J. Math. Soc. Japan
Vol. 19, No. 3, 1967

Harmonic forms and Betti numbers of certain contact Riemannian manifolds

By Shûkichi TANNO*

(Received July 7, 1966)

Let M be a compact regular contact manifold, then M can be considered as a prinicipal circle bundle over the symplectic manifold $B=M/\xi$. And we can define a Riemannian metric in M so that M is a K-contact Riemannian manifold. More special K-contact Riemannian manifold is a normal contact Riemannian manifold or Sasakian manifold. With respect to this Riemannian metric we study the properties of harmonic forms on M. When M is a compact regular K-contact Riemannian manifold the fibering $M\rightarrow B$ gives the standard model to study the Betti numbers and harmonic forms. In particular if M is a compact regular normal contact Riemannian manifold, $M\rightarrow B$ is the most typical, since B is kählerian and a Hodge manifold. By these models, we get some results on harmonic forms and Betti numbers of K-contact Riemannian manifolds which are not necessarily regular. In § 1, we give the fundamental relations satisfied by the structure tensors, and next the relations of the Betti numbers. In § 3 and § 4, the relations of harmonic forms on M and B are given. In § 5, we devide K-contact Riemannian manifolds into two classes according as M is an η-Einstein space or not, and study respective cases. Manifolds are assumed to be of class C^{∞} and connected.

§ 1. Preliminary.

We denote by ϕ, ξ, η, w and g the structure tensors of an m-dimensional ($m\geqq 3$) contact Riemannian manifold M, where ϕ, ξ, η and w are tensor fields on M of type (1,1), (1,0), (0,1) and (0,2) respectively and g is the metric tensor. They satisfy the following relations:

(1.1) $$\phi\xi=0\,,\qquad \eta(\xi)=1\,,\qquad \phi^2u=-u+\eta(u)\xi\,,$$

(1.2) $$\eta(u)=g(u,\xi)\,,\qquad g(\phi u,\phi v)=g(u,v)-\eta(u)\eta(v)\,,$$

(1.3) $$d\eta(u,v)=2g(u,\phi v)=2w(u,v)$$

for any vector fields u and v on M. If we denote by ∇ and δ the Riemannian

* The author is partially supported by The Sakkokai Foundation.

connection and codifferentiation with respect to g, we have

$$\nabla_{\xi}\phi=0, \tag{1.4}$$

$$\delta\eta=0, \qquad \delta w=(m-1)\eta. \tag{1.5}$$

If ξ is a Killing vector field, i.e. M is a K-contact Riemannian manifold, we have

$$\nabla_u\xi=-\phi u, \tag{1.6}$$

$$R_1(u,\xi)=(m-1)\eta(u), \qquad R(u,\xi)v=(\nabla_u\phi)v, \tag{1.7}$$

$$g(R(u,\xi)v,\xi)=g(u,v)-\eta(u)\eta(v), \tag{1.8}$$

where $R(u,v)x=\nabla_{[u,v]}x-\nabla_u\nabla_v x+\nabla_v\nabla_u x$ and R_1 is the Ricci curvature tensor. Further M is normal, if the relation

$$(\nabla_u w)(v,x)=\eta(v)g(u,x)-\eta(x)g(u,v) \tag{1.9}$$

holds good ([**6**], [**9**], [**10**], [**11**]).

Suppose that M is a compact regular contact manifold and let $\pi: M\to B=M/\xi$ be the fibering of M ([**1**], [**14**]). We define an almost complex structure J and almost Hermitian metric h on B satisfying

$$d\eta=2\pi^*\Omega, \qquad \Omega(X,Y)=h(X,JY) \tag{1.10}$$

for any vector fields X and Y on B, where Ω is the fundamental 2-form. Then by the metric g and (1,1)-tensor ϕ defined by $g=\pi^*h+\eta\otimes\eta$, $\phi X^*=(JX)^*$, M is a K-contact Riemannian manifold, where X^* is the horizontal lift of X with respect to the infinitesimal connection η. This almost kählerian structure on B is kählerian if and only if the contact metric structure is normal ([**5**]).

