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On generalized graded Lie algebras and geometric structures I

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Introduction.

The main purpose of the present paper is to establish a new prolongation theorem for certain linear group structures which are of infinite type and even not elliptic, and is to apply this theorem to the geometry of differential systems (distributions in the sense of Chevalley) and the geometry of real submanifolds in complex manifolds. (A linear group G is called elliptic if the Lie algebra $\mathfrak{g}$ of G contains no matrix of rank 1, and a G-structure is called elliptic if the linear group G is elliptic. A recent work of Ochiai [4] has shown that a G-structure is elliptic if and only if the defining equation of infinitesimal automorphisms of the G-structure forms an elliptic system of linear differential equations).

First of all, we introduce the notion of a generalized graded Lie algebra (Def. 2.1). Let $\mathfrak{g}$ be a Lie algebra, and let $(\mathfrak{g}_p)_{p\in \boldsymbol{Z}}$ be a family of subspaces of $\mathfrak{g}$, $\boldsymbol{Z}$ being the additive group of integers, which satisfies the following conditions:

1) $\mathfrak{g}=\sum_p \mathfrak{g}_p$ (direct sum);
2) $\dim \mathfrak{g}_p<\infty$;
3) $[\mathfrak{g}_p, \mathfrak{g}_q]\subset \mathfrak{g}_{p+q}$.

Under these conditions, we say that the direct sum $\mathfrak{g}=\sum_p \mathfrak{g}_p$ is a generalized graded Lie algebra or simply a graded Lie algebra. (Note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and the mappings $\mathfrak{g}_0\times\mathfrak{g}_p\ni(X_0, X_p)\rightarrow[X_0, X_p]\in\mathfrak{g}_p$ define representations ρ_p of the Lie algebra $\mathfrak{g}_0$ on the vector spaces $\mathfrak{g}_p$.)

Now consider a graded Lie algebra of the form $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$, where we must think $\mathfrak{g}_p$ ($p<-2$ or $p>0$) of vanishing, and assume the following conditions:

1° $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$;
2° the representation ρ_{-1} of the Lie algebra $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_{-1}$ is faithful.

It is shown that there corresponds to such a graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ a suitable graded Lie algebra $\mathfrak{g}=\sum_{p=-2}^{\infty}\mathfrak{g}_p$ (the prolongation of $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$) such that the given graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ will be a subalgebra of $\mathfrak{g}$ (see § 2). One finds that this situation is just analogous to the usual prolongation of a subalgebra $\mathfrak{g}_0$ of the general linear Lie algebra $\mathfrak{gl}(\mathfrak{g}_{-1})$ of a vector space $\mathfrak{g}_{-1}$ (cf. Singer-Sternberg [**5**]).

From now on, we shall consider a fixed graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ (over the field $\boldsymbol{R}$ of real numbers) satisfying conditions 1° and 2° and its prolongation $\mathfrak{g}=\sum_{p=-2}^{\infty}\mathfrak{g}_p$. We set $\mathfrak{m}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$, $m=\dim\mathfrak{m}$, $n'=\dim\mathfrak{g}_{-2}$ and $n=\dim\mathfrak{g}_{-1}$. We have $[\mathfrak{g}_0,\mathfrak{m}]\subset\mathfrak{m}$ and hence $\mathfrak{g}_0$ may be identified with a subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{m})$ by condition 2°. Let G_0 be the group of $\mathfrak{g}_0$, being a connected Lie subgroup of the general linear group $GL(\mathfrak{m})$ of $\mathfrak{m}$. The elements σ of G_0 may be represented by the matrices of the form

$$\begin{pmatrix}\rho_{-2}(\sigma) & 0\\ 0 & \rho_{-1}(\sigma)\end{pmatrix},$$

where $\rho_p(p=-2,-1)$ denotes the representation of G_0 on $\mathfrak{g}_p$ corresponding to the representation ρ_p of $\mathfrak{g}_0$ on $\mathfrak{g}_p$. Let $G_0^{\#}$ denote the subgroup of $GL(\mathfrak{m})$ consisting of all the linear automorphisms of $\mathfrak{m}$ represented by matrices of the form

$$\begin{pmatrix}\rho_{-2}(\sigma) & 0\\ v & \rho_{-1}(\sigma)\end{pmatrix},$$

where $\sigma\in G_0$ and v are linear mappings of $\mathfrak{g}_{-2}$ to $\mathfrak{g}_{-1}$.

$G_0^{\#}$ being a Lie subgroup of $GL(\mathfrak{m})$, we have the notion of a $G_0^{\#}$-structure (Def. 1.2). A $G_0^{\#}$-structure on a manifold M of dimension m is defined to be a pair $(P_0^{\#},\omega^{(0)})$ formed by a principal fiber bundle $P_0^{\#}$ over the base space M with structure group $G_0^{\#}$ and an $\mathfrak{m}$-valued 1-form $\omega^{(0)}$ (the basic form) on $P_0^{\#}$ satisfying certain conditions. We denote by $\omega_p^{(0)}$ $(p=-2,-1)$ the $\mathfrak{g}_p$-component of $\omega^{(0)}$ with respect to the decomposition $\mathfrak{m}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$. Moreover, we define the notion of a pseudo-G_0-structure. A pseudo-G_0-structure on a manifold M of dimension m is a pair $(P_0,\theta^{(0)})$ formed by a principal fiber bundle P_0 over the base space M with structure group G_0 and a collection $\theta^{(0)}=\{\theta_{-2}^{(0)},\theta_{-1}^{(0)}\}$ as follows: $\theta_{-2}^{(0)}$ is a $\mathfrak{g}_{-2}$-valued 1-form on P_0, while $\theta_{-1}^{(0)}$ is a $\mathfrak{g}_{-1}$-valued 1-form defined only for vectors in a differential system D_0 on P_0. Moreover the collection $\theta^{(0)}$ must satisfy certain conditions. We show that to every $G_0^{\#}$-structure $(P_0^{\#},\omega^{(0)})$ on a manifold M there is associated a pseudo-G_0-structure $(P_0,\theta^{(0)})$ on M in such a way that the assignment $(P_0^{\#},\omega^{(0)})\to(P_0,\theta^{(0)})$ will be compatible with the respective isomorphisms (Th. 4.1).

The main theorem (Ths. 4.1, 4.2 and Cor. 1) in this paper may roughly be stated as follows: (1) Let $(P_0^\#, \omega^{(0)})$ be a $G_0^\#$-structure on a manifold M which satisfies the follwing condition

$$C_0^\#: d\omega_{-2}^{(0)}+\frac{1}{2}[\omega_{-1}^{(0)}, \omega_{-1}^{(0)}]\equiv 0 \qquad (\mathrm{mod}\ \omega_{-2}^{(0)}),$$

and let $(P_0, \theta^{(0)})$ be the corresponding pseudo-G_0-structure on M. Then we have a sequence

$$(P)\quad \cdots \rightarrow (P_k, \theta^{(k)}) \underset{\varpi_k}{\rightarrow} (P_{k-1}, \theta^{(k-1)}) \rightarrow \cdots \rightarrow (P_1, \theta^{(1)}) \underset{\varpi_1}{\rightarrow} (P_0, \theta^{(0)})$$

(the prolongation of $(P_0, \theta^{(0)})$ or $(P_0^\#, \omega^{(0)})$) as follows:

1) P_k is a principal fiber bundle over the base space P_{k-1} with a certain group G_k as structure group, and ϖ_k is the projection of P_k onto P_{k-1}, where G_k is an abelian Lie group of dimension equal to $\dim \mathfrak{g}_k$, and it is constructed from the prolongation $\mathfrak{g}=\sum_{p=-2}^{\infty}\mathfrak{g}_p$ of $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ (see § 3);

2) $\theta^{(k)}$ is a collection $(\theta_p^{(k)})_{-2\leqq p\leqq k-1}$ as follows: $\theta_p^{(k)}$ $(-2\leqq p<k-1)$ is a $\mathfrak{g}_p$-valued 1-form on P_k, while $\theta_{k-1}^{(k)}$ is a $\mathfrak{g}_{k-1}$-valued 1-form defined only for vectors in a differential system D_k on P_k;

3) the pair $(P_k, \theta^{(k)})$ defines a "pseudo-G_k-structure" on P_{k-1} (Def. 4.2);

(2) the assignment $(P_0^\#, \omega^{(0)})\rightarrow(P)$ is compatible with the various isomorphisms.

It should be noted that, in the case where $n'=0$, we have $(P_0^\#, \omega^{(0)})=(P_0, \theta^{(0)})$ and the sequence (P) reduces to the usual prolongation of the $G_0^\#$-structure $(P_0^\#, \omega^{(0)})$. The construction (see §§ 5 and 6) of the sequence (P) is much more complicated and delicate compared with the construction (cf. Singer-Sternberg [**5**] and E. Cartan [**1**]) of the usual prolongation of a linear group structure.

As an immediate consequence of the above prolongation theorem, we have the following finiteness theorem for $G_0^\#$-structures (Cor. 3 to Ths. 4.1 and 4.2): Let $(P_0^\#, \omega^{(0)})$ be a $G_0^\#$-structure on a manifold M satisfying condition $C_0^\#$. If the Lie algebra $\mathfrak{g}$ is finite dimensional, then the Lie algebra $\mathfrak{a}$ of all the infinitesimal automorphisms of $(P_0^\#, \omega^{(0)})$ is finite dimensional and of dimension $\leqq \dim \mathfrak{g}$. It should be emphasized that, in the case where $n'\neq 0$, the linear group $G_0^\#$ is not elliptic and hence is of infinite type.

The first application will be concerned with the geometry of differential systems (see § 7). Let M be a manifold of dimension $n+n'$ and let D be a differential system on M of dimension n. Our main theorem is only and well applied for the case where $n'\leqq\frac{1}{2}n(n-1)$. We define the "torsion" T of the differential system D, and, under suitable conditions on T, we show that there corresponds to D a graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ satisfying conditions 1° and

2° together with a G_0-structure $(P_0^\sharp, \omega^{(0)})$ on M satisfying condition $C_0^\sharp$, where $\dim \mathfrak{g}_{-2} = n'$ and $\dim \mathfrak{g}_{-1} = n$. The above-mentioned finiteness theorem proves that the Lie algebra of all the infinitesimal automorphisms of D is generally finite dimensional in the following two cases: $n' = \frac{1}{2}n(n-1)-1$ (n even $\geqq 4$) and $n' = \frac{1}{2}n(n-1)$ ($n \geqq 3$).

