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On $\aleph_0$-complete cardinals

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In [4], D. Scott proved that, if we assume $V=L$ and the existence of a measurable cardinal number in the set theory Σ^* of [1], then we have a contradiction.

The main purpose of this parer is to investigate on the problem concerning to certain kind of constructibility and the existence of $\aleph_0$-complete cardinal numbers (2-valued measurable cardinal numbers). In view of this point, we first remark that if the system Σ^*, $\exists xT(x)$ is consistent, then the system

$$\Sigma^*, \exists y(T(y) \wedge \exists x(V=L_x \wedge Od_x``x \subset 2^y))$$

is consistent, where $T(y)$ is the statement that there is a non-principal $\aleph_0$-complete ultrafilter over the set y whose character is cardinal number y, and L_x is the class constructed from the set x by Lévy's method in [2].

In this paper we prove the following several results:

1) The system Σ^*, $\exists y(T(y) \wedge \exists x(V=L_x \wedge Od_x``x \subset y))$ is not consistent.

2) Let $\Phi(a)$ be a standard defining postulate defined later. Then the system Σ^*, $\exists x(T(x) \wedge \Phi(x))$ is not consistent.

Remark that, as is well known, all of the defining postulates of the following cardinals are standerd: $\aleph_0, \aleph_1, \cdots, \aleph_\omega, \cdots$; the first one of weakly inaccessible cardinal, strongly inaccessible cardinal, hyper-inaccessible cardinal; the first cardinal α such that α is hyper-inaccessible of type α; and so on.

Concerning to this kind of results, I would like to propose the following problem: For what kind of formula $A(a)$, is the system Σ^*, $\exists x(T(x) \wedge A(a))$ not consistent? Especially what will happen for the formulas $\exists x(V=L_x \wedge \sup(Od_x``x) < 2^{\overline{\overline{a}}})$ or $\exists x(V=L_x \wedge \sup(Od_x``x) < a^+)$ where a^+ is the smallest cardinal number strictly greater than a.

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1. We shall begin by introducing several notations and the terminology.

DEFINITION. An ultrafilter $\mathcal{F}$ is said to be $\aleph_\alpha$-complete, if the following condition is satisfied:

$$\text{if } A_\nu \in \mathcal{F} \text{ for each } \nu \in I, \text{ then } \bigcap_{\nu \in I} A_\nu \in \mathcal{F}, \text{ where } \overline{\overline{I}} \leqq \aleph_\alpha .$$

A cardinal number $\aleph_\lambda$ is said to be $\aleph_\alpha$-complete, if there exists a non-principal ultrafilter $\mathcal{F}_{\aleph_\lambda}$ over $\aleph_\lambda$ such that $\mathcal{F}_{\aleph_\lambda}$ is $\aleph_\alpha$-complete. A cardinal number $\aleph_\alpha$ is said to be the character of a non-principal ultrafilter $\mathcal{F}$, if $\aleph_\alpha$ is the least cardinal number such that $\mathcal{F}$ is not $\aleph_\alpha$-complete.

The character of a non-principal ultrafilter $\mathcal{F}$ is sometimes written as $ch(\mathcal{F})$.

CONVENTIONS. A set of the form

$$\{\langle x_0, 0\rangle, \langle x_1, 1\rangle, \cdots, \langle x_\nu, \nu\rangle, \cdots\}$$

is sometimes written as

$$(x_0, x_1, \cdots, x_\nu, \cdots).$$

Let $\mathcal{F}_{\aleph_\tau}$ be an ultrafilter over a cardinal number $\aleph_\tau$ and let $a, b \in V^{\aleph_\tau}$, where V is the universe of Σ^*. Then $a \in {}^*b$, $a = {}^*b$ and $a < {}^*b$ are defined by

$$a \in {}^*b \equiv \{\alpha : a`\alpha \in b`\alpha\} \in \mathcal{F}_{\aleph_\tau},$$

$$a = {}^*b \equiv \{\alpha : a`\alpha = b`\alpha\} \in \mathcal{F}_{\aleph_\tau},$$

$$a < {}^*b \equiv \{\alpha : a`\alpha < b`\alpha\} \in \mathcal{F}_{\aleph_\tau}.$$

Now, we have the following lemmata.

LEMMA 1. *There is a function G in Σ^* which gives the* 1—1 *correspondence between the class V, and the class On, consisting of all ordinal numbers of Σ^*, and it has the property that if $\alpha < \beta$, then $R`G`\alpha \leqq R`G`\beta$, where $R`x$ is the rank of the set x.*

This is well-known.

LEMMA 2. *There is a class K such that $V = L_K$.*

PROOF. Let K be the class defined by the following postulate:

$$\langle x\alpha\rangle \in K \equiv x \in G`\alpha.$$

Then the class K has the required property.