As the Betti numbers of M are topological property, they do not depend on the Riemannian metric. Therefore throughout this peper we consider contact manifolds with the *associated* Riemannian metric, although the word 'Riemannian' is unnecessary in the statements of the properties of the Betti numbers.

Now the exact sequence of Gysin for our circle bundle is:

$$\begin{aligned} &0\longrightarrow H^1(B;R)\xrightarrow{\pi^*}H^1(M;R)\longrightarrow H^0(B;R)\xrightarrow{L_0} \\ &\xrightarrow{L_0}H^2(B;R)\xrightarrow{\pi^*}H^2(M;R)\longrightarrow H^1(B;R)\xrightarrow{L_1}\cdots \\ &\cdots\xrightarrow{L_{p-2}}H^p(B;R)\xrightarrow{\pi^*}H^p(M;R)\longrightarrow H^{p-1}(B;R)\xrightarrow{L_{p-1}}\cdots \end{aligned} \tag{1.11}$$

where $H^p(M;R)$ is the p-th cohomology group of M with real coefficients and L_p sends $\alpha\in H^p(B;R)$ to $\Omega\wedge\alpha\in H^{p+2}(B;R)$ ([**1**], [**2**], [**12**]). We denote by $b_p(M)=\dim H^p(M;R)$ the p-th Betti number of M. As L_0 is an isomorphism, $\pi^*: H^1(B;R)\to H^1(M;R)$ is an onto isomorphism. Thus we have:

In a compact regular K-contact Riemannian manifold, we have

$$b_1(M)=b_1(B)\,. \tag{1.12}$$

If the structure in M is normal, B is kählerian and L_p is an into isomorphism for $p\leqq(m-3)/2$, and π^* is onto, so by this and the duality we have:

In a compact regular normal contact Riemannian manifold,

$$b_1(M)=b_1(B)\,, \tag{1.12)'$$

$$b_p(M)=b_p(B)-b_{p-2}(B)\,,\qquad 2\leqq p\leqq(m-1)/2\,, \tag{1.13}$$

$$b_p(M)=b_{p-1}(B)-b_{p+1}(B)\,,\qquad (m+1)/2\leqq p<m\,. \tag{1.14}$$

Especially, as $H^0(B;R)\cong R$, we have

$$b_2(M)=b_2(B)-1\,. \tag{1.15}$$

As for (1.12)′, (1.13) and (1.14), it is known that $b_p(B)$ is even, if p is odd, and so we have:

If p is odd ($\leqq(m-1)/2$), $b_p(M)$ is even (or 0).

If p is even ($\geqq(m+1)/2$), $b_p(M)$ is even (or 0).

These two can be easily shown even if the structure is not necessarily regular utilizing the last remark in [**13**], and also see [**3**].

§ 2. A lemma.

Let M be a compact regular K-contact Riemannian manifold. We denote by $(\ ,\ )$ the local inner product of forms in M and B, and by $\langle\ ,\ \rangle_M$ or $\langle\ ,\ \rangle_B$ the global inner product in M or B respectively. Then we have

$$\langle\ ,\ \rangle_M=c_1\int_M(\ ,\ )\eta\wedge(d\eta)^n\,, \tag{2.1}$$

$$\langle\ ,\ \rangle_B=c_2\int_B(\ ,\ )\Omega^n\,, \tag{2.2}$$

where c_1, c_2 are constant and $m=2n+1$.

LEMMA 2.1. *For any r-forms λ and μ on B, we have*

$$\langle\pi^*\lambda,\pi^*\mu\rangle_M=c\langle\lambda,\mu\rangle_B\,,\qquad c=constant\,. \tag{2.3}$$