The second application will be concerned with the geometry of real submanifolds in complex manifolds (see § 8). This geometry turns out to be just similar to the geometry of differential systems. In the following, we assume the differentiability of class C^ω. Let M (resp. M') be a real submanifold in a complex manifold $\tilde{M}$ (resp. $\tilde{M}'$). A real analytic homeomorphism φ is called an isomorphism of M onto M' if it can be extended to a complex analytic homeomorphism $\tilde{\varphi}$ of a neighbourhood of M onto a neighbourhood of M'. Let M be a real submanifold in a complex manifold $\tilde{M}$. The tangent vector space $T_x(\tilde{M})$ to $\tilde{M}$ at any point $x \in M$ is a complex vector space and the tangent vector space $T_x(M)$ to M at x is a real subspace of the complex vector space $T_x(\tilde{M})$. This being said, we say that M is of type (n, n') if we have $\dim M = 2n+n'$ and if the complex dimension of the maximum complex subspace of $T_x(\tilde{M})$ contained in $T_x(M)$ is equal to n. We shall see that our geometry is essentially reduced to the case of real submanifolds M of type (n, n') in complex manifolds $\tilde{M}$ of complex dimension $n+n'$.

Now let M be a real submanifold of type (n, n') in a complex manifold $\tilde{M}$ of complex dimension $n+n'$. Our main theorem is only and well applied for the case where $n' \leqq n^2$. We define the "torsion" or the "Levi form" T of the real submanifold M, and, under suitable conditions on T, we show that there corresponds to M a graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ satisfying conditions 1° and 2° together with a $G_0^\sharp$-structure $(P_0^\sharp, \omega^{(0)})$ on M satisfying condition $C_0^\sharp$, where $\dim \mathfrak{g}_{-2} = n'$ and $\mathfrak{g}_{-1}$ is equipped with a structure of a complex vector space of complex dimension n. From the above-mentioned finiteness theorem, it follows that the Lie algebra of all the infinitesimal automorphisms of M is generally finite dimensional in the following three cases: $n' = 1$ ($n \geqq 1$), $n' = n^2-1$ ($n \geqq 2$) and $n' = n^2$ ($n \geqq 1$). We notice that the geometry of real submanifolds M of type (n, n') in complex manifolds $\tilde{M}$ of complex dimension $n+n'$ has an intimate relationship with the geometry of Siegel domains D of second kind developed by Pyatetski-Shapiro and others. The relation is given through the Silov boundaries of the domains D. We want to take up this problem at another occasion.

In the subsequent paper [8], we shall introduce the notion of a generalized filtered Lie algebra, and, as an application, prove a structure theorem for certain homogeneous spaces whose linear isotropy representations are reducible.

Furthermore, we shall make a more profound study of the equivalence of two $G_0^{\#}$-structures $(P_0^{\#}, \omega^{(0)})$ satisfying condition $C_0^{\#}$.

Finally, we add that our theory may be extended to more general theories in which graded Lie algebras of the form $\mathfrak{g}=\sum_{p=-k}^{\infty}\mathfrak{g}_p$ $(k\geqq 3)$ will play important roles (cf. Remarks at the ends of §§ 7 and 8).

§ 1. Preliminaries: Differential forms defined on differential systems, principal fiber bundles and linear group structures.

Throughout this paper except § 8, we always assume the differentiability of class C^∞. The manifolds to be considered are manifolds satisfying the second axiom of countability.

Let M be an m-dimensional manifold. An n-dimensional differential system on M is, by definition, a differentiable mapping D which maps every point $x\in M$ to an n-dimensional contact element D_x to M at x.

DEFINITION 1.1. Let D be an n-dimensional differential system on an m-dimensional manifold M. A p-form on (M, D) is a differentiable mapping α which maps every point $x\in M$ to an anti-symmetric p-form α_x on the vector space D_x.

A 0-form α on (M, D) is nothing but a function on M. And if $n=m$, a p-form α on (M, D) reduces to a usual p-form on M. Let α (resp. β) be a p-form (resp. q-form) on (M, D). Then we define the exterior product $\alpha\wedge\beta$, being a $(p+q)$-form on (M, D) in the same manner as in usual forms on M. Given a finite dimensional vector space V, we define a V-valued p-form α on (M, D) in a trivial manner.

Let α be a V-valued p-form on (M, D), and let D' be a differential subsytem of D, i. e., $D'_x\subset D_x$ at each $x\in M$. Then $\alpha|D'$, being a V-valued p-form on (M, D'), will denote the restriction of α to D'. Let φ be a mapping of a manifold M' to M. We assume that φ is of maximal rank, i. e., the differential $\varphi_{*x'}$ of φ at x' is surjective at each $x'\in M'$. $D'=\varphi^*D$ will denote the differential system on M' defined by $D'_{x'}=(\varphi_{*x'})^{-1}D_x$ at each x', where $x=\varphi(x')$. Given a V-valued p-form α on (M, D), $\alpha'=\varphi^*\alpha$ will denote the V-valued p-form on (M', D') defined by $\alpha'_{x'}=(\varphi_{*x'})^*\alpha_x$ at each x'.

Let α (resp. α') be a V-valued p-form (resp. V'-valued p'-form) on (M, D). Let (v_i) (resp. (v'_j)) be a base of V (resp. V') and express α (resp. α') as $\alpha=\sum_i \alpha_i v_i$ (resp. $\alpha'=\sum_j \alpha'_j v'_j$). Let $c(x\to c_x)$ be a mapping of M to the vector space of all the bilinear mappings of $V\times V'$ to a finite dimensional vector space W. Then the notation $c(\alpha, \alpha')$ will denote the W-valued $(p+p')$-form on (M, D) defined by $c(\alpha, \alpha')=\sum_{i,j} c(v_i, v'_j)\alpha_i\wedge\alpha'_j$, where $c(v_i, v'_j)$ means the mappings $M \ni x \to c_x(v_i, v'_j) \in W$. Analogous but more general notations may be defined in the same way.

Let $\beta = \sum_{l=1}^{t} \beta_l w_l$ be a W-valued q-form on (M, D) and let $\alpha = \sum_{i=1}^{s} \alpha_i v_i$ (resp. $\alpha' = \sum_{j=1}^{s'} \alpha'_j v'_j$, resp. $\alpha'' = \sum_{k=1}^{s''} \alpha''_k v''_k$) be a V-valued p-form (resp. V'-valued p'-form, resp. V''-valued p''-form) on (M, D). By the notation $\beta \equiv 0 \pmod{\alpha, \alpha' \cdot \alpha''}$, we shall mean the following: $\beta_l \equiv 0 \pmod{\alpha_i, \alpha'_j \wedge \alpha''_k \ (1 \leqq i \leqq s, 1 \leqq j \leqq s', 1 \leqq k \leqq s'')}$ for all l $(1 \leqq l \leqq t)$, where the meaning of mod should be considered in the algebra of all the forms on (M, D). Analogous but more general notations may be defined in the same way.

A V-valued 1-form $\omega = \sum_i \omega_i v_i$ on (M, D) is called V-independent if the linear mapping $D_x \ni X \to \omega(X) \in V$ is surjective at each $x \in M$. It is clear that ω is V-independent if and only if (ω_i) is linearly independent at each $x \in M$. Let α be a W-valued 2-form on (M, D) and let ω be a V-valued 1-form on (M, D) which is V-independent. If $\alpha \equiv 0 \pmod{\omega^2}$, then there is a unique mapping c of M to the vector space of all the anti-symmetric bilinear mappings of $V \times V$ to W such that $\alpha = c(\omega, \omega)$. Analogous results in the same line can be also obtained.

Let P be a principal fiber bundle over a manifold M with a Lie group G as structure group. Such a principal fiber bundle P will be symbolically written as $P(M, G)$. A tangent vector Y to P is called a vertical vector in $P(M, G)$ if we have $\pi_* Y = 0$, π being the projection of P onto M. For each $\sigma \in G$, the transformation $R(\sigma): P \ni z \to z\sigma \in P$ is called the right translation induced by σ. Let $\mathfrak{g}$ be the Lie algebra of G. For each $X \in \mathfrak{g}$, let $r(X)$ be the vector field on P induced by the one parameter group $R(\text{expt } X)$. This vector field $r(X)$ is called the vertical vector field induced by X. Now, let N be a closed normal subgroup of G. We define an equivalence relation $\sim$ in P as follows: Let $z, z' \in P$. Then $z \sim z'$ if and only if there is a $\sigma \in N$ such that $z' = z\sigma$. The quotient space P/N by the equivalence relation $\sim$ is naturally a manifold. Furthermore P (resp. P/N) may be considered as a principal fiber bundle over the base space P/N (resp. M) with structure group N (resp. G/N).

Let us now define the notion of G-structures. Given a finite dimensional vector space V over a field K, $GL(V)$ (resp. $\mathfrak{gl}(V)$) will denote the group (resp. the Lie algebra) of all the automorphisms (resp. all the endomorphisms) of the vector space V.

DEFINITION 1.2. Let $\mathfrak{m}$ be an m-dimensional vector space over the field $\boldsymbol{R}$ of real numbers and let G be a Lie subgroup of $GL(\mathfrak{m})$. Let M be an m-dimensional manifold and let P be a principal fiber bundle over the base space M with structure group G. Let θ be an $\mathfrak{m}$-valued 1-form on P. Then the pair (P, θ) is called a G-structure on M if it satisfies the following conditions:

1) Let Z be a tangent vector to P. Then $\theta(Z)=0$ if and only if Z is a vertical vector in the principal fiber bundle $P(M, G)$;

2) $R(\sigma)^*\theta=\sigma^{-1}\theta$ for all $\sigma \in G$.

Let (P, θ) (resp (P', θ')) be a G-structure on a manifold M (resp. M'). A bundle isomorphism φ of P onto P' is called an isomorphism of (P, θ) onto (P', θ') if we have $\varphi^*\theta'=\theta$. An isomorphism φ of (P, θ) onto (P', θ') is called an equivalence if $M=M'$ and if φ induces the identity transformation of M.

It is clear that the 1-form θ is $\mathfrak{m}$-independent.

As an important example of a G-structure, we shall now explain the frame bundle F of any manifold M of dimension m. We take a fixed m-dimensional vector space $\mathfrak{m}$ over $\boldsymbol{R}$, and denote by F the set of all the linear isomorphisms z of $\mathfrak{m}$ onto the tangent vector spaces $T_x(M)$ to M at $x \in M$, where x runs through any points of M. Then F is naturally a manifold, and the group $GL(\mathfrak{m})$ acts on F, as a Lie transformation group, by the mapping $F \times GL(\mathfrak{m}) \ni (z, \sigma) \rightarrow z\sigma = z \circ \sigma \in F$. Thus we get a principal fiber bundle F over the base space M with structure group $GL(\mathfrak{m})$ with projection $\pi: F \ni z \rightarrow x \in M$. This principal fiber bundle F is called the frame bundle of M. Now define an $\mathfrak{m}$-valued 1-form ω on F by $z \cdot \omega(Z)=\pi_* Z$ for all $Z \in T_z(F)$. Then the pair (F, ω) is clearly a $GL(\mathfrak{m})$-structure on M.

