: *Delete every proposition in any sequent of* $\Pi[\mathbf{T}]$ *which has any part of the form* $R(\xi', x)$ *for a bound variable* x, *and replace every part of the form* $R(x', w)$ *of any other proposition in* $\Pi[\mathbf{T}]$ *for a free variable* w *(*w *is surely listed!) by the proposition* $R[w]$.

For the proof-figure $\Pi[\mathbf{T}]$, *for any sequent* $[\mathfrak{b}]$ *for any sequence* Γ *of propositions, and for any proposition* $\mathfrak{G}$ *in* $\Pi[\mathbf{T}]$, *let us denote the figure, the sequent, the sequence of propositions, and the proposition obtained by the above replacement from* $\Pi[\mathbf{T}]$, $[\mathfrak{b}]$, Γ *and* $\mathfrak{G}$ *by* $\Pi^*[\mathbf{T}]$, $[\mathfrak{b}]^*$, Γ^*, *and* $\mathfrak{G}^*$, *respectively*. Then, I assert that $\Pi^*[\mathbf{T}]$ *is a right proof-figure in* **LJ**.

To show this, I will remark at first that *any sequent* $[\mathfrak{c}]$ *in* $\Pi[\mathbf{T}]$ *for a maximal index* $\mathfrak{c}$ *of* $\mathbf{T}$ *has no proposition containing a part of the form* $R(\xi', x)$ *for a bound variable* x. For, $[\mathfrak{c}]$ must be a fundamental sequent of the form $\mathfrak{M}\vdash\mathfrak{M}$ if $\mathfrak{c}$ is a maximal index of $\mathbf{T}$. Because $\Pi[\mathbf{T}]$ is cut-free, there must be a proposition among the propositions in $\sum\vdash\mathfrak{A}$ having a sub-formula of the forms $\mathfrak{M}\to\mathfrak{N}$, $\mathfrak{N}\to\mathfrak{M}$, or $\to\mathfrak{M}$. However, this is impossible for any $\mathfrak{M}$

having a part of the form $R(\xi', x)$ for a bound variable x, because $\sum$ consists of propositions of the form $(\zeta)(\exists x)(R(\xi', x) \equiv \mathfrak{F}(\zeta))$ for $\mathfrak{F}(\zeta)$ with no occurrence of R and the proposition $\mathfrak{A}$ with no occurrence of R. Accordingly, $[\mathfrak{c}]^*$ is also a fundamental sequent.

It can be easily checked by making use of Lemma 4 that *any inference of the form* $[\mathfrak{b}]$ *from* $[\mathfrak{b}A]$ *as well as any inference of the form* $[\mathfrak{b}]$ *from* $[\mathfrak{b}B]$ *and* $[\mathfrak{b}C]$ *except for such* $\mathfrak{b}$ *satisfying* $\mathfrak{b}=|w|$ *for any listed variable* w *is transformed into a right inference* $[\mathfrak{b}]^*$ *from* $[\mathfrak{b}A]^*$ *or* $[\mathfrak{b}]^*$ *from* $[\mathfrak{b}B]^*$ *and* $[\mathfrak{b}C]^*$ *of the same type.*

For any listed free variable w, the inference

$$|w| : \Gamma, (\exists x)(R(\xi', x) \equiv \mathfrak{F}(\eta)) \vdash \Delta ,$$

$$|w| A : \Gamma, R(\xi', w) \equiv \mathfrak{F}(\eta) \vdash \Delta$$

is transformed into

$$|w| : \Gamma^* \vdash \Delta^* ,$$

$$|w| A : \Gamma^*, \mathfrak{F}(\eta) \equiv \mathfrak{F}(\eta) \vdash \Delta^* ,$$

which forms a right inference in **LJ** because $\mathfrak{F}(\eta) \equiv \mathfrak{F}(\eta)$ is provable in it.

Consequently, $\Pi^*[\mathbf{T}]$ is a right proof-figure of the transformed sequent of $\sum \vdash \mathfrak{A}$ which is surely $\vdash \mathfrak{A}$. Thus, $\vdash \mathfrak{A}$ is provable in Gentzen's **LJ**.