PROOF. We denote by S_i, $i=1,\cdots,k$ the disjoint images of local cross sections such that, πS_i are open and connected, $\pi S_i\cap\pi S_j=\emptyset$ if $i\neq j$ and $\bigcup_i\overline{\pi S_i}=B$. For each i we take an arbitrary trajectory T_i of ξ which intersects S_i. The length l of T_i is the same constant for each trajectory on M. Then

$$\begin{aligned}\langle\pi^*\lambda,\pi^*\mu\rangle_M&=2^nc_1\int_M(\pi^*\lambda,\pi^*\mu)\eta\wedge(\pi^*\Omega)^n\\&=2^nc_1\sum_i\int_{S_i}\int_{T_i}\eta\wedge(\pi^*\lambda,\pi^*\mu)(\pi^*\Omega)^n\end{aligned}$$

$$= 2^n c_1 l \sum_i \int_{S_i} \pi^*((\lambda, \mu)\Omega^n)$$

$$= 2^n c_1 c_2^{-1} l \langle \lambda, \mu \rangle_B ,$$

completing the proof for $c = 2^n c_1 c_2^{-1} l$.

§ 3. A compact regular K-contact Riemannian manifold.

In this section corresponding to (1.12), we prove

PROPOSITION 3.1. *Let $\pi: M \to B$ be the fibering of a compact regular K-contact Riemannian manifold M. Then for any harmonic 1-form λ on B, $\pi^*\lambda$ is harmonic. Conversely any harmonic 1-form α on M is written as $\alpha = \pi^*\lambda$ for some harmonic 1-form λ on B.*

PROOF. Let λ be a harmonic 1-form on B, then $\alpha = \pi^*\lambda$ is closed. To prove that α is coclosed, let X be the vector field in B associated with λ by the Riemannian metric h. And denote by X^* the horizontal lift of X, then X^* is easily seen to be associated with α with respect to g. As λ is coclosed, X is incompressible vector field, i.e. $L(X)(\Omega^n) = 0$, where $L(X)$ is the Lie derivation by X. Now we have

$$L(X^*)(\eta \wedge (d\eta)^n) = (i(X^*)d\eta) \wedge (d\eta)^n + 2^n \eta \wedge (L(X^*)(\pi^*\Omega)^n) .$$

The first term of the right hand side is an m-form and

$$i(\xi)[(i(X^*)d\eta) \wedge (d\eta)^n] = 0$$

holds, so it vanishes. As for the second term, we have

$$L(X^*)(\pi^*\Omega^n) = \pi^*(L(X)\Omega^n) = 0 ,$$

and X^* is an incompressible vector field, namely α is coclosed and hence harmonic.

Conversely let α be a harmonic 1-form on M, then $L(\xi)\alpha = 0$, since ξ is a Killing vector field. By the Theorem below, α is orthogonal to η, and we have a 1-form λ on B such that $\alpha = \pi^*\lambda$. By the fact that π^* is an isomorphism for differential forms, we have $d\lambda = 0$ from $\pi^* d\lambda = d\pi^*\lambda = d\alpha = 0$. Next by Lemma 2.1, we have

$$c\langle \delta\lambda, \mu \rangle_B = c\langle \lambda, d\mu \rangle_B = \langle \pi^*\lambda, \pi^* d\mu \rangle_M$$

$$= \langle \pi^*\lambda, d\pi^*\mu \rangle_M = \langle \delta\pi^*\lambda, \pi^*\mu \rangle_M$$

$$= \langle \delta\alpha, \pi^*\mu \rangle_M = 0$$

for any 0-form μ on B. Thus $\delta\lambda = 0$ and λ is harmonic.

THEOREM 3.2. *Any harmonic 1-form α in a compact K-contact Riemannian manifold is orthogonal to η, i.e. $i(\xi)\alpha = 0$.*

PROOF. As α is a harmonic form and ξ is a Killing vector field, $i(\xi)\alpha$ is constant. Put $\gamma=\alpha-(i(\xi)\alpha)\eta$ and operate $\Delta=d\delta+\delta d$ to γ, then one gets $\Delta\gamma=-(i(\xi)\alpha)\Delta\eta$. By (1.5) we get $\Delta\eta=\delta d\eta=2(m-1)\eta$. Therefore we have

$$\Delta\gamma=-2(m-1)(i(\xi)\alpha)\eta\,. \tag{3.1}$$

However γ is orthogonal to η, and so $\langle\Delta\gamma,\gamma\rangle=0$. This implies that γ is harmonic and (3.1) shows $i(\xi)\alpha=0$.