Let G be a Lie subgroup of $GL(\mathfrak{m})$. Let P be a G-subbundle of F and denote by θ the restriction of ω to P. Then the pair (P, θ) forms a G-structure on M. The 1-form θ is called the basic form of P. Conversely, we can prove the followings: (1) Let (P, θ) be a G-structure on M. Then there is a unique base preserving injecteve homomorphism ι of P to F such that $\theta=\iota^*\omega$.

(2) Let (P, θ) and (P', θ') be two G-structures on M, and let ι and ι' be the corresponding homomorphisms of P and P' to F respectively. Then $\iota(P) =\iota'(P')$ if and only if there is a unique equivalence φ of (P, θ) onto (P', θ') such that $\iota' \circ \varphi=\iota$.

§2. Prolongation of generalized graded Lie algebras $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$.

In this section, the vector spaces and the Lie algebras to be considered are those over a fixed field K of characteristic zero. $\boldsymbol{Z}$ will denote the additive group of integers.

Definition 2.1 ([7]). Let $\mathfrak{g}$ be a Lie algebra and let $(\mathfrak{g}_p)_{p \in \boldsymbol{Z}}$ be a family of subspaces of $\mathfrak{g}$. Then the pair $\{\mathfrak{g}, (\mathfrak{g}_p)\}$ is called a generalized graded Lie algebra or simply a graded Lie algebra if it satisfies the following conditions:

1) $\mathfrak{g}=\sum_{p \in \boldsymbol{Z}} \mathfrak{g}_p$ (direct sum);

2) $\dim \mathfrak{g}_p < \infty$;

3) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$.

It is clear that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}$ be a Lie algebra and let $(\mathfrak{g}_p)_{k<p<l}$ be a family of subspaces of $\mathfrak{g}$. We shall say that $\mathfrak{g} = \sum_{k<p<l} \mathfrak{g}_p$ is a graded Lie algebra if $\{\mathfrak{g}, (\mathfrak{g}_p)\}$ forms a graded Lie algebra by setting $\mathfrak{g}_p = \{0\}$ for $p \leqq k$ or $p \geqq l$.

Now, consider a graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ which satisfies the following conditions:

(2.1) 1) $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$;

2) if $X_0 \in \mathfrak{g}_0$ and if $[X_0, \mathfrak{g}_{-1}] = \{0\}$, then $X_0 = 0$.

We shall show that to such a graded Lie algebra there is associated a "maximal" family $(\mathfrak{g}_p)_{p \geqq 1}$ of vector spaces such that

1° the direct sum $\mathfrak{g} = \sum_{p=-2}^{\infty} \mathfrak{g}_p$ is a graded Lie algebra;

2° let p be any integer $\geqq 0$. If $X_p \in \mathfrak{g}_p$ and if $[X_p, \mathfrak{g}_{-1}] = \{0\}$, then we have $X_p = 0$;

3° the given graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g} = \sum_p \mathfrak{g}_p$.

First we define vector spaces $\mathfrak{g}_p$ together with bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_{-1} \ni (X_p, X_{-1}) \to [X_p, X_{-1}] \in \mathfrak{g}_{p-1}$ and $\mathfrak{g}_p \times \mathfrak{g}_{-2} \ni (X_p, X_{-2}) \to [X_p, X_{-2}] \in \mathfrak{g}_{p-2}$ ($p \geqq 0$) inductively as follows: For $p=0$, these things have already been defined. Suppose that we have defined vector spaces $\mathfrak{g}_p$ together with bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_{-1} \ni (X_p, X_{-1}) \to [X_p, X_{-1}] \in \mathfrak{g}_{p-1}$ and $\mathfrak{g}_p \times \mathfrak{g}_{-2} \ni (X_p, X_{-2}) \to [X_p, X_{-2}] \in \mathfrak{g}_{p-2}$ ($0 \leqq p \leqq k-1$) in such a way that we have

$$
\begin{aligned}
(2.2) \qquad & [[X_p, X_{-1}], Y_{-1}] - [[X_p, Y_{-1}], X_{-1}] = [X_p, [X_{-1}, Y_{-1}]], \\
& [[X_p, X_{-2}], Y_{-1}] = [[X_p, Y_{-1}], X_{-2}]
\end{aligned}
$$

for all $X_p \in \mathfrak{g}_p$ ($1 \leqq p \leqq k-1$), $X_{-1}, Y_{-1} \in \mathfrak{g}_{-1}$ and $X_{-2} \in \mathfrak{g}_{-2}$. Then we define $\mathfrak{g}_k$ to be the vector space of all the linear mappings X_k of $\mathfrak{g}_{-1}$ to $\mathfrak{g}_{k-1}$ satisfying the following conditions: There are linear mappings X'_k of $\mathfrak{g}_{-2}$ to $\mathfrak{g}_{k-2}$ such that

$$
\begin{aligned}
(2.3) \qquad & [X_k(X_{-1}), Y_{-1}] - [X_k(Y_{-1}), X_{-1}] = X'_k([X_{-1}, Y_{-1}]), \\
& [X'_k(X_{-2}), Y_{-1}] = [X_k(Y_{-1}), X_{-2}]
\end{aligned}
$$

for all $X_{-1}, Y_{-1} \in \mathfrak{g}_{-1}$ and $X_{-2} \in \mathfrak{g}_{-2}$. We set $[X_k, X_{-1}] = X_k(X_{-1})$ for all $X_{-1} \in \mathfrak{g}_{-1}$. Since we have $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, we see that X'_k is uniquely determined by X_k. This being said, we set $[X_k, X_{-2}] = X'_k(X_{-2})$ for all $X_{-2} \in \mathfrak{g}_{-2}$. Thus we get bilinear mappings $\mathfrak{g}_k \times \mathfrak{g}_{-1} \ni (X_k, X_{-1}) \to [X_k, X_{-1}] \in \mathfrak{g}_{k-1}$ and $\mathfrak{g}_k \times \mathfrak{g}_{-2} \ni (X_k, X_{-2}) \to [X_k, X_{-2}] \in \mathfrak{g}_{k-2}$. We have clearly equalities (2.2) with $p=k$, completing our inductive definition.

From $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ and (2.2), we get easily

$$[[X_k, X_{-2}], Y_{-2}]=[[X_k, Y_{-2}], X_{-2}] \tag{2.4}$$

for all $X_k \in \mathfrak{g}_k$ $(k \geqq 0)$ and $X_{-2}, Y_{-2} \in \mathfrak{g}_{-2}$.

We set $[X_{-1}, X_p]=-[X_p, X_{-1}]$ and $[X_{-2}, X_p]=-[X_p, X_{-2}]$ for all $X_p \in \mathfrak{g}_p$ $(p \geqq 1)$, $X_{-1} \in \mathfrak{g}_{-1}$ and $X_{-2} \in \mathfrak{g}_{-2}$. Let us now define bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_q \ni (X_p, X_q) \to [X_p, X_q] \in \mathfrak{g}_{p+q}$ $(p, q \geqq 0)$ inductively as follows: For $p=q=0$, these things have already been defined. Suppose that we have defined bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_q \ni (X_p, X_q) \to [X_p, X_q] \in \mathfrak{g}_{p+q}$ $(p, q \geqq 0, p+q<k)$ in such a way that we have

$$\begin{aligned} [[X_p, X_q], X_{-1}] &= [[X_p, X_{-1}], X_q]+[X_p, [X_q, X_{-1}]], \\ [[X_p, X_q], X_{-2}] &= [[X_p, X_{-2}], X_q]+[X_p, [X_q, X_{-2}]] \end{aligned} \tag{2.5}$$

for all $X_p \in \mathfrak{g}_p$, $X_q \in \mathfrak{g}_q$ $(p, q \geqq 0, p+q<k)$, $X_{-1} \in \mathfrak{g}_{-1}$ and $X_{-2} \in \mathfrak{g}_{-2}$. We take any $X_p \in \mathfrak{g}_p$ and $X_q \in \mathfrak{g}_q$ $(p, q \geqq 0, p+q=k)$ and define linear mappings X_k and X_k' of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ to $\mathfrak{g}_{k-1}$ and $\mathfrak{g}_{k-2}$ respectively by

$$\begin{aligned} X_k(X_{-1}) &= [[X_p, X_{-1}], X_q]+[X_p, [X_q, X_{-1}]], \\ X_k'(X_{-2}) &= [[X_p, X_{-2}], X_q]+[X_p, [X_q, X_{-2}]] \end{aligned}$$

for all $X_{-1} \in \mathfrak{g}_{-1}$ and $X_{-2} \in \mathfrak{g}_{-2}$. Then we see that X_k and X_k' satisfy (2.3) with $p=k$. Hence we have $X_k \in \mathfrak{g}_k$. This being said, we define $[X_p, X_q]$ to be X_k. Thus we get bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_q \ni (X_p, X_q) \to [X_p, X_q] \in \mathfrak{g}_{p+q}$ $(p, q \geqq 0, p+q=k)$. We have clearly equalities (2.5) with p and q $(p, q \geqq 0, p+q=k)$, completing our inductive definition. Note that $[X_p, X_q]=-[X_q, X_p]$ for all $X_p \in \mathfrak{g}_p$ and $X_q \in \mathfrak{g}_q$ $(p, q \geqq 0)$.

By induction, we can also prove

$$[[X_p, X_q], X_r]+[[X_q, X_r], X_p]+[[X_r, X_p], X_q]=0 \tag{2.6}$$

for all $X_p \in \mathfrak{g}_p$, $X_q \in \mathfrak{g}_q$ and $X_r \in \mathfrak{g}_r$ $(p, q, r \geqq 0)$.

Finally we set $\mathfrak{g}=\sum_{p=-2}^{\infty} \mathfrak{g}_p$ (direct sum). By (2.2), (2.4), (2.5) and (2.6), we know that the bracket operation in $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ and the bilinear mappings $\mathfrak{g}_p \times \mathfrak{g}_q \ni (X_p, X_q) \to [X_p, X_q] \in \mathfrak{g}_{p+q}$, thus defined, give a structure of Lie algebra on $\mathfrak{g}$ in such a way that the direct sum $\mathfrak{g}=\sum_p \mathfrak{g}_p$ satisfies conditions 1°—3° stated above. The graded Lie algebra $\mathfrak{g}=\sum_p \mathfrak{g}_p$ is called the prolongation of the graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$.