LEMMA 3. *Let $\aleph_\tau$ be an $\aleph_0$-complete cardinal number and $\mathcal{F}_{\aleph_\tau}$ be a non-principal $\aleph_0$-complete ultrafilter over $\aleph_\tau$. Then there is a class H such that*

$$H \subset On^{\aleph_\tau} \wedge \forall a(a \in On^{\aleph_\tau} \to \exists b(b \in H \wedge a = {}^*b))$$

$$\wedge \forall a \forall b(a \in H \wedge b \in H \wedge a = {}^*b \to a = b).$$

Moreover, the class H is well-ordered by the relation $<^*$.

PROOF. Similarly to [4] we can prove that the class H is well-ordered by the relation $<^*$. We show the existence of the class H. By Lemma 1, there is a enumeration function G. We consider a function defined by

$$On^{\aleph_\tau} \rceil G.$$

We define a function A by the following postulate:

$$\langle x\alpha\rangle \in A \cdot \equiv \cdot \alpha \in \mathfrak{D}(On^{\aleph_\tau} \upharpoonright G) \wedge \forall\beta(\beta < \alpha \to \{\delta : ((On^{\aleph_\tau} \upharpoonright G)`\beta)`\delta$$
$$= ((On^{\aleph_\tau} \upharpoonright G)`\alpha)`\delta\} \notin \mathscr{F}_{\aleph_\tau} \wedge x = (On^{\aleph_\tau} \upharpoonright G)`\alpha\,.$$

Then $\mathfrak{W}(A)$ is the required class.

In order to show this, we consider any $a \in On^{\aleph_\tau}$. By the definition of G, there is an α such that

$$a = G`\alpha\,.$$

If for all β less than α,

$$\{\delta : ((On^{\aleph_\tau} \upharpoonright G)`\beta)`\delta = a`\delta\} \notin \mathscr{F}_{\aleph_\tau}\,,$$

then $\langle G`\alpha, \alpha\rangle \in A$ and so $a = G`\alpha \in \mathfrak{W}(A)$.

In the case where there is a β less than α such that

$$\{\delta : ((On^{\aleph_\tau} \upharpoonright G)`\beta)`\delta = a`\delta\} \in \mathscr{F}_{\aleph_\tau}\,,$$

we consider the least such ordinal number β. Then $\langle (On^{\aleph_\tau} \upharpoonright G)`\beta, \beta\rangle \in A$. Hence there is a set b such that

$$\{\delta : a`\delta = b`\delta\} \in \mathscr{F}_{\aleph_\tau} \quad \text{and} \quad b \in \mathfrak{W}(A)\,.$$

Next, we shall show that

$$\forall a \forall b (a \in \mathfrak{W}(A) \wedge b \in \mathfrak{W}(A) \wedge \{\delta : a`\delta = b`\delta\} \in \mathscr{F}_{\aleph_\tau} \to a = b)\,.$$

Let $a = A`\alpha$, $b = A`\beta$ and $\{\delta : a`\delta = b`\delta\} \in \mathscr{F}_{\aleph_\tau}$. If $\alpha < \beta$, then $\langle b\beta\rangle \notin A$ by the definition of A, which is a contradiotion. By the symmetry of the reason, we see $a = b$. Thus we complete the proof of the lemma.

DEFINITION. By α^* we denote the α-th element of H by the well-ordering $<^*$.

LEMMA 4. *If the character of the ultrafilter $\mathscr{F}_{\aleph_\tau}$ is $\aleph_\alpha$, then*

$$\gamma^* = {}^*(\gamma, \gamma, \cdots, \gamma, \cdots) \text{ for every } \gamma \text{ less than } \aleph_\alpha\,.$$

This is proved by the induction on γ.

DEFINITION. Let N, K_1, K_2, J be the functions defined by the same method as 9.1, 9.24 in [1] except that the constant 9 is replaced by 10 so that the following condition is satisfied:

$$\alpha = J`\langle N`\alpha, K_1`\alpha, K_2`\alpha\rangle \qquad (N`\alpha = 0, 1, \cdots, 9)\,.$$

Given any class K, we define the function F_K in the same way as in [2], Dfn. 1.1, where the functions $J_0^*, \cdots, J_9^*, K_1^*, K_2^*$ are replaced by $J`\langle 0, *, *\rangle, \cdots, J`\langle 9, *, *\rangle, K_1, K_2$ respectively. The class L_K is defined by $L_K = F_K``On$ as in [2].

We define N^*, K_1^*, K_2^* and J^* as follows:

Let a be a function of the form $a = (\alpha_1, \alpha_2, \cdots, \alpha_\nu, \cdots)$. We consider a function defined by $(N`\alpha_1, N`\alpha_2, \cdots, N`\alpha_\nu, \cdots)$. By the property of the class H, there is $b \in H$ uniquely such that

$$\{\nu : b'\nu = N'\alpha_\nu\} \in \mathcal{F}_{\aleph_\tau}.$$

Then, we put $N^{*}{}'a = b$. K_1^*, K_2^* and J^* are defined similarly.