§ 4. A compact regular normal contact Riemannian manifold.

S. Tachibana got the following ([**13**])

PROPOSITION 4.1. *For any harmonic r-form α on a compact normal contact Riemannian manifold M, we have $i(\xi)\alpha=0$ if $r\leqq(m-1)/2$.*

Corresponding to (1.13), we have

PROPOSITION 4.2. *Let $\pi: M\rightarrow B$ be the fibering of a compact regular normal contact Riemannian manifold M, then for any harmonic r-form α on M there exists a harmonic r-form λ on B such that $\alpha=\pi^*\lambda$, provided $r\leqq(m-1)/2$.*

PROOF. The second part of the proof of Proposition 3.1 is valid if we replace Theorem 3.2 by Proposition 4.1.

§ 5. K-contact η-Einstein spaces.

A contact Riemannian manifold M is said to be an η-Einstein space if the Ricci curvature tensor R_1 is of the form:

$$R_1=ag+b\eta\otimes\eta \tag{5.1}$$

for some scalar field a and b. When an η-Einstein space ($m>3$) is a K-contact Riemannian manifold, a and b become always constant.

Denoting by R the scalar curvature, first we have

THEOREM 5.1. *In a compact K-contact η-Einstein space, if $a>0$ or equivalently $R>m-1$ then $b_1(M)=0$.*

PROOF. By Theorem 3.2 any harmonic 1-form α on a compact K-contact Riemannian manifold is orthogonal to η. So $R_1(\alpha,\alpha)=ag(\alpha,\alpha)$ holds. By the theory of harmonic forms (for example [**16**]) we see that if R_1 is positive definite for α there is no harmonic 1-form on a compact orientable Riemannian manifold. Thus we have $b_1(M)=0$, if $a>0$. Transvecting (5.1) with g^{-1} and ξ, we have $a=(R-m+1)(m-1)^{-1}$.

We generalize this in the case $m>3$ to the

THEOREM 5.2. *In a compact K-contact η-Einstein space M, if $a=contact>-2$, then $b_1(M)=0$.*

PROOF. If we can change the Riemannian metric g to $\bar{g}$ such that the Ricci curvature tensor $\bar{R}_1$ for $\bar{g}$ is positive definite, we have $b_1(M)=0$. Thus Theorem 5.2 follows from

PROPOSITION 5.3. *Any K-contact η-Einstein space ($a=constant>-2$) is considered as an Einstein space by changing the metric.*

PROOF. For a constant $\beta>-1$, put

$$\bar{g}=g+\beta\eta\otimes\eta, \tag{5.2}$$

then denoting by W the difference of the components of the Riemmanian connections by $\bar{g}$ and g, we have

$$W=-\beta(\phi\otimes\eta+\eta\otimes\phi) \tag{5.3}$$

by ([**15**], (4.6)) and by using the relations in § 1. And by ([**15**], (6.3)) the Ricci curvature tensors are given by

$$\bar{R}_1=R_1-2\beta g+(2m\beta+(m-1)\beta^2)\eta\otimes\eta.$$

Thus by (5.1) and (5.2), we obtain

$$\bar{R}_1=(a-2\beta)\bar{g}+(b+(2m-a)\beta+(m+1)\beta^2)\eta\otimes\eta.$$

Contracting (5.1) with ξ, we have $a+b=m-1$. Therefore if $\beta=(a-m+1)(m+1)^{-1}$, we have $\bar{R}_1=(a+2)(m-1)(m+1)^{-1}\bar{g}$. Here note that this β satisfies the required condition $\beta>-1$, since $a+2>0$.

When a is not constant (then $m=3$), we assume regularity and get Theorem 5.5 below.