We shall now state an important proposition concerning a graded Lie algebra $\mathfrak{g}=\sum_{p=-2}^{\infty} \mathfrak{g}_p$. Given such a graded Lie algebra, we denote by ρ_p the representation of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_p$ defined by $\rho_p(X_0)X_p=[X_0, X_p]$ for all $X_0\in\mathfrak{g}_0$ and $X_p\in\mathfrak{g}_p$.

PROPOSITION 2.1 *Let* $\mathfrak{g}=\sum_{p=-2}^{\infty}\mathfrak{g}_p$ *be a graded Lie algebra satisfying the following conditions:*

a) $\mathfrak{g}$ *is finite dimensional;*

b) $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]\neq\{0\}$;

c) *let* p *be any integer* $\geqq 0$. *If* $X_p\in\mathfrak{g}_p$ *and if* $[X_p, \mathfrak{g}_{-1}]=\{0\}$, *then we have* $X_p=0$;

d) *both the representations* ρ_{-2} *and* ρ_{-1} *are irreducible;*

e) $\mathfrak{g}_1\neq\{0\}$.

Then we have:

1) $\mathfrak{g}$ *is simple;*

2) $\mathfrak{g}_p=\{0\}$ *for any* $p>2$, *and* $\dim\mathfrak{g}_p=\dim\mathfrak{g}_{-p}$ *for any* p.

A stronger theorem will be proved in the subsequent paper [8].

§ 3. The groups N_k, $G_k^{\#}$ and G_k.

Let $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ be a graded Lie algebra over $\boldsymbol{R}$ satisfying condition (2.1) and let $\mathfrak{g}=\sum_{p=-2}^{\infty}\mathfrak{g}_p$ be its prolongation. We have $[\mathfrak{g}_0, \mathfrak{g}_{-2}]\subset\mathfrak{g}_{-2}$, $[\mathfrak{g}_0, \mathfrak{g}_{-1}]\subset\mathfrak{g}_{-1}$ and hence $[\mathfrak{g}_0, \mathfrak{g}_{-2}+\mathfrak{g}_{-1}]\subset\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$. By condition (2.1), 2), we know that the representation of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$ is faithful. Therefore $\mathfrak{g}_0$ may be identified with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-2}+\mathfrak{g}_{-1})$. We denote by G_0 the connected Lie subgroup of $GL(\mathfrak{g}_{-2}+\mathfrak{g}_{-1})$ generated by the subalgebra $\mathfrak{g}_0$ of $\mathfrak{gl}(\mathfrak{g}_{-2}+\mathfrak{g}_{-1})$. It is clear that $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are G_0-invariant subspaces and that the bracket operation in $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$ is also G_0-invariant.

We set $n'=\dim\mathfrak{g}_{-2}$, $\mathfrak{m}_k=\sum_{p=-2}^{\infty}\mathfrak{g}_p$ and $m_k=\dim\mathfrak{m}_k$, and denote by $\mathcal{L}_k^{-2}$ the vector space of all the linear mappings v of $\mathfrak{g}_{-2}$ to $\mathfrak{g}_k$. Given a vector $X\in\mathfrak{g}$, X_p will denote the $\mathfrak{g}_p$-component of X.

The groups N_k ($k\geqq 0$). For each $v\in\mathcal{L}_{k-1}^{-2}$ ($k\geqq 0$), we define an element $B_k(v)$ of $GL(\mathfrak{m}_{k-1})$ by

$$B_k(v)X=X+v(X_{-2})$$

for all $X\in\mathfrak{m}_{k-1}$. We have

$$B_k(v_1)B_k(v_2)=B_k(v_1+v_2)\,.$$

We denote by N_k the closed connected abelian subgroup of $GL(\mathfrak{m}_{k-1})$ consisting of all the elements $B_k(v)$ ($v\in\mathcal{L}_{k-1}^{-2}$). For each $v\in\mathcal{L}_{k-1}^{-2}$ ($k\geqq 0$), we define an element $b_k(v)$ of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ by $b_k(v)X=v(X_{-2})$ for all $X\in\mathfrak{m}_{k-1}$. Then we have $B_k(v)=\exp b_k(v)$. The Lie algebra of N_k is given by the subalgebra $\mathfrak{n}_k$ of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ consisting of all the elements $b_k(v)$ ($v\in\mathcal{L}_{k-1}^{-2}$).

The groups $G_k^\#$ $(k \geqq 0)$. For any $a \in G_0$ and $v \in \mathcal{L}_{-1}^{-2}$, we have

$$B_0(v)a = aB_0(v^a),$$

where v^a is the element of $\mathcal{L}_{-1}^{-2}$ defined by $v^a(X_{-2}) = a^{-1}v(aX_{-2})$ for all $X_{-2} \in \mathfrak{g}_{-2}$. For each $X_k \in \mathfrak{g}_k$ $(k \geqq 1)$, we define an element $S_k(X_k)$ of $GL(\mathfrak{m}_{k-1})$ by

$$S_k(X_k)Y = Y + [X_k, Y_{-1}] + [X_k, Y_{-2}]$$

for all $Y \in \mathfrak{m}_{k-1}$. Then we have

$$S_k(X_k)B_k(v) = B_k(v)S_k(X_k)$$

and we have: If $k=1$, then

$$S_1(X_1)S_1(Y_1) = S_1(X_1+Y_1)B_1(\gamma(X_1, Y_1)),$$

where $\gamma(X_1, Y_1)$ denote the element of $\mathcal{L}_0^{-2}$ defined by $\gamma(X_1, Y_1)X_{-2} = [X_1, [Y_1, X_{-2}]]$ for all $X_{-2} \in \mathfrak{g}_{-2}$, and if $k>1$, then

$$S_k(X_k)S_k(Y_k) = S_k(X_k+Y_k).$$

With these preparations, let us define connected Lie subgroups $G_k^\#$ $(k \geqq 0)$ as follows: $G_0^\#$ is defined to be the product $G_0 \cdot N_0$, and $G_k^\#$ $(k \geqq 1)$ to be the subgroup of $GL(\mathfrak{m}_{k-1})$ consisting of all the elements $S_k(X_k)B_k(v)$ $(X_k \in \mathfrak{g}_k, v \in \mathcal{L}_{k-1}^{-2})$.

For each $X_k \in \mathfrak{g}_k$ $(k \geqq 1)$, we define an element $s_k(X_k)$ of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ by

$$s_k(X_k)Y = [X_k, Y_{-1}] + [X_k, Y_{-2}]$$

for all $Y \in \mathfrak{m}_{k-1}$. We have: If $k=1$, then

$$\exp s_1(X_1) = S_1(X_1)B_1\Big(\frac{1}{2}\gamma(X_1, X_1)\Big)$$

and if $k>1$, then

$$\exp s_k(X_k) = S_k(X_k).$$

We denote by $\mathfrak{g}_k^\#$ the Lie algebra of $G_k^\#$. We have $\mathfrak{g}_0^\# = \mathfrak{g}_0 + \mathfrak{n}_0$ and we know that $\mathfrak{g}_k^\#$ $(k \geqq 1)$ is given by the subalgebra of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ consisting of all the elements $s_k(X_k) + b_k(v)$ $(X_k \in \mathfrak{g}_k, v \in \mathcal{L}_{k-1}^{-2})$.

The groups G_k $(k \geqq 0)$. N_k is a closed normal subgroup of $G_k^\#$. The factor group $G_0^\#/N_0$ may be identified with the subgroup G_0 of $G_0^\#$. For each $k \geqq 1$, we denote by G_k the factor group $G_k^\#/N_k$. Every element $\bar{\sigma}$ of G_k $(k \geqq 1)$ is represented by a unique element of $G_k^\#$ of the form $S_k(X_k)$. This $\bar{\sigma}$ will be denoted by $\bar{S}_k(X_k)$. We have $\bar{S}_k(X_k)\bar{S}_k(Y_k) = \bar{S}_k(X_k+Y_k)$. Hence G_k $(k \geqq 1)$ is an abelian group. If $k>1$, we may identify G_k with a closed subgroup of $G_k^\#$ in such a way that we have $\bar{S}_k(X_k) = S_k(X_k)$. We denote by $\bar{\mathfrak{g}}_k$ the Lie algebra of G_k, i.e., $\bar{\mathfrak{g}}_k = \mathfrak{g}_k^\#/\mathfrak{n}_k$. We have $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$. Every element $\bar{X}$ of $\bar{\mathfrak{g}}_k$ $(k \geqq 1)$ is represented by a unique element of $\mathfrak{g}_k^\#$ of the form $s_k(X_k)$. This $\bar{X}$ is denoted by $\bar{s}_k(X_k)$.

§ 4. Prolongation of pseudo-G_0-structures.

In this section, we shall use the same notations as the previous sections. The group $G_k^\#$ being a Lie subgroup of $GL(\mathfrak{m}_{k-1})$, we have the notion of a $G_0^\#$-structure $(P_k^\#, \omega^{(k)})$ on a manifold M_{k-1} of dimension m_{k-1}. By the definition, $\omega^{(k)}$ is an $\mathfrak{m}_{k-1}$-valued 1-form on $P_k^\#$. Given a $\mathfrak{g}$-valued form α on a manifold P, α_p will denote the $\mathfrak{g}_p$-component of α.

DEFINITION 4.1. Let $(P_k^\#, \omega^{(k)})$ be a $G_k^\#$-structure on a manifold M_{k-1} of dimension m_{k-1} $(k \geqq 0)$. We say that the $G_k^\#$-structure $(P_k^\#, \omega^{(k)})$ satisfies condition $C_k^\#$ if we have, for $k=0$,

$$d\omega_{-2}^{(0)} + \frac{1}{2}[\omega_{-1}^{(0)}, \omega_{-1}^{(0)}] \equiv 0 \qquad (\mathrm{mod}\ \omega_{-2}^{(0)})$$

and if we have, for $k>0$,

$$d\omega_{-2}^{(k)} + \frac{1}{2}[\omega_{-1}^{(k)}, \omega_{-1}^{(k)}] \equiv 0 \qquad (\mathrm{mod}\ \omega_{-2}^{(k)}),$$

$$d\omega_{k-2}^{(k)} + [\omega_{k-1}^{(k)}, \omega_{-1}^{(k)}] \equiv 0 \qquad (\mathrm{mod}\ \omega_{-2}^{(k)}, (\sum_{p=-1}^{k-2} \omega_p^{(k)})^2).$$

We now give the following

DEFINITION 4.2. (1) Let M_{k-1} be a maniold of dimension m_{k-1} $(k \geqq 0)$ and let P_k be a principal fiber bundle over the base space M_{k-1} with structure group G_k. Let $\theta^{(k)} = (\theta_p^{(k)})_{-2 \leqq p \leqq k-1}$ be a collection which satisfies the following conditions:

1) For each p $(-2 \leqq p < k-1)$, $\theta_p^{(k)}$ is a $\mathfrak{g}_p$-valued 1-form on P_k;

2) $\theta_{-2}^{(k)}$ is $\mathfrak{g}_{-2}$-independent, i.e., the mapping $T_y(P_k) \ni Y \rightarrow \theta_{-2}^{(k)}(Y) \in \mathfrak{g}_{-2}$ is surjective at each $y \in P_k$.