We consider the following two functions:

$$f : \alpha \to \alpha^*$$

$$g : \alpha \to J^{*}{}'\langle (N'\alpha)^*, (K_1'\alpha)^*, (K_2'\alpha)^* \rangle .$$

Since they are order-preserving onto-mappings, we obtain

$$\alpha^* = J^{*}{}'\langle (N'\alpha)^*, (K_1'\alpha)^*, (K_2'\alpha)^* \rangle .$$

Hence $N^{*}{}'\alpha^* = (N'\alpha)^*$, $K_1^{*}{}'\alpha^* = (K_1'\alpha)^*$ and $K_2^{*}{}'\alpha^* = (K_2'\alpha)^*$ by the definitions. Next, we take a function F_K^* defined on the class $On^{\aleph_\tau}$ as follows:

$$F_K^{*}{}'a = (F_K'(a'0), F_K'(a'1), \cdots, F_K'(a'\nu), \cdots).$$

2. Let $\mathcal{F}_{\aleph_\tau}$ be a non-principal $\aleph_0$-complete ultrafilter over cardinal number $\aleph_\tau$ which has the character $\aleph_\tau$.

We define a function σ by the following:

$$\begin{cases} \sigma(F_K^{*}{}'0^*) = \phi , & \text{for } \alpha = 0 , \\ \sigma(F_K^{*}{}'\alpha^*) = \{\sigma(F_K^{*}{}'\beta^*) : F_K^{*}{}'\beta^* \in {}^*F_K^{*}{}'\alpha^* \text{ and } \beta^* <{}^* \alpha^*\}, & \text{for } \alpha > 0 . \end{cases}$$

The class

$$\{\sigma(F_K^{*}{}'\alpha^*) : \{\delta : (F_K^{*}{}'\alpha^*)'\delta \in K\} \in \mathcal{F}_{\aleph_\tau}\}$$

is abbreviated by $\sigma(F_K^{*}{}'(K^*))$.

Lemma 5. *We have $\sigma(F_K^{*}{}'\alpha^*) = F_U'\alpha$, where U is $\sigma(F_K^*(K^*))$.*

Proof. This is proved by the induction on α. In the case where $\alpha = 0$, we have $\sigma(F_K^{*}{}'0^*) = \phi = F_U'0$.

The case where $\alpha > 0$, is divided into several subcases. Since other cases are treated similarly, we treat only the cases where $N'\alpha = 5$ and $N'\alpha = 9$. To do this, we note that $\sigma(F_K^{*}{}'\beta^*) \in \sigma(F_K^{*}{}'\alpha^*) \leftrightarrow F_K^{*}{}'\beta^* \in^* F_K^{*}{}'\alpha^*$ and $\sigma(F_K^{*}{}'\beta^*) = \sigma(F_K^{*}{}'\alpha^*) \leftrightarrow F_K^{*}{}'\beta^* =^* F_K^{*}{}'\alpha^*$.

In the case where $N'\alpha = 5$, we have the followings:

$$\begin{aligned} \sigma(F_K^{*}{}'\alpha^*) &= \sigma(F_K^{*}{}'J^{*}{}'\langle 5^*, (K_1'\alpha)^*, (K_2'\alpha)^* \rangle) \\ &= \{\sigma(F_K^{*}{}'\beta^*) : F_K^{*}{}'\beta^* \in^* F_K^{*}{}'(K_1'\alpha)^* \text{ and there exist } F_K^{*}{}'\delta_1^* \\ &\quad \text{and } F_K^{*}{}'\delta_2^* \text{ such that } \langle F_K^{*}{}'\delta_1^*, F_K^{*}{}'\delta_2 \rangle \in^* F_K^{*}{}'(K_2'\alpha)^* \\ &\quad \text{and } F_K^{*}{}'\delta_2^* =^* F_K^{*}{}'\beta^*\} \\ &= \{F_U'\beta : F_U'\beta \in F_U'K_1'\alpha \text{ and there exist } F_U'\delta_1 \text{ and } F_U'\delta_2 \\ &\quad \text{such that } \langle F_U'\delta_1, F_U'\delta_2 \rangle \in F_U'K_2'\alpha \text{ and } F_U'\delta_2 = F_U'\beta\} \end{aligned}$$

$$=F_U`J`\langle 5, K_1`\alpha, K_2`\alpha\rangle$$
$$=F_U`\alpha\,.$$

In the case where $N`\alpha=9$, we have the followings:

$$\begin{aligned}\sigma(F^*_K`\alpha^*)&=\sigma(F^*_K`J^*`\langle 9^*, (K_1`\alpha)^*, (K_2`\alpha)^*\rangle)\\&=\{\sigma(F^*_K`\beta^*):\{\delta:(F^*_K`\beta^*)`\delta\in K\}\in\mathcal{F}_{\aleph_\tau}\text{ and } F^*_K`\beta^*\in^* F^*_K`(K_1`\alpha)^*\}\\&=\{F_U`\beta:F_U`\beta\in U\text{ and } F_U`\beta\in F_U`K_1`\alpha\}\\&=F_U`J`\langle 9, K_1`\alpha, K_2`\alpha\rangle\\&=F_U`\alpha\,.\end{aligned}$$

Thus, the proof of the lemma is established.

Note that, if K is a set k, then

$$U=\{\sigma(F^*_K`\alpha^*):\{\delta:(F^*_K`\alpha^*)`\delta\in k\}\in\mathcal{F}_{\aleph_\tau}\}$$

is a set.