PROPOSITION 5.4. *In the fibering $\pi: M\to B$ of a regular K-contact Riemannian manifold M, M is an η-Einstein space if and only if B is an Einstein almost kählerian manifold.*

PROOF. First we notice the following identities:

$$[X^*, Y^*]=[X, Y]^*+\eta([X^*, Y^*])\xi, \tag{5.4}$$

$$\nabla_{X^*}Y^*=('\nabla_X Y)^*+2^{-1}\eta([X^*, Y^*])\xi, \tag{5.5}$$

$$\nabla_{X^*}\xi=\nabla_\xi X^*, \qquad ([X^*, \xi]=0), \tag{5.6}$$

where $'\nabla$ is the Riemannian connection by h ([8]). Then we have

$$\begin{aligned} R(X^*, Y^*)Z^* = {} & ('R(X, Y)Z)^*+2w(X^*, Y^*)\phi Z^* \\ & +w(X^*, Z^*)\phi Y^*-w(Y^*, Z^*)\phi X^* \\ & -2^{-1}\{\eta([X^*, ('\nabla_Y Z)^*]-[Y^*, ('\nabla_X Z)^*]) \\ & +X^*\cdot\eta([Y^*, Z^*])-Y^*\cdot\eta([X^*, Z^*])-\eta([[X, Y]^*, Z^*])\}\xi. \end{aligned} \tag{5.7}$$

From (5.7) and (1.8), it follows that

$$g(R(X^*, Z^*)Y^*, Z^*)=h('R(X, Z)Y, Z)\cdot\pi-3\Omega(X, Z)\Omega(Y, Z)\cdot\pi, \tag{5.8}$$

$$g(R(X^*, \xi)Y^*, \xi)=h(X, Y)\cdot\pi. \tag{5.9}$$

One can derive (5.8) also from the curvature relation in [7]. Now take locally an orthonormal basis $e_1, \cdots, e_n$, $e_{n+1}=Je_1, \cdots, e_{2n}=Je_n$ in B, put $Z^*=e_i^*$ in (5.8), sum for i and add the result to (5.9), then we get

$$R_1(X^*, Y^*) = {}'R_1(X, Y)\cdot\pi - 2h(X, Y)\cdot\pi . \tag{5.10}$$

Here notice that if $m=3$ a is not necessarily constant on M, but we see $L(\xi)a=0$, since ξ is a Killing vector field. In fact, the Lie derivatives of (5.1) are $(L(\xi)a)g+(L(\xi)b)\eta\otimes\eta=0$. Transvecting this with a vector which is orthogonal to ξ we get $L(\xi)a=0$. Thus if M is an η-Einstein space (5.1), then ${}'R_1=(a+2)h$ holds in B. Conversely if ${}'R_1=(m-1)^{-1}{}'Rh$, then we have $R_1(X^*, Y^*)=(m-1)^{-1}({}'R-2m+2)\ h(X, Y)\cdot\pi$, where ${}'R$ is the scalar curvature of ${}'\nabla$. This and (1.7) give

$$R_1=(m-1)^{-1}({}'R-2m+2)g+(m-1)^{-1}(m^2-1-{}'R)\eta\otimes\eta . \tag{5.11}$$

From (5.11) we have the relation of the scalar curvatures:

$$R={}'R-(m-1). \tag{5.12}$$

Theorem 5.5. *If M is a compact regular K-contact η-Einstein space and if $a>-2$, then $b_1(M)=0$.*

Proof. By $R_1=(a+2)h$, we have $b_1(B)=0$. And by (1.12), $b_1(M)=0$.

This theorem is needed only when a is not constant $(m=3)$.

In a compact kählerian manifold B, if the scalar curvature ${}'R$ is constant, ${}'H$ defined by ${}'H(X, Y)={}'R_1(X, JY)$ for vector fields X and Y on B is harmonic. And if B is not Einstein we have $b_2(B)\geqq 2$. This fact and Proposition 5.4 suggest us the following

Theorem 5.6. *Let M be a compact normal contact Riemannian manifold $(m>3)$ which is not η-Einstein. If the scalar curvature R is constant, then $b_2(M)\geqq 1$.*

Proof. We define a tensor H by

$$H(u, v)=(m-1)R_1(u, \phi v)+(m-1-R)w(u, v) \tag{5.13}$$

for any vector fields u and v on M. We can deduce from ([**9**], Lemma 4.4) that H is harmonic. And $H=0$ if and only if M is an η-Einstein space.