3) Denote by D_k the differential system on P_k (of codimension n') defined by the equation $\theta_{-2}^{(k)} = 0$. Then $\theta_{k-1}^{(k)}$ is a $\mathfrak{g}_{k-1}$-valued 1-form on (P_k, D_k).

4) Let Y be a tangent vector to P_k. $\theta_p^{(k)}(Y) = 0$ for all p $(-2 \leqq p \leqq k-1)$ if and only if Y is a vertical vector in the principal fiber bundle $P_k(M_{k-1}, G_k)$.

5) For each $\sigma \in G_k$, denote by $R(\sigma)$ the right translation of P_k induced by σ. If $k=0$, then we have

$$R(\sigma)^* \theta_{-2}^{(0)} = \sigma^{-1} \theta_{-2}^{(0)},$$

$$R(\sigma)^* \theta_{-1}^{(0)} = \sigma^{-1} \theta_{-1}^{(0)}$$

for all $\sigma \in G_0$, and if $k>0$, then we have

$$R(\sigma)^* \theta_p^{(k)} = \theta_p^{(k)} \qquad (-2 \leqq p < k-2),$$

$$R(\sigma)^* \theta_{k-2}^{(k)} = \theta_{k-2}^{(k)} - [X_k, \theta_{-2}^{(k)}],$$

$$R(\sigma)^* \theta_{k-1}^{(k)} = \theta_{k-1}^{(k)} - [X_k, \theta_{-1}^{(k)} | D_k]$$

for all $\sigma=\bar{S}_k(X_k)\in G_k(X_k\in\mathfrak{g}_k)$.

Under the above conditions, we say that the pair $(P_k, \theta^{(k)})$ is a pseudo-G_k-structure on M_{k-1}.

(2) Let $(P_k, \theta^{(k)})$ (resp. $(P'_k, \theta'^{(k)})$) be a pseudo-G_k-structure on a manifold M_{k-1} (resp. M'_{k-1}). A bundle isomorphism φ of P_k onto P'_k is called an isomorphism of $(P_k, \theta^{(k)})$ onto $(P'_k, \theta'^{(k)})$ if it satisfies the following conditions:

$$\varphi^*\theta'^{(k)}_p=\theta^{(k)}_p \qquad (-2\leqq p\leqq k-1).$$

It is clear that the 1-form $\sum_{p=-1}^{k-2}\theta^{(k)}_p|D_k+\theta^{(k)}_{k-1}$ on (P_k, D_k) is $\sum_{p=-1}^{k-1}\mathfrak{g}_p$-independent.

DEFINITION 4.3. Let $(P_k, \theta^{(k)})$ be a pseudo-G_k-structure on a manifold M_{k-1} of dimension m_{k-1} $(k\geqq 0)$. We say that the pseudo-G_k-structure $(P_k, \theta^{(k)})$ satisfies condition C_k if we have, for $k=0$,

$$d\theta^{(0)}_{-2}|D_0+\frac{1}{2}[\theta^{(0)}_{-1}, \theta^{(0)}_{-1}]=0$$

and if we have, for $k>0$,

$$d\theta^{(k)}_{-2}+\frac{1}{2}[\theta^{(k)}_{-1}, \theta^{(k)}_{-1}]\equiv 0 \qquad (\mathrm{mod}\ \theta^{(k)}_{-2}),$$

$$d\theta^{(k)}_{k-2}|D_k+[\theta^{(k)}_{k-1}, \theta^{(k)}_{-1}|D_k]\equiv 0 \qquad (\mathrm{mod}\ (\sum_{p=-1}^{k-2}\theta^{(k)}_p|D_k)^2).$$

The main purpose of this paper is to prove the following two theorems.

THEOREM 4.1. (1) *To every $G^\#_k$-structure $(P^\#_k, \omega^{(k)})$ on a manifold M_{k-1} $(k\geqq 0)$, there is associated, in a natural way, a pseudo-G_k-structure $(P_k, \theta^{(k)})$ on M_{k-1} having the following properties:*

1) *The principal fiber bundle $P_k(M_{k-1}, G_k)$ is just equal to the principal fiber bundle $P^\#_k/N_k(M_{k-1}, G_k)$, the quotient of $P^\#_k$ by the normal subgroup N_k of $G^\#_k$.*

2) *Denote by β_k the projection of $P^\#_k$ onto $P_k=P^\#_k/N_k$. Then we have*

$$\beta^*_k\theta^{(k)}_p=\omega^{(k)}_p \qquad (-2\leqq p<k-1),$$

$$\beta^*_k\theta^{(k)}_{k-1}=\omega^{(k)}_{k-1}|D^\#_k,$$

where $D^\#_k$ denotes the differential system (of codimension n') on $P^\#_k$ defined by the equation $\omega^{(k)}_{-2}=0$.

Moreover, if $(P^\#_k, \omega^{(k)})$ satisfies condition $C^\#_k$, then $(P_k, \theta^{(k)})$ satisfies condition C_k.

(2) *The assignment $(P^\#_k, \omega^{(k)})\to(P_k, \theta^{(k)})$ is compatible with the respective isomorphisms: Let $(P^\#_k, \omega^{(k)})$ (resp. $(P'^\#_k, \omega'^{(k)})$) be a $G^\#_k$-structure on a manifold M_{k-1} (resp. M'_{k-1}) and let $(P_k, \theta^{(k)})$ (resp. $(P'_k, \theta'^{(k)})$) be the corresponding pseudo-G_k-structure on M_{k-1} (resp. M'_{k-1}).*

i) *Every isomorphism* $\varphi_k^\#$ *of* $(P_k^\#, \omega^{(k)})$ *onto* $(P_k'^\#, \omega'^{(k)})$ *induces a unique isomorphism* φ_k *of* $(P_k, \theta^{(k)})$ *onto* $(P_k', \theta'^{(k)})$.

ii) *If* φ_k *is an isomorphism of* $(P_k, \theta^{(k)})$ *onto* $(P_k', \theta'^{(k)})$, *then there corresponds to* φ_k *a unique isomorphism* $\varphi_k^\#$ *of* $(P_k^\#, \omega^{(k)})$ *onto* $(P_k'^\#, \omega'^{(k)})$ *which induces the given* φ_k.

THEOREM 4.2. (1) *To every pseudo-*G_{k-1}*-structure* $(P_{k-1}, \theta^{(k-1)})$ *on a manifold* M_{k-2} $(k \geqq 1)$ *satisfying condition* C_{k-1}, *there is associated, in a canonical manner, a* $G_k^\#$*-structure* $(P_k^\#, \omega^{(k)})$ *on* P_{k-1} *satisfying condition* $C_k^\#$ *which has the following properties: Denote by* α_k *the projection of* $P_k^\#$ *onto* P_{k-1}. *Then we have*

$$\alpha_k^* \theta_p^{(k-1)} = \omega_p^{(k)} \qquad (-2 \leqq p < k-2),$$

$$\alpha_k^* \theta_{k-2}^{(k-1)} = \omega_{k-2}^{(k)} | D_k^\# ,$$

$D_k^\#$ *being just as in Th.* 4.1.

(2) *The assignment* $(P_{k-1}, \theta^{(k-1)}) \rightarrow (P_k^\#, \omega^{(k)})$ *is compatible with the respective isomorphisms: Let* $(P_{k-1}, \theta^{(k-1)})$ *(resp.* $(P_{k-1}', \theta'^{(k-1)})$*) be a pseudo-*$G_{k-1}$*-structure on a manifold* M_{k-2} *(resp.* M_{k-2}'*) satisfying condition* C_{k-1}, *and let* $(P_k^\#, \omega^{(k)})$ *(resp.* $(P_k'^\#, \omega'^{(k)})$*) be the corresponding* $G_k^\#$*-structure on* P_{k-1} *(resp.* P_{k-1}'*).*

i) *If* φ_{k-1} *is an isomorphism of* $(P_{k-1}, \theta^{(k-1)})$ *onto* $(P_{k-1}', \theta'^{(k-1)})$, *there corresponds to* φ_{k-1} *a unique isomorphism* $\varphi_k^\#$ *of* $(P_k^\#, \omega^{(k)})$ *onto* $(P_k'^\#, \omega'^{(k)})$ *which induces the given* φ_{k-1}.

ii) *Every isomorphism* $\varphi_k^\#$ *of* $(P_k^\#, \omega^{(k)})$ *onto* $(P_k'^\#, \omega'^{(k)})$ *induces a unique isomorphism* φ_{k-1} *of* $(P_{k-1}, \theta^{(k-1)})$ *onto* $(P_{k-1}', \theta'^{(k-1)})$.

COROLLARY 1. (1) *To every pseudo-*G_0*-structure* $(P_0, \theta^{(0)})$ *on a manifold* M_{-1} *satisfying condition* C_0, *there is associated, in a canonical manner, a sequence of pseudo-*G_k*-structures*

$$(P) \quad \cdots \rightarrow (P_k, \theta^{(k)}) \underset{\varpi_k}{\rightarrow} (P_{k-1}, \theta^{(k-1)}) \rightarrow \cdots \rightarrow (P_1, \theta^{(1)}) \underset{\varpi_1}{\rightarrow} (P_0, \theta^{(0)})$$

as follows:

1) *For each* $k \geqq 1$, $(P_k, \theta^{(k)})$ *is a pseudo-*G_k*-structure on* P_{k-1} *and* ϖ_k *is the projection of* P_k *onto* P_{k-1}.