In fact, let k^* be $(k, k, \cdots, k, \cdots)$. Then we have $U=\{\sigma(F^*_K`\beta^*):F^*_K`\beta^*\in^* k^*\}$. By the fact that k is a set, there is an ordinal number γ such that

$$Od_k``k\subset\gamma\quad\text{and}\quad N`\gamma=0\,.$$

We consider an element θ^* of H such that $\theta^*=^*(\gamma,\gamma,\cdots,\gamma,\cdots)$. We have $U=\sigma(F^*_K`J^*`\langle 9^*, \theta^*, 0^*\rangle)$. Thus U is a set.

LEMMA 6. *For every γ less than $\aleph_\tau$, we have $F_K`\gamma=F_U`\gamma$, where $U=\sigma(F^*_K(K^*))$.*

PROOF. We prove this by the induction on γ. It is clear that $F_K`0=\phi=F_U`0$. We assume that the lemma is true for all β less than γ. Namely we assume that $F_K`\beta=F_U`\beta$ for all β less than γ. If $F_K`\beta\in F_K`\gamma$, then we have $F^*_K`\beta^*\in^* F^*_K`\gamma^*$ by Lemma 4. Hence, using Lemma 5, we have

$$F_K`\beta=F_U`\beta=\sigma(F^*_K`\beta^*)\in\sigma(F^*_K`\gamma^*)=F_U`\gamma\,.$$

Therefore, we see $F_U`\gamma\supset F_K`\gamma$. On the other hand. If $F_U`\beta\in F_U`\gamma$, then we have $F^*_K`\beta^*\in^* F^*_K`\gamma^*$ by Lemma 5 and the definition of σ. Using Lemma 4, and the hypothesis of the induction

$$F_U`\beta=F_K`\beta\in F_K`\gamma\,.$$

Therefore, we see $F_U`\gamma\subset F_K`\gamma$. Thus, we have $F_K`\gamma=F_U`\gamma$.

LEMMA 7. *Let $\mathcal{F}_{\aleph_\tau}$ be a non-principal ultrafilter over $\aleph_\tau$ such that $ch(\mathcal{F}_{\aleph_\tau})=\aleph_\tau>\aleph_0$. Then*

$$\theta^*=^*(\aleph_\tau, \aleph_\tau, \cdots, \aleph_\tau, \cdots)\quad\textit{implies}\quad 2^{\aleph_\tau}<\theta\,.$$

PROOF. There is $(\alpha_1, \alpha_2, \cdots, \alpha_\nu, \cdots)$ such that $\aleph_\tau^*=^*(\alpha_1, \alpha_2, \cdots, \alpha_\nu, \cdots)$ where every α_ν is less than $\aleph_\tau$. We consider a function f_{α_ν} for each α_ν such that f_{α_ν} is a 1—1 correspondence between $\mathfrak{P}(\alpha_\nu)$ and $2^{\bar{\bar{\alpha}}_\nu}$, where $\mathfrak{P}(\alpha_\nu)$ is the power-set of α_ν.

We consider a function t defined by

$$t`a=(f_{\alpha_1}`(a\cap\alpha_1), \cdots, f_{\alpha_\nu}`(a\cap\alpha_\nu), \cdots) \text{ for any } a\subset\aleph_\tau .$$

Then, for any $a\subset\aleph_\tau$, we have

$$t`a<^*(2^{\bar{\bar{\alpha}}_1}, \cdots, 2^{\bar{\bar{\alpha}}_\nu}, \cdots) .$$

Let $a\subset\aleph_\tau$, $b\subset\aleph_\tau$ and $a\neq b$. Because, there is a $\nu_0<\aleph_\tau$ such that

$$a\cap\nu\neq b\cap\nu \quad \text{for all } \nu>\nu_0 ,$$

we obtain

$$\{\nu:(t`a)`\nu\neq(t`b)`\nu\}\supset\{\nu:\alpha_\nu>\nu_0\}\in\mathscr{F}_{\aleph_\tau} .$$

Hence, if $\beta^*=^*(2^{\bar{\bar{\alpha}}_1}, \cdots, 2^{\bar{\bar{\alpha}}_\nu}, \cdots)$, then $2^{\aleph_\tau}\leqq\beta$.

But $(2^{\bar{\bar{\alpha}}_1}, \cdots, 2^{\bar{\bar{\alpha}}_\nu}, \cdots)<^*(\aleph_\tau, \cdots, \aleph_\tau, \cdots)=^*\theta^*$, from which we obtain $2^{\aleph_\tau}\leqq\beta<\theta$. Thus we complete the proof of the lemma.

3. We consider the model Δ_X determined by the class X. Namely, we consider the model whose sets are the members of L_X whose classes are the X-constructible classes and whose ε-relation is the ε-relation of set theory (cf. [2]).

DEFINITION. A formula $\Phi(a_1, \cdots, a_n)$ is called normal if it has no class variable.