Conversely, let us assume that $\mathfrak{A}$ *is provable in* **LJ**. Then, $\mathfrak{A}$ must be provable by making use of inference rules **F**, **I***, **I**, **U***, **U**, **C***, **C**, **D***, **D**, **E***, **E**, and **N**. Accordingly, $\mathfrak{A}^{[R]}$ must be provable if it is admitted to use **F**, **I***, **I**, **U***, **U**, **C*R**, **CR**, **D*R**, **DR**, **E*R**, **ER**, and **NR** as inference rules for R-transforms of propositions. On the other hand, any inference of this kind is admitted in **LO** for R-closed propositions in **LO** as has been remarked before, and the R-transforms of propositions are surely R-closed. Hence, $\mathfrak{A}^{[R]}$ is provable in **LO**.

THEOREM 2. *Any proposition* $\mathfrak{A}$ *is provable in* **LK** *if and only if* $\mathfrak{A}^{[R]}$ *is provable in* **LO** *for any proposition symbol* (0*-ary relation symbol*) R *which does not occur in* $\mathfrak{A}$.

PROOF. Let $\mathfrak{A}$ be any proposition and R be any proposition symbol which does not occur in $\mathfrak{A}$.

I will prove at first that $\mathfrak{A}$ *is provable in* **LK** *if* $\mathfrak{A}^{[R]}$ *is provable in* **LO**. Because **LK** can be regarded as a logic stronger than **LO**, the proposition $\mathfrak{A}^{[R]}$ is provable in **LK**. On the other hand, we can prove by virtue of Lemma 2 that $\mathfrak{A}^{[R]}$ *and* $\mathfrak{A}$ *are mutually equivalent if we assume* 人$^{[R]} \equiv R$. However, we can easily prove this in **LK**, so $\mathfrak{A}$ *is provable in* **LK**.

Next, let us assume conversely that $\mathfrak{A}$ *is provable in* **LK**. Then, $\mathfrak{A}$ must be provable by making use of inference rules **F**, **I***, **I**, **U***, **U**, **C***, **C**, **D***, **D**, **E***,

E, **N**, and **P**. So, $\mathfrak{A}^{[R]}$ must be provable if it is admitted to use **F**, **I***, **I**, **U***, **U**, **C*R**, **CR**, **D*R**, **DR**, **E*R**, **ER**, **NR**, and **PR** as inference rules for R-transforms of propositions. On the other hand, any inference of this kind is admitted for R-closed propositions in **LO** and the R-transforms of propositions are surely R-closed. Hence, $\mathfrak{A}^{[R]}$ *must be provable in* **LO**.

The positive logics **LP** and **LQ** can be regarded as sub-logics of **LJ** and **LK**, respectively. Since **LJ** and **LK** can be faithfully interpreted in **LO** as has been shown in the preceding theorems, we see that *the positive logics* **LP** *and* **LQ** *can be also interpreted faithfully in* **LO**.

The minimal logics **LM** and **LN** can be regarded as the positive logics **LP** and **LQ**, respectively, when we regard 人 as a proposition constant and define $\neg\mathfrak{F}$ by $\mathfrak{F}\to$人. Hence, we can see also that **LM** *and* **LN** *can be faithfully interpreted in* **LO**.

(3) Remarks.

REMARK 1. If we restrict ourselves to a formal system having the primitive notions $\{\cdots, A_i, \cdots\}$ (A_i being n_i-ary relation symbol, $n_i \leqq n$) and unify these primitive notions into an $(n+1)$-ary single relation R by taking

$$(x_1)\cdots(x_n)(A_i(x_1, \cdots, x_{n_i}) \equiv R(a_i, x_1, \cdots, x_{n_i}, \cdots, x_n)),$$

we can give faithful interpretations of logical systems standing on **J**-series logics to a logical system standing on the primitive logic **LO**. In this case, any elementary proposition of the original system can be regarded as R-closed and $\underset{R}{\wedge}$, $\underset{R}{\vee}$, $(\underset{R}{\exists}\ )$, and $\underset{R}{\neg}$ behave exactly as *conjunction, disjunction, existential quantification*, and *negation* of the intuitionistic logic **LJ**, respectively.

I have adopted this device in my work [2]. Although it was not necessary to introduce the device for reducing **J**-series logics to **LO**, we can make use of this method to adjust the theory of my work [5] for **J**-series logics.