2) *For each* $k \geqq 1$, *we have*

$$\theta_p^{(k)} = \varpi_k^* \theta_p^{(k-1)} \qquad (-2 \leqq p < k-2),$$

$$\theta_{k-2}^{(k)} | D_k = \varpi_k^* \theta_{k-2}^{(k-1)} .$$

(2) *The assignment* $(P_0, \theta^{(0)}) \rightarrow (P)$ *is compatible with the various isomorphisms.*

PROOF. This follows immediately from Ths. 4.1 and 4.2. Indeed, let $(P_{k-1}, \theta^{(k-1)})$ be a pseudo-G_{k-1}-structure on a manifold M_{k-2} satisfying condition C_{k-1} $(k \geqq 1)$. By Th. 4.2, there is attached to $(P_{k-1}, \theta^{(k-1)})$ a $G_k^\#$-structure

$(P_k^\#, \omega^{(k)})$ on P_{k-1} satisfying condition $C_k^\#$. By Th. 4.1, $(P_k^\#, \omega^{(k)})$ gives rise to a pseudo-G_k-structure $(P_k, \theta^{(k)})$ on P_{k-1} satisfying condition C_k. Let α_k (resp. β_k, resp. ϖ_k) denote the projection of $P_k^\#$ (resp. $P_k^\#$, resp. P_k) onte P_{k-1} (resp. P_k, resp. P_{k-1}). We have $\varpi_k \circ \beta_k = \alpha_k$. Therefore we have $\beta_k^* \varpi_k^* \theta_p^{(k-1)} = \alpha_k^* \theta_p^{(k-1)} = \omega_p^{(k)} = \beta_k^* \theta_p^{(k)}$ for any p $(-2 \leqq p < k-2)$, whence $\varpi_k^* \theta_p^{(k-1)} = \theta_p^{(k)}$. In particular, it follows that $\varpi_k^* D_{k-1} = D_k$. Moreover we have $D_k^\# = \alpha_k^* D_{k-1} = \beta_k^* D_k$. Hence we get $\beta_k^* \varpi_k^* \theta_{k-2}^{(k-1)} = \alpha_k^* \theta_{k-2}^{(k-1)} = \omega_{k-2}^{(k)} | D_k^\# = \beta_k^* (\theta_{k-2}^{(k)} | D_k)$, whence $\varpi_k^* \theta_{k-2}^{(k-1)} = \theta_{k-2}^{(k)} | D_k$. The iterative applications of the assignment $(P_{k-1}, \theta^{(k-1)}) \to (P_k, \theta^{(k)})$ yield the desired sequence (P).

The sequence (P) is called the prolongation of the pseudo-G_0-structure $(P_0, \theta^{(0)})$.

COROLLARY 2. *Let $(P_0^\#, \omega^{(0)})$ be a $G_0^\#$-structure on a connected manifold M_{-1} satisfying condition $C_0^\#$. If $\dim \mathfrak{g} < \infty$, then the group $\Phi_0^\#$ of all the automorphisms of $(P_0^\#, \omega^{(0)})$ becomes a Lie group of dimension $\leqq \dim \mathfrak{g}$ with respect to the natural topology (in such a way that $\Phi_0^\#$ is a Lie transformation group on M_{-1}).*

PROOF. By Th. 4.1 applied for $k=0$, there is attached to $(P_0^\#, \omega^{(0)})$ a pseudo-G_0-structure $(P_0, \theta^{(0)})$ on M_{-1} satisfying condition C_0. Let us consider the sequence (P) in Cor. 1 which is attached to $(P_0, \theta^{(0)})$. Denote by Φ_k the group of all the automorphisms of $(P_k, \theta^{(k)})$. By Cor. 1, the group Φ_k is naturally isomorphic with the group Φ_{k-1} for any $k \geqq 1$. Let l be the smallest k with $k \geqq 0$ and $\mathfrak{g}_k = \{0\}$. Then we have $G_l = \{e\}$ and $G_{l+1} = \{e\}$. Hence we have $P_{l+1} = P_l = P_{l-1}$, where we put $P_{-1} = M_{-1}$. We have $\theta_l^{(l+1)} = 0$ and we find that the linear mapping $T_z(P_{l+1}) \ni Z \to \xi(Z) = \sum_{p=-2}^{l-1} \theta_p^{(l+1)}(Z) \in \mathfrak{m}_{l-1} = \mathfrak{g}$ gives an isomorphism at each $z \in P_{l+1}$. Furthermore we find that the group $\Phi = \Phi_{l+1}$ is composed of all the transformations φ of P_{l+1} which leave the 1-form ξ invariant. Therefore a theorem of Kobayashi [3] shows that Φ becomes a Lie group of dimension $\leqq \dim \mathfrak{g}$ in such a way that it is a Lie transformation group on P_{l+1}. By Cor. 1, it follows that Φ may be considered as a Lie transformation group on M_{-1}. By Th. 4.1 applied for $k=0$, we have a natural isomorphism of Φ_0 onto $\Phi_0^\#$, and we know from the above remark that Φ is naturally isomorphic with the group Φ_0. Consequently, we have shown that $\Phi_0^\#$ becomes a Lie group of dimension $\leqq \dim \mathfrak{g}$ so that it is a Lie transformation group on M_{-1}.

Cor. 2 remains true in its local form. Namely we have the following

COROLLARY 3. *We use the same notation as in Cor. 2. If $\dim \mathfrak{g} < \infty$, then the Lie algebra of all the infinitesimal automorphisms of $(P_0^\#, \omega^{(0)})$ is finite dimensional and of dimension $\leqq \dim \mathfrak{g}$.*

The proof of Cor. 3 is just analogous to that of Cor. 2, but it is base on the local property[1] of the assignment $(P_0^\#, \omega^{(0)}) \to (P)$.

§ 5. Proof of Theorem 4.1.

Let $(P_k^\#, \omega^{(k)})$ be a $G_k^\#$-structure on a manifold $M_{k-1}(k \geqq 0)$. We denote by α_k the projection of $P_k^\#$ onto M_{k-1} and by $D_k^\#$ the differential system on $P_k^\#$ (of codimension n') defined by the equation $\omega_{-2}^{(k)}=0$. We set $P_k = P_k^\#/N_k$. Then $P_k^\#$ is a principal fiber bundle over the base space P_k with structure group N_k. We denote by β_k the projection of $P_k^\#$ onto P_k. Moreover, P_k is a principal fiber bundle over the base space M_{k-1} with structure group $G_k = G_k^\#/N_k$. We denote by ϖ_k the projection of P_k onto M_{k-1}. We have $\alpha_k = \varpi_k \circ \beta_k$. For each $\sigma \in G_k^\#$ (resp. $\sigma \in G_k$), $R^\#(\sigma)$ (resp. $R(\sigma)$) will denote the right translation of $P_k^\#(M_{k-1}, G_k^\#)$ (resp. $P_k(M_{k-1}, G_k)$) induced by σ. Since $(P_k^\#, \omega^{(k)})$ is a $G_k^\#$-structure on M_{k-1}, we have

1) $R(\sigma)^*\omega^{(k)} = \sigma^{-1}\omega^{(k)}$ for all $\sigma \in G_k^\#$.

2) Let Z be a tangent vector to $P_k^\#$. Then $\omega^{(k)}(Z)=0$ if and only if Z is a vertical vector in $P_k^\#(M_{k-1}, G_k^\#)$.

LEMMA 5.1. (1) *For each $p(-2 \leqq p < k-1)$, there is a unique $\mathfrak{g}_p$-valued 1-form $\theta_p^{(k)}$ such that*

$$\beta_k^*\theta_p^{(k)} = \omega_p^{(k)} .$$

(2) *Denote by D_k the differential system on P_k (of codimension n') defined by the equation $\theta_{-2}^{(k)}=0$. Then there is a unique $\mathfrak{g}_{k-1}$-valued 1-form $\theta_{k-1}^{(k)}$ on (P_k, D_k) such that*

$$\beta_k^*\theta_{k-1}^{(k)} = \omega_{k-1}^{(k)} | D_k^\# .$$

PROOF. (1) For any $\sigma \in N_k$, we have $R^\#(\sigma)^*\omega_p^{(k)} = \omega_p^{(k)}$ $(-2 \leqq p < k-1)$. Since a vertical vertor in $P_k^\#(P_k, N_k)$ is also a vertical vector in $P_k^\#(M_{k-1}, G^\#)$, we have $\omega_p^{(k)}(Z)=0$ $(-2 \leqq p < k-1)$ for any vertical vector Z in $P_k^\#(P_k, N_k)$. Therefore, for each $p(-2 \leqq p < k-1)$, there is a unique $\mathfrak{g}_p$-valued 1-form $\theta_p^{(k)}$ on P_k such that $\beta_k^*\theta_p^{(k)} = \omega_p^{(k)}$.

(2) For any $\sigma \in N_k$, we have $R^\#(\sigma)^*\omega_{k-1}^{(k)} \equiv \omega_{k-1}^{(k)}$ $(\text{mod}\, \omega_{-2}^{(k)})$. We have $R^\#(\sigma)^*D_k^\# = D_k^\#$. It follows that $R^\#(\sigma)^*(\omega_{k-1}^{(k)}|D_k^\#) = \omega_{k-1}^{(k)}|D_k^\#$. The vertical vectors in $P_k^\#(P_k, N_k)$ are contained in $D_k^\#$ and we have $\omega_{k-1}^{(k)}(Z)=0$ for any vertical vector Z in $P_k^\#(P_k, N_k)$. We have $D_k^\# = \beta_k^* D_k$. Therefore there is a unique $\mathfrak{g}_{k-1}$-valued 1-form $\theta_{k-1}^{(k)}$ on (P_k, D_k) such that $\beta_k^*\theta_{k-1}^{(k)} = \omega_{k-1}^{(k)}|D_k^\#$.

$\theta_p^{(k)}$ being as in Lemma 5.1, we set $\theta^{(k)} = (\theta_p^{(k)})_{-2 \leqq p \leqq k-1}$. We shall show that the pair $(P_k, \theta^{(k)})$ is a pseuco-G_k-structure on M_{k-1}. Since $\omega_{-2}^{(k)} = \beta_k^*\theta_{-2}^{(k)}$ is $\mathfrak{g}_{-2}$-independent, so is $\theta_{-2}^{(k)}$. Now, let Y be a tangent vector to P_k. We take a

1) From the proofs of Ths. 4.1 and 4.2 given §§ 5 and 6, we shall see that the assignment $(P^\#_0, \omega^{(0)}) \to (P)$ has a local property necessary for Cor. 3 to be valid.

fixed tangent vector Z to $P_k^\#$ such that $\beta_{k*}Z=Y$. Since $\alpha_k=\varpi_k\circ\beta_k$, we see that Y is vertical in $P_k(M_{k-1}, G_k)$ if and only if Z is vertical in $P_k^\#(M_{k-1}, G_k^\#)$. It follows that Y is vertical in $P_k(M_{k-1}, G_k)$ if and only if we have $\theta_p^{(k)}(Y)=0$ $(-2\leqq p\leqq k-1)$. We have thereby proved $(P_k, \theta^{(k)})$ satisfies conditions 2) and 4) in Def. 4.2. It remains to prove that it satisfies also condition 5) in Def. 4.2.