LEMMA 8. *Let $\Phi(a_1, \cdots, a_n)$ be a normal formula. Then for any class K such that $V=L_K$, we have the following equivalence:*

$$\Phi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n)\equiv\{\delta:\Phi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta)\}\in\mathscr{F}_{\aleph_\tau} .$$

where $\Phi_{\Delta U}(a_1, \cdots, a_n)$ is the relativization of $\Phi(a_1, \cdots, a_n)$ to the model Δ_U and $U=\sigma(F_K^*(K^*))$.

PROOF. We prove the lemma by the induction on the number of logical symbols of $\Phi(a_1, \cdots, a_n)$. In the case where the outermost symbol of $\Phi(a_1, \cdots, a_n)$ is $\in$ or $=$, the lemma is easily proved by Lemma 5. If the outermost symbol is $\neg$, $\vee$, $\wedge$ or $\rightarrow$, then the proof is clear. Therefore, we prove only the case the outermost symbol of $\Phi(a_1, \cdots, a_n)$ is $\exists$.

First, we shall prove that

$$\exists x(x\in F_U``On\wedge\Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, x))$$
$$\rightarrow\{\delta:\exists x(\Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, x))\}\in\mathscr{F}_{\aleph_\tau} .$$

We assume that

$$\exists x(x\in F_U``On\wedge\Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, x)) .$$

Then there is an ordinal number β such that

$$\Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, F_U`\beta).$$

By the hypothesis of the induction, we obtain

$$\{\delta : \Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, (F_K^*`\beta^*)`\delta)\} \in \mathscr{F}_{\aleph_\tau},$$

which implies

$$\{\delta : \exists x(\Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, x))\} \in \mathscr{F}_{\aleph_\tau}.$$

Next, we shall show that

$$\{\delta : \exists x(\Psi(F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, x))\} \in \mathscr{F}_{\aleph_\tau}$$
$$\rightarrow \exists x(x \in F_U``On \wedge \Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, x)).$$

We assume that

$$\{\delta : \exists x(\Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, x))\} \in \mathscr{F}_{\aleph_\tau}.$$

By $V = L_K$, there is a function $a \in On^{\aleph_\tau}$ such that

$$\{\delta : \Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, (F_K^*`a)`\delta)\} \in \mathscr{F}_{\aleph_\tau}.$$

Therefore by the property of the class H, there is an ordinal number α such that

$$\{\delta : a`\delta = \alpha^*`\delta\} \in \mathscr{F}_{\aleph_\tau}.$$

Hence we obtain

$$\{\delta : \Psi((F_K^*`\alpha_1^*)`\delta, \cdots, (F_K^*`\alpha_n^*)`\delta, (F_K^*`\alpha^*)`\delta)\} \in \mathscr{F}_{\aleph_\tau}.$$

By the hypothesis of the induction, we have

$$F_U`\alpha \in F_U``On \wedge \Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, F_U`\alpha),$$

which implies

$$\exists x(x \in F_U``On \wedge \Psi_{\Delta U}(F_U`\alpha_1, \cdots, F_U`\alpha_n, x)).$$

Thus, the lemma is proved.

DEFINITION. Let $a \sim b$ be an abbreviation of the formula $\exists f(\mathfrak{Un}_2(f) \wedge \mathfrak{W}(f) = a \wedge \mathfrak{D}(f) = b)$. Let $T(a)$ be a normal formula satisfying the following conditions:

1) $T(a)$ and $a \sim b$ imply $T(b)$.

2) $T(a)$ implies that there is a non-principal $\aleph_0$-complete ultrafilter $\mathscr{F}_a$ over the set a such that the character of the filter $\mathscr{F}_a$ is $\overline{\overline{a}}$.

3) $T(a)$ and $\overline{\overline{b}} < \overline{\overline{a}}$ imply $\neg T(b)$.

For example, the statement '$\overline{\overline{a}}$ is the first $\aleph_0$-complete cardinal', satisfies the above conditions 1) to 3).

LEMMA 9. *We have* $T(\aleph_\tau) \wedge \alpha \leqq 2^{\aleph_\tau} \rightarrow \neg T_{\Delta U}(F_U`\alpha)$ *in* Σ^*, *where* U

$=\sigma(F^*_K(K^*))$ *and* $V=L_K$.

In fact, we assume $T(\aleph_\tau)$. Then there is an ultrafilter $\mathcal{F}_{\aleph_\tau}$ such that $ch(\mathcal{F}_{\aleph_\tau})=\aleph_\tau$. We take the class H (cf. Lemma 3) determined by this ultrafilter. Let α be an ordinal number such that $\alpha \leqq 2^{\aleph_\tau}$. We easily see $\{\delta : \neg T((F^*_K{}'\alpha^*)'\delta)\} \supset \{\delta : (F^*_K{}'\alpha^*)'\delta < \aleph_\tau\} \supset \{\delta : \alpha^*{}'\delta < \aleph_\tau\}$. By lemma 7, we have $\{\delta : \alpha^*{}'\delta < \aleph_\tau\} \in \mathcal{F}_{\aleph_\tau}$. Then we have $\{\delta : \neg T((F^*_K{}'\alpha^*)'\delta)\} \in \mathcal{F}_{\aleph_\tau}$, which implies $\neg T_{\Delta_U}(F_U{}'\alpha)$ by Lemma 8.