Corollary 5.7. *In a compact normal contact Riemannian manifold M $(m>3)$, if the scalar curvature is constant and $b_2(M)=0$, then M is an η-Einstein space.*

For a regular normal contact Riemannian manifold, which is of positive curvature and of constant scalar curvature, see [**4**].

Corresponding to (1.15) we state

Proposition 5.8. *Assume that $\pi: M\to B$ is the fibering of a compact regular normal contact Riemannian manifold M $(m>3)$ which is not an η-Einstein space. If the scalar curvature R is constant, we have $b_2(B)\geqq 2$. $b_2(B)=2$ holds if and only if $b_2(M)=1$, and in this case any harmonic 2-form α on M is of the form rH, for real number r.*

REMARK 5.9. Assume $m=3$ and that the scalar curvature R is constant in a compact normal contact Riemannian manifold M. Then M is an η-Einstein space.

PROOF. Even if $m=3$, H defined by (5.13) is harmonic. Clearly $H(u, \xi)=0$ and $i(\xi)H=0$. Let $*H$ be the dual 1-form determined by the volume element, then we must have $i(\xi)(*H)\neq 0$. However $*H$ is a harmonic form and we have $i(\xi)(*H)=0$ since $1\leqq(3-1)/2$. So $H=0$, and M is an η-Einstein space.

However precisely we have

THEOREM 5.10. *Assume $m=3$, then any K-contact Riemannian manifold M is an η-Einstein space.*

PROOF. Let p be an arbitrary point of M and take a small coordinate neighborhood U so that ξ is regular in U and we have a fibering $U\rightarrow U/\xi$. U/ξ is a 2-dimensional Riemannian manifold and hence always an Einstein space by a classical result. So, if we apply Proposition 5.4, U and also M is an η-Einstein space.

Mathematical Institute
Tôhoku University

References

[1] W.M. Boothby and H.C. Wang, On contact manifolds, Ann. of Math., **68** (1958), 721–734.

[2] S.S. Chern and E. Spanier, The homology structure of sphere bundles, Proc. Nat. Acad. Sci. U.S.A., **36** (1950), 248–255.

[3] T. Fujitani, Complex-valued differential forms on a normal contact manifold, Tôhoku Math. J., **18** (1966), 349–361.

[4] S. Goldberg, Rigidité de variétés de contact à coubure positive, C.R. Acad. Sci. Paris, **261** (1965), 1936–1939.

[5] Y. Hatakeyama, Some notes on differentiable manifolds with almost contact structures, Tôhoku Math. J., **15** (1963), 176–181.

[6] Y. Hatakeyama, Y. Ogawa and S. Tanno, Some properties of manifolds with contact metric structure, Tôhoku Math. J., **15** (1963), 42–48.

[7] S. Kobayoshi, Topology of positively pinched Kaehler manifolds, Tôhoku Math. J., **15** (1963), 121–139.

[8] K. Ogiue, On fiberings of almost contact manifolds, Kōdai Math. Sem. Rep., **17** (1965), 53–61.

[9] M. Okumura, Some remarks on space with a certain contact structure, Tôhoku Math. J., **14** (1962), 135–145.

[10] S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure, I, Tôhoku Math. J., **12** (1960), 459–476.

[11] S. Sasaki, Almost contact manifolds, Lecture note, Tôhoku Univ., 1965.

[12] J.P. Serre, Homologie singulière des espaces fibrés, Ann. of Math., **54** (1951), 425–505.

[13] S. Tachibana, On harmonic tensors in compact Sasakian spaces, Tôhoku Math. J., **17** (1965), 271–284.

[14] S. Tanno, A theorem on regular vector fields and its applications to almost contact structure, Tôhoku Math. J., 17 (1965), 235–238.

[15] S. Tanno, Partially conformal transformations with respect to $(m-1)$-dimensional distributions of m-dimensional Riemannian manifolds, Tôhoku Math. J., 17 (1965), 358–409.

[16] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies 32, Princeton, 1953.