The case $k=0$. Let σ be any element of G_0. We have $R(\sigma)\circ\beta_0=\beta_0\circ R_k^\#(\sigma)$. Therefore we have

$$\beta_0^*(R(\sigma)^*\theta_{-2}^{(0)})=R^\#(\sigma)^*\beta_0^*\theta_{-2}^{(0)}=R^\#(\sigma)^*\omega_{-2}^{(0)}=\sigma^{-1}\omega_{-2}^{(0)}=\beta_0^*(\sigma^{-1}\theta^{(0)}),$$

whence $R(\sigma)^*\theta_{-2}^{(0)}=\sigma^{-1}\theta_{-2}^{(0)}$. We have $R(\sigma)^*D_0=D_0$, $R^\#(\sigma)^*D_0^\#=D_0^\#$ and

$$\begin{aligned}\beta_0^*(R(\sigma)^*\theta_{-1}^{(0)})&=R^\#(\sigma)\beta_0^*\theta_{-1}^{(0)}=R^\#(\sigma)^*(\omega_{-1}^{(0)}\,|\,D_0^\#)\\&=\sigma^{-1}\omega_{-1}^{(0)}\,|\,D_0^\#=\beta_0^*(\sigma^{-1}\theta_{-1}^{(0)}),\end{aligned}$$

whence $R(\sigma)^*\theta_{-1}^{(0)}=\sigma^{-1}\theta_{-1}^{(0)}$.

The case $k>0$. Let $\bar\sigma$ be any element of G_k and express it as $\bar S_k(X_k)(X_k\in\mathfrak{g}_k)$. We set $\sigma=S_k(X_k)$. We have $R(\bar\sigma)\circ\beta_k=\beta_k\circ R^\#(\sigma)$. Therefore we have, for each $p(-2\leqq p<k-2)$,

$$\beta_k^*(R(\bar\sigma)^*\theta_p^{(k)})=R^\#(\sigma)^*\beta_k^*\theta_p^{(k)}=R^\#(\sigma)^*\omega_p^{(k)}=\omega_p^{(k)}=\beta_k^*\theta_p^{(k)},$$

whence $R(\bar\sigma)^*\theta_p^{(k)}=\theta_p^{(k)}$. We have

$$\begin{aligned}\beta_k^*(R(\bar\sigma)^*\theta_{k-2}^{(k)})&=R^\#(\sigma)^*\beta_k^*\theta_{k-2}^{(k)}=R^\#(\sigma)^*\omega_{k-2}^{(k)}\\&=\omega_{k-2}^{(k)}-[X_k, \omega_{-2}^{(k)}]=\beta_k^*(\theta_{k-2}^{(k)}-[X_k, \theta_{-2}^{(k)}]),\end{aligned}$$

whence $R(\bar\sigma)^*\theta_{k-2}^{(k)}=\theta_{k-2}^{(k)}-[X_k, \theta_{-2}^{(k)}]$. We have $R(\bar\sigma)^*D_k=D_k$, $R^\#(\sigma)^*D_k^\#=D_k^\#$ and

$$\begin{aligned}\beta_k^*(R(\bar\sigma)^*\theta_{k-1}^{(k)})&=R^\#(\sigma)^*\beta_k^*\theta_{k-1}^{(k)}=R^\#(\sigma)^*(\omega_{k-1}^{(k)}\,|\,D_k^\#)\\&=\omega_{k-1}^{(k)}\,|\,D_k^\#-[X_k, \omega_{-1}^{(k)}\,|\,D_k^\#]=\beta_k^*[\theta_{k-1}^{(k)}-[X_k, \theta_{-1}^{(k)}\,|\,D_k]),\end{aligned}$$

whence $R(\bar\sigma)^*\theta_{k-1}^{(k)}=\theta_{k-1}^{(k)}-[X_k, \theta_{-1}^{(k)}\,|\,D_k]$. We have thus shown that $(P_k, \theta^{(k)})$ becomes a pseudo-G_k-structure on M_{k-1}.

The notation being as above, we have easily: If $(P_k^\#, \omega^{(k)})$ satisfies condition $C_k^\#$, then $(P_k, \theta^{(k)})$ satisfies condition C_k.

Let us now show that the assignment $(P_k^\#, \omega^{(k)})\to(P_k, \theta^{(k)})$, thus obtained, is compatible with the respective isomorphisms. Let $(P_k^\#, \omega^{(k)})$ (resp. $(P_k'^\#, \omega'^{(k)})$) be a $G_k^\#$-structure on a manifold M_{k-1} (resp. M'_{k-1}), and let $(P_k, \theta^{(k)})$ (resp. $(P'_k, \theta'^{(k)})$) be the corresponding G_k-structure on M_{k-1} (resp. M'_{k-1}). We shall write as A' the quantity in $(P_k'^\#, \omega'^{(k)})$ or $(P'_k, \theta'^{(k)})$ which corresponds to a quantity A in $(P^\#, \omega^{(k)})$ or $(P_k, \theta^{(k)})$.

Let $\varphi^\#$ be an isomorphism of $(P_k^\#, \omega^{(k)})$ onto $(P_k'^\#, \omega'^{(k)})$. Since we have $\varphi^\#(z\sigma)=\varphi^\#(z)\sigma$ for all $z\in P_k^\#$ and $\sigma\in G_k^\#$, $\varphi^\#$ induces a bundle isomorphism φ of $P_k(M_{k-1}, G_k)$ onto $P'_k(M'_{k-1}, G_k)$. We have $\varphi\circ\beta_k=\beta'_k\circ\varphi^\#$. Since we have

$\varphi^{\#}\omega'^{(k)}=\omega^{(k)}$, we have $\varphi^{*}\theta'^{(k)}_{p}=\theta^{(k)}_{p}$ $(-2\leqq p\leqq k-1)$ just in the same way as above. Therefore, φ gives an isomorphism of $(P_k, \theta^{(k)})$ onto $(P'_k, \theta'^{(k)})$.

Let φ be an isomorphism of $(P_k, \theta^{(k)})$ onto $(P'_k, \theta'^{(k)})$. Let ψ be a bundle isomorphism of $P^{\#}_k$ $(M_{k-1}, G^{\#}_k)$ onto $P'^{\#}_k(M'_{k-1}, G^{\#}_k)$ such that $\beta'_k\circ\psi=\varphi\circ\beta_k$. Then we have easily the following two equalities:

$$\psi^{*}\omega'^{(k)}_{p}=\omega^{(k)}_{p} \qquad (-2\leqq p< k-1), \tag{5.1}$$

$$\psi^{*}(\omega'^{(k)}_{k-1}\,|\,D'^{\#}_{k})=\omega^{(k)}_{k-1}\,|\,D^{\#}_{k}\,.$$

Now, let ψ' be a second bundle isomorphism of $P^{\#}_k(M_{k-1}, G^{\#}_k)$ onto $P'^{\#}_k(M'_{k-1}, G^{\#}_k)$ such that $\beta'_k\circ\psi'=\varphi\circ\beta_k$. Then we can find a unique mapping K of $P^{\#}_k$ to N_k such that

$$\psi'(z)=\psi(z)\cdot K(z)$$

for all $z\in P^{\#}_k$. Let Z be a tangent vector to $P^{\#}_k$ at $z\in P^{\#}_k$. Then we have

$$\psi'_{*}Z=R^{\#}(K(z))_{*}\psi_{*}Z+W\,,$$

where W is a suitable vertical vector in $P^{\#}_k(P_k, N_k)$. It follows that

$$(\psi'^{*}\omega'^{(k)})(Z)=K(z)^{-1}(\psi^{*}\omega'^{(k)})(Z)\,.$$

We now take a unique mapping $v(z\to v_z)$ of $P^{\#}_k$ to $\mathcal{L}^{-2}_{k-1}$ such that $K(z)=B_k(v_z)$ for all $z\in P^{\#}_k$. Since we have $\psi^{*}\omega'^{(k)}_{-2}=\omega^{(k)}_{-2}$, we get the equality:

$$\psi'^{*}\omega'^{(k)}=\psi^{*}\omega'^{(k)}-v(\omega_{-2})\,. \tag{5.2}$$

Assume now that both ψ and ψ' are isomorphisms of $(P^{\#}_k, \omega^{(k)})$ onto $(P'^{\#}_k, \omega'^{(k)})$. Since we have $\psi'^{*}\omega'^{(k)}=\psi^{*}\omega'^{(k)}=\omega^{(k)}$, it follows from (5.2) that $v(\omega_{-2})=0$. Therefore we get $v_z=0$, i.e., $K(z)=e$ for all $z\in P^{\#}_k$. Hence ψ' and ψ coincide.

φ being as above, we must finally show that there is an isomorphism $\varphi^{\#}$ of $(P^{\#}_k, \omega^{(k)})$ onto $(P'^{\#}_k, \omega'^{(k)})$ which induces the given φ. From the uniqueness of $\varphi^{\#}$ just proved, we may assume without loss of generality that both $P^{\#}_k(M_{k-1}, G^{\#}_k)$ and $P'^{\#}_k(M'_{k-1}, G^{\#}_k)$ are trivial. Then we can find at least one bundle isomorphism, say ψ, of $P^{\#}_k(M_{k-1}, G^{\#}_k)$ onto $P'^{\#}_k(M'_{k-1}, G^{\#}_k)$ such that $\varphi\circ\beta_k=\beta'_k\circ\psi$. By (5.1), we have $\psi^{*}\omega'^{(k)}_{p}=\omega^{(k)}_{p}$ $(-2\leqq p<k-1)$ and $\psi^{*}\omega'^{(k)}_{k-1}-\omega^{(k)}_{k-1}\equiv 0$ $(\mathrm{mod}\,\omega^{(k)}_{-2})$. Therefore there is a unique mapping $v(z\to v_z)$ of $P^{\#}_k$ to $\mathcal{L}^{-2}_{k-1}$ such that

$$\psi^{*}\omega'^{(k)}-\omega^{(k)}=v(\omega^{(k)}_{-2})\,. \tag{5.3}$$

LEMMA 5.2. *We set* $K(z)=B_k(v_z)$. *Then we have*

$$K(z\sigma)=\sigma^{-1}K(z)\sigma$$

for all $z\in P^{\#}_k$ *and* $\sigma\in G^{\#}_k$.