LEMMA 10. *Let K be any class. Then $V \neq L_U$ under Σ^*, $V=L_K$, $T(\aleph_\tau)$ where $U=\sigma(F^*_K(K^*))$.*

To prove this, assume $V=L_U$, then $T_{\Delta_U}(F_U{}'\aleph_\tau)$ would be equivalent to $T(F_U{}'\aleph_\tau)$. By Lemma 9, we have $\neg T_{\Delta_U}(F_U{}'\aleph_\tau)$. But $F_U{}'\aleph_\tau$ has the cardinality $\aleph_\tau$, so we have $T(F_U{}'\aleph_\tau)$, which contradicts to the above. Thus we have $V \neq L_U$.

4. Now, we have the following theorems.

THEOREM 1. *Let k be any set. The we have $\neg(Od_k{}''k \subset \aleph_\tau)$ under Σ^*, $T(\aleph_\tau)$, $V=L_k$.*

PROOF. We assume Σ^*, $T(\aleph_\tau)$, $V=L_k$ and $Od_k{}''k \subset \aleph_\tau$. Then by Lemma 6, we have

$$k=F_U{}'J'\langle 9, \aleph_\tau, 0\rangle ,$$

where $U=\{\sigma(F^*_K{}'\alpha^*) : F^*_K{}'\alpha^* \in^* k^*\}$. Therefore, we have $k \in F_U{}''On$ which means $V=L_U$. But this contradicts to Lemma 10.

DEFINITION. Let Δ_1 and Δ_2 be two models of set theory Σ^*. We say that Δ_1 is a complete inner model of Δ_2 (denoted by $\Delta_1 \subset \Delta_2$), if the following conditions are satisfied:

1) $\mathfrak{Cls}_{\Delta_1}(X)$ implies $\mathfrak{Cls}_{\Delta_2}(X)$.

2) $\mathfrak{M}_{\Delta_1}(X)$ implies $\mathfrak{M}_{\Delta_2}(X)$.

3) $X \in_{\Delta_1} Y$ is equivalent to $\mathfrak{M}_{\Delta_1}(X) \wedge \mathfrak{Cls}_{\Delta_1}(Y) \wedge X \in_{\Delta_2} Y$.

4) $X =_{\Delta_1} Y$ is equivalent to $\mathfrak{Cls}_{\Delta_1}(X) \wedge \mathfrak{Cls}_{\Delta_1}(Y) \wedge X =_{\Delta_2} Y$.

5) $X \in_{\Delta_2} Y \wedge \mathfrak{M}_{\Delta_1}(Y)$ implies $\mathfrak{M}_{\Delta_1}(X)$.

6) The class On_{Δ_1} of all ordinal numbers of Δ_1 coincides with the class of ordinal numbers On_{Δ_2} of Δ_2.

Moreover, if $\mathfrak{Cls}_{\Delta_2}(X)$, $\mathfrak{M}_{\Delta_2}(X)$, $X \in_{\Delta_2} Y$ and $X =_{\Delta_2} Y$ are equivalent to $\mathfrak{Cls}(X)$, $\mathfrak{M}(X)$, $X \in Y$ and $X=Y$ respectively, then Δ_1 is called a complete inner model of set theory Σ^*.

THEOREM 2. *If there is a model of $\exists x T(x)$ and Σ^*, then there are countably many complete inner models $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \supset \cdots$ of Σ^*, such that the following conditions are satisfied:*

1) *$\exists x(V=L_x)$, $\exists x T(x)$, Σ^* are satisfied in every Δ_i.*

2) *Let a_n be the initial ordinal such that $T_{\Delta_n}(a_n)$. Then we have the fol-*

lowing inequality:

$$a_1 < (2^{a_1})_{\Delta_1} < a_2 < (2^{a_2})_{\Delta_2} < \cdots < a_n < (2^{a_n})_{\Delta_n} < \cdots .$$

PROOF. As mentioned in the introduction, there is an complete inner model Δ_1 of Σ^* for the system of axioms

$$\exists x(V = L_x), \exists x T(x), \Sigma^* .$$

We assume that the complete inner models $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n$ are already defined and they have the following properties:

1) $\exists x(V = L_x), \exists x T(x), \Sigma^*$ are satisfied in every Δ_i $(i = 1, \cdots, n)$.

2) Let a_i be the initial ordinal such that $T_{\Delta_i}(a_i)$. Then we have the following inequality:

$$a_1 < (2^{a_1})_{\Delta_1} < a_2 < (2^{a_2})_{\Delta_2} < \cdots < a_n < (2^{a_n})_{\Delta_n} .$$

Now we consider the model Δ_n. Since Δ_n is a model for $\exists x(V = L_x) \wedge \exists x T(x)$, there are k and $\aleph_\rho$ such that $V = L_k$ and $T(\aleph_\rho)$ in Δ_n. As in Lemma 5, we can define the function σ and a set k_1 so that $\sigma(F_k^*{}'\alpha^*) = F_{k_1}{}'\alpha$ for all α. Δ_{n+1} is defined to be the inner model defined by this set k_1. Then by Lemma 8, we see that $V = L_{k_1}$ and $T(\sigma(F_K^*{}'\beta^*))$ in Δ_{n+1} where $\beta^* = (\delta, \delta, \cdots, \delta, \cdots)$ and $\aleph_\rho = F_k{}'\delta$. Therefore there is a complete inner model Δ_{n+1} of Δ_n such that

1) $\exists x(V = L_x), \exists x T(x), \Sigma^*$ are satisfied in the model Δ_{n+1}.