If we defiine proposition $\mathfrak{F}$ by $\mathfrak{F}\equiv(\xi)R(\xi)$ and define $\underset{\mathfrak{F}}{\wedge}$, $\underset{\mathfrak{F}}{\vee}$, $(\underset{\mathfrak{F}}{\exists}\ )$, and $\underset{\mathfrak{F}}{\neg}$ in **LO**, we can not regard every elementary propostion of the original system R-closed. Howener, $\underset{\mathfrak{F}}{\wedge}$, $\underset{\mathfrak{F}}{\vee}$, $(\underset{\mathfrak{F}}{\exists}\ )$, and $\underset{\mathfrak{F}}{\neg}$ behave exactly as *conjunction, disjunction, existential quantification*, and *negation* of the classical logic **LK** with respect to propositions constructed from $\mathfrak{F}$-transforms of elementary propositions of the original system.

REMARK 2. In my work [3], I have described a way of defining logical constants $\underset{R}{\wedge}$, $\underset{R}{\vee}$, $(\underset{R}{\exists}\ )$, and $\underset{R}{\neg}$ for any formal system having just one primitive notion R. In fact, we can define these logical constants so that they satisfy the inference rules of **LJ**. However, I am not yet certain about that these logical constants really behave just as the corresponding logical constants of

LJ.

For example, let us suppose a formal system **SA** having $\underset{0}{\in}$ as its sole primitive notion and standing on the primitive logic **LO**. If we define $\in$, $\wedge$, $\vee$, $(\exists\ )$, and 人 by

$$x \in y \equiv x \underset{0}{\in} y\,,$$

$$\mathfrak{A} \wedge \mathfrak{B} \equiv (x)(y)((\mathfrak{A} \to (\mathfrak{B} \to x \underset{0}{\in} y)) \to x \underset{0}{\in} y)\,,$$

$$\mathfrak{A} \vee \mathfrak{B} \equiv (x)(y)((\mathfrak{A} \to x \underset{0}{\in} y) \to ((\mathfrak{B} \to x \underset{0}{\in} y) \to x \underset{0}{\in} y))\,,$$

$$(\exists t)\mathfrak{A}(t) \equiv (x)(y)((t)(\mathfrak{A}(t) \to x \underset{0}{\in} y) \to x \underset{0}{\in} y)\,,$$

$$\text{人} \equiv (x)(y) x \underset{0}{\in} y\,,$$

we would have a formal system which behave similarly as the formal system **SJ** having $\in$ as its sole primitive notion and standing on the intuitionistic logic **LJ**. However, **SA** *might be stronger than* **SJ**. It must be an interesting problem to decide whether **SA** is really stronger than **SJ** or not.

On the other hand, we would have a formal system **SB** which behave just as the formal system **SK** having $\in$ as its sole primitive notion and standing on the classical logic **LK**, if we define $\in$, $\wedge$, $\vee$, $(\exists\ )$, and 人 by

$$x \in y \equiv ((x \underset{0}{\in} y \to (u)(v) u \underset{0}{\in} v) \to (u)(v) u \underset{0}{\in} v)\,,$$

$$\mathfrak{A} \wedge \mathfrak{B} \equiv ((\mathfrak{A} \to (\mathfrak{B} \to (u)(v) u \underset{0}{\in} v)) \to (u)(v) u \underset{0}{\in} v)\,,$$

$$\mathfrak{A} \vee \mathfrak{B} \equiv ((\mathfrak{A} \to (u)(v) u \underset{0}{\in} v) \to \mathfrak{B})\,,$$

$$(\exists t)\mathfrak{A}(t) \equiv ((t)(\mathfrak{A}(t) \to (u)(v) u \underset{0}{\in} v) \to (u)(v) u \underset{0}{\in} v)\,,$$

$$\text{人} \equiv (u)(v) u \underset{0}{\in} v\,.$$

REMARK 3. In my work [4], I have described a way of unifying a finite number of primitive notions into a single one, deeming that any formal system having just one primitive notion has certain superiority according to my work [3]. However, it is still dubious as mentioned in the previous remark whether such system can boast to have such superiority in establishing formal theories standing on the intuitionistic predicate logic **LJ**. The device itself is naturally valid.

Mathematical Institute
Nagoya University

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