2) Let a_{n+1} be the first ordinal number such that $T_{\Delta_{n+1}}(a_{n+1})$. Then $a_n < (2^{a_n})_{\Delta_n} < a_{n+1}$.

Thus we complete the proof of the theorem.

DEFINITION. A formula of the form $\Phi(a)$ is said to be a postulate, if the following conditions are satisfied:

1) $\Phi(a)$ is a normal formula.

2) $\forall a \forall b(\Phi(a) \wedge \Phi(b) \to a = b)$.

A postulate $\Phi(a)$ is said to be 'standard', if the following conditions are satisfied:

1) $\Phi(a)$ implies $\mathfrak{Ord}(a)$.

2) Let $\Phi_{\Delta_i}(a)$ be the relativization of the formula $\Phi(a)$ to the model Δ_i. Then we have that if $\Delta_1 \subset \Delta_2$, $\Phi_{\Delta_1}(a)$ and $\Phi_{\Delta_2}(b)$ then $a \leqq b$.

THEOREM 3. *The system* $\Sigma^*, \exists x(T(x) \wedge \Phi(x))$ *is not consistent, where* $\Phi(a)$ *is a standard postulate.*

PROOF. We assume $\Sigma^*, \exists x(T(x) \wedge \Phi(x))$. By Lemma 2, we have a class K such that $V = L_K$. By $\exists x(T(x) \wedge \Phi(x))$, there is an ordinal number α such that $T(F_K{}'\alpha)$ and $\Phi(F_K{}'\alpha)$.

By the property of the formula $\Phi(a)$, we have

$$\Phi(F_K{}'\alpha) \text{ implies } \mathfrak{Ord}(F_K{}'\alpha) .$$

We put $\aleph_\tau = \overline{\overline{F_K`\alpha}}$. Then we have $T(\aleph_\tau)$ by the property 1) of T. Moreover by the property 2) we have an $\aleph_0$-complete ultrafilter $\mathcal{F}_{\aleph_\tau}$ over $\aleph_\tau$ such that $ch(\mathcal{F}_{\aleph_\tau}) = \aleph_\tau$. And we also consider the class H (cf. Lemma 3) determined by this ultrafilter. Let $\beta^* = {}^*(\alpha, \alpha, \cdots, \alpha, \cdots)$. Then by Lemma 8, we obtain

$$T_{\Delta_U}(F_U`\beta) \wedge \Phi_{\Delta_U}(F_U`\beta) \wedge \mathfrak{Ord}_{\Delta_U}(F_U`\beta),$$

where $U = \sigma(F_K^*(K^*))$. Since $\mathfrak{Ord}$ is absolute, we have $\mathfrak{Ord}(F_U`\beta)$. By the property of the standard postulate $\Phi(a)$, we obtain

$$\Phi(F_K`\alpha) \wedge \Phi_{\Delta_U}(F_U`\beta) \text{ implies } F_U`\beta \leqq F_K`\alpha .$$

Since $F_U`\beta$ is an ordinal number such that $F_U`\beta \leqq F_K`\alpha < \aleph_{\tau+1}$, it is constructible from the class U with the ordinal less than $\aleph_{\tau+1}$ (cf. [2]). Namely, we see that

$$F_U`\beta = F_U`\gamma \text{ for some } \gamma \text{ less than } \aleph_{\tau+1} .$$

Let $\gamma^* = {}^*(\gamma_1, \gamma_2, \cdots, \gamma_\nu, \cdots)$. Then, by Lemma 7, we obtain $\{\nu : \gamma_\nu < \aleph_\tau\} \in F_{\aleph_\tau}$, and hence, $\{\delta : \neg T((F_K^*`\gamma^*)`\delta\} \in \mathcal{F}_{\aleph_\tau}$. By using Lemma 8, we obtain $\neg T_{\Delta_U}(F_U`\gamma)$, i.e. $\neg T_{\Delta_U}(F_U`\beta)$ which contradicts to $T_{\Delta_U}(F_U`\beta)$ and $\Phi_{\Delta_U}(F_U`\beta)$. Thus we complete the proof of the theorem.

Note that Theorem 3 means that for any cardinal number $\aleph_\tau$ defined by a standard defining postulate $\Phi(a)$, we have

$$\Sigma^*, \Phi(\aleph_\tau) \rightarrow \neg T(\aleph_\tau) .$$

For example, the cardinal number $\aleph_\tau$ such that $\Phi(\aleph_\tau)$ is not the first $\aleph_0$-complete cardinal number.

DEFINITION. A model Δ of Σ^* is called an absolute cardinal model, if for any complete inner model Δ_1 of Δ,

$$\mathfrak{Card}_{\Delta_1}(a) \rightarrow \mathfrak{Card}_{\Delta}(a)$$

where $\mathfrak{Card}(a)$ means that a is a cardinal number.

Cleary a complete inner model of the system $V = L$, Σ^* is an absolute cardinal model.

THEOREM 4. *Let Δ be any absolute cardinal model. Then $\exists x T(x)$ is not satisfied in the model Δ.*

PROOF. We assume that $\exists x T(x)$ is satisfied in an absolute cardinal model Δ. In the proof of this theorem, discussion will be done in the model Δ. We omit the subscript Δ which expresses the relativization to the model Δ. Let $\aleph_\tau$ be a cardinal number such that $T(\aleph_\tau)$. By Lemma 2, there is a class K such that $V = L_K$. We consider a complete inner model Δ_U defined by the class

$$U = \{\sigma(F_K^*`\alpha^*) : \{\delta : (F_K^*`\alpha^*)`\delta \in K\} \in \mathcal{F}_{\aleph_\tau}\} .$$

We now consider an ordinal number η such that

$$\eta^* = {}^*(\aleph_\tau, \aleph_\tau, \cdots, \aleph_\tau, \cdots).$$

Let $\aleph_\gamma$ be the least cardinal number such that $2^{\aleph_\tau} < \aleph_\gamma$. Then we have $2^{\aleph_\tau} < \eta < \aleph_\gamma$, by Lemma 7. Let $Ord_K{}'\aleph_\tau = \sigma < \aleph_\gamma$, and put $\beta^* = {}^*(\alpha, \alpha, \cdots, \alpha, \cdots)$. Then we have

$$2^{\aleph_\tau} < \beta < \aleph_\gamma.$$

By $\mathfrak{Card}(\aleph_\tau)$, we obtain

$$\{\delta : \mathfrak{Card}((F_K^*{}'\beta^*)'\delta)\} \in \mathscr{F}_{\aleph_\tau}.$$

Hence, by Lemma 8, we obtain $\mathfrak{Card}_{\Delta_U}(F_U{}'\beta)$. By the definition of β, we have

$$2^{\aleph_\tau} < F_U{}'\beta < \aleph_\gamma.$$

We use here the absolute cardinality of the model. Then we obtain $\mathfrak{Card}(F_U{}'\beta)$. This contradicts to the fact that $\aleph_\gamma$ is the least cardinal number such that $2^{\aleph_\tau} < \aleph_\gamma$.

NOTICE. Let $\Psi(a)$ be a normal formula such that $\Psi(\aleph_\tau)$ means that '$\aleph_\tau$ is the least srtongly inaccessible cardinal for which $2^{\aleph_\tau} > \aleph_{\tau+1}$'. Then we have $\Sigma^*, \exists x(T(x) \wedge \Psi(x))$ is not consistent.

PROOF. We assume $\Sigma^*, \exists x(T(x) \wedge \Psi(x))$. Then there is a cardinal number $\aleph_\tau$ such that

$$T(\aleph_\tau) \text{ and } \Psi(\aleph_\tau).$$

Since $\aleph_\tau$ is a cardinal number, we obtain that $\{\nu : \mathfrak{Card}(\aleph_\tau{}^*{}'\nu)\} \in \mathscr{F}_{\aleph_\tau}$. Let $\aleph_\tau{}^* = {}^*(\aleph_{\alpha_1}, \aleph_{\alpha_2}, \cdots, \aleph_{\alpha_\nu}, \cdots)$. Then by $T(\aleph_\tau)$, we have

$$\{\delta : \aleph_{\alpha_\delta} \text{ is strongly inaccessible}\} \in \mathscr{F}_{\aleph_\tau}.$$

Therefore by the property of the formula $\Psi(\aleph_\tau)$, we have

$$\{\delta : 2^{\aleph_{\alpha_\delta}} = \aleph_{\alpha_{\delta+1}}\} \in \mathscr{F}_{\aleph_\tau}.$$

We shall now consider an ordinal number such that

$$\eta^* = {}^*(\aleph_{\alpha_1+1}, \cdots, \aleph_{\alpha_\nu+1}, \cdots).$$

Since $\eta^* = {}^*(2^{\aleph_{\alpha_1}}, \cdots, 2^{\aleph_{\alpha_\nu}}, \cdots)$, we have $\eta \geqq 2^{\aleph_\tau} > \aleph_{\tau+1}$, by the proof of the lemma 7. We shall now consider $\aleph^*_{\tau+1}$, clearly there are cardinal numbers such that $\aleph^*_{\tau+1} = {}^*(\aleph_{\beta_1}, \cdots, \aleph_{\beta_\nu}, \cdots)$. But then we have

$$\{\delta : \aleph_{\alpha_\delta} < \aleph_{\beta_\delta} < \aleph_{\alpha_\delta+1}\} \in \mathscr{F}_{\aleph_\tau}.$$

which is a contradiction.

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