PROOF. Let σ be any element of $G^{\#}_k$. Then we have

$$R^{\#}(\sigma)(v(\omega_{-2}^{(k)}))=(R^{\#}(\sigma)^{*}v)(R^{\#}(\sigma)^{*}\omega_{-2}^{(k)}).$$

We have $R'^{\#}(\sigma)\circ\psi=\psi\circ R^{\#}(\sigma)$ and

$$\begin{aligned}R^{\#}(\sigma)^{*}(v(\omega_{-2}^{(k)}))&=R^{\#}(\sigma)^{*}\psi^{*}\omega'^{(k)}-R^{\#}(\sigma)^{*}\omega^{(k)}\\&=\psi^{*}R'^{\#}(\sigma)^{*}\omega'^{(k)}-\sigma^{-1}\omega^{(k)}\\&=\sigma^{-1}(\psi^{*}\omega'^{(k)}-\omega^{(k)})=\sigma^{-1}v(\omega_{-2}^{(k)}).\end{aligned}$$

Hence we have the equality:

$$(R^{*}(\sigma)^{*}v)(R^{\#}(\sigma)\omega_{-2}^{(k)})=\sigma^{-1}v(\omega_{-2}^{(k)}).$$

If $k=0$, we have $R^{\#}(\sigma)^{*}\omega_{-2}^{(0)}=(\sigma_{-2})^{-1}\omega_{-2}^{(0)}$, where, for each $\sigma\in G_0^{\#}$, σ_{-2} denotes the linear automorphism of $\mathfrak{g}_{-2}$ defined by $\sigma_{-2}X_{-2}\equiv\sigma X_{-2}\ (\mathrm{mod}\ \mathfrak{g}_{-1})$ for all $X_{-2}\in\mathfrak{g}_{-2}$. If $k>0$, we have $R^{\#}(\sigma)^{*}\omega_{-2}^{(k)}=\omega_{-2}^{(k)}$ and $\sigma^{-1}v(\omega_{-2}^{(k)})=v(\omega_{-2}^{(k)})$. Therefore we have: If $k=0$, $v_{z\sigma}(X_{-2})=\sigma^{-1}v_z(\sigma_{-2}X_{-2})$ for all $X_{-2}\in\mathfrak{g}_{-2}$; if $k>0$, $v_{z\sigma}(X_{-2})=v_z(X_{-2})$ for all $X_{-2}\in\mathfrak{g}_{-2}$. From these equalities, we easily get the equality in Lemma 5.2.

We set $\psi'(z)=\psi(z)\cdot K(z)$ for all $z\in P_k^{\#}$. By Lemma 5.2, we see that the mapping $\psi' : P_k^{\#}\ni z\to\psi'(z)\in P_k'^{\#}$ gives a bundle isomorphism of $P_k^{\#}(M_{k-1}, G_k^{\#})$ onto $P_k'^{\#}(M'_{k-1}, G_k^{\#})$. Since $K(z)\in N_k$, we have $\beta'_k\circ\psi'=\varphi\circ\beta_k$. Therefore we get $\psi'^{*}\omega'^{(k)}=\psi^{*}\omega'^{(k)}-v(\omega_{-2}^{(k)})=\omega^{(k)}$ by (5.2) and (5.3). We have thus proved ψ' to be an isomorphism of $(P_k^{\#}, \omega^{(k)})$ onto $(P_k'^{\#}, \omega'^{(k)})$.

§ 6. Proof of Theorem 4.2.

Before proceeding to the proof, we shall explain the notations that will be needed hereafter.

We set $\mathfrak{d}_k=\sum_{p=-1}^{k}\mathfrak{g}_p$. We denote by $\mathcal{L}_k^{-1}$ the vector space of all the linear mapping u of $\mathfrak{d}_{k-1}$ to $\mathfrak{g}_k$, by $\mathcal{T}_k^{-1}$ the vector space of all the anti-symmetric bilinear mappings c of $\mathfrak{d}_k\times\mathfrak{d}_k$ to $\mathfrak{g}_k$, and by $\mathcal{T}_k^{-2}$ the vector space of all the bilinear mappings c' of $\mathfrak{g}_{-2}\times\mathfrak{d}_{k+1}$ to $\mathfrak{g}_k$. We set $\mathcal{L}_k=\mathcal{L}_k^{-1}\times\mathcal{L}_{k-1}^{-2}$ and $\mathcal{T}_k=\mathcal{T}_k^{-1}\times\mathcal{T}_{k-1}^{-2}$. We define a linear mapping ∂ of $\mathcal{L}_k$ to $\mathcal{T}_{k-1}$ $(k\geqq 0)$ as follows: For any $(u, u')\in\mathcal{L}_k$, $\partial(u, u')=(c, c')$ is defined to be

$$c(X, Y)=[u(X), Y_{-1}]-[u(Y), X_{-1}]-u'([X_{-1}, Y_{-1}]),$$

$$c'(Z_{-2}, Y)=[u'(Z_{-2}), Y_{-1}]-[u(Y), Z_{-2}]$$

for all $X, Y\in\mathfrak{d}_{k-1}$ and $Z_{-2}\in\mathfrak{g}_{-2}$.

We have easily

LEMMA 6.1. *Let* (u, u') *be an element of* $\mathcal{L}_{k-1}$ $(k\geqq 1)$. $\partial(u, u')=0$ *if and only if we have*

1) $u(Y)=0$ *for all* $Y\in\sum_{p=0}^{k-2}\mathfrak{g}_p$;

2) *there is a unique element* X_k *of* $\mathfrak{g}_k$ *such that*

$$u(Y_{-1})=[X_k, Y_{-1}],$$

$$u'(Y_{-2})=[X_k, Y_{-2}]$$

for all $Y_{-1}\in\mathfrak{g}_{-1}$ *and* $Y_{-2}\in\mathfrak{g}_{-2}$.

For each $(u, u')\in\mathcal{L}_{k-1}$ $(k\geqq 1)$, we define an element $A_k(u, u')$ of $GL(\mathfrak{m}_{k-1})$ by

$$A_k(u, u')X=X+u(\sum_{p=-1}^{k-2}X_p)+u'(X_{-2})$$

for all $X\in\mathfrak{m}_{k-1}$. We have

$$A_k(u, u')B_k(v)=B_k(v)A_k(u, u'),$$

$$A_k(u_1, u_1')A_k(u_2, u_2')=A_k(u_1+u_2, u_1'+u_2')B_k(u_1\circ u_2').$$

We denote by H_k $(k\geqq 1)$ the closed subgroup of $GL(\mathfrak{m}_{k-1})$ consisting of all the elements $A_k(u, u')B_k(v)((u, u')\in\mathcal{L}_{k-1}, v\in\mathcal{L}_{k-1}^{-2})$. Let (u, u') be an element of $\mathcal{L}_{k-1}$ with $\partial(u, u')=0$, and let X_k be the element of $\mathfrak{g}_k$ determined by (u, u') (see Lemma 6.1). Then we have clearly $A_k(u, u')=S_k(X_k)$.

Therefore we have

LEMMA 6.2. *The group* $G_k^{\#}$ $(k\geqq 1)$ *consists of all the elements* $A_k(u, u')B_k(v)$, *where* $(u, u')\in\mathcal{L}_{k-1}$, $v\in\mathcal{L}_{k-1}^{-2}$ *and* $\partial(u, u')=0$.

For each $(u, u')\in\mathcal{L}_{k-1}$, we define an element $a_k(u, u')$ of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ by

$$a_k(u, u')Y=u(\sum_{p=-1}^{k-2}Y_p)+u'(Y_{-2})$$

for all $Y\in\mathfrak{m}_{k-1}$. The Lie algebra of H_k is given by the subalgebra $\mathfrak{h}_k$ of $\mathfrak{gl}(\mathfrak{m}_{k-1})$ consisting of all the elements $a_k(u, u')+b_k(v)((u, u')\in\mathcal{L}_{k-1}, v\in\mathcal{L}_{k-1}^{-2})$.

From now on, we shall consider a fixed pseudo-G_{k-1}-structure $(P_{k-1}, \theta^{(k-1)})$ on a manifold M_{k-2} of dimension m_{k-2} $(k\geqq 1)$. We denote by D_{k-1} the differential system on P_{k-1} defined by the equation $\theta_{-2}^{(k-1)}=0$. For each $\sigma\in G_{k-1}$ (resp. $X\in\bar{\mathfrak{g}}_{k-1}$), $R(\sigma)$ (resp. $r(X)$) will denote the right translation of P_{k-1} (resp. the vertical vector field on P_{k-1}) induced by σ (resp. X). Let us define a linear mapping Φ of $\mathfrak{g}_{k-1}$ to the vector space $\mathfrak{X}(P_{k-1})$[2] of all the vector fields on P_{k-1} as follows: If $k=1$,

$$\Phi(X_0)=r(X_0)$$

for all $X_0\in\mathfrak{g}_0=\bar{\mathfrak{g}}_0$; if $k>1$,

2) In general, let M be a manifold. $\mathfrak{X}(M)$ will denote the vector space of all the vector fields on M.

$$\Phi(X_{k-1})=r(\bar{s}_{k-1}(X_{k-1}))$$

for all $X_{k-1}\in\mathfrak{g}_{k-1}$.

P_{k-1} is a manifold of dimension m_{k-1}. By using the vector space $\mathfrak{m}_{k-1}$, we define the frame bundle $F(P_{k-1}, GL(\mathfrak{m}_{k-1}))$ of P_{k-1} as in § 1. Let π denote the projection of F onto P_{k-1}. We denote by F_k the subset of F which is composed of all the elements z satisfying the following equalities:

$$\theta_q^{(k-1)}(zX_{-2})=\delta_{q,-2}X_{-2} \qquad (-2\leqq q\leqq k-3),$$
$$\theta_q^{(k-1)}(zX_p)=\delta_{q,p}X_p \qquad (-1\leqq p\leqq k-2,\ -2\leqq q\leqq k-2),$$
$$zX_{k-1}=\Phi(X_{k-1})_{\pi(z)}$$

for all $X_p\in\mathfrak{g}_p(-2\leqq p\leqq k-1)$, where $(\delta_{p,q})$ denotes the Kronecker's symbol.

LEMMA 6.3. (1) $\pi(F_k)=P_{k-1}$.

(2) *Let* $z\in F_k$ *and* $\sigma\in GL(\mathfrak{m}_{k-1})$. *Then* $z\sigma\in F_k$ *if and only if* $\sigma\in H_k$.

PROOF. (1) We only remark the following things: Let x be a point of P_{k-1}. 1) The linear mapping $T_x(P_{k-1})\ni X\to\theta_{-2}^{(k-1)}(X)\in\mathfrak{g}_{-2}$ is surjective and its kernel is $(D_{k-1})_x$. 2) The linear mapping $(D_{k-1})_x\ni X\to\sum_{p=-1}^{k-2}\theta_p^{(k-2)}(X)\in\mathfrak{d}_{k-2}$ is surjective and its kernel is the vector space V_x of all the vertical vectors at x in $P_{k-1}(M_{k-2}, G_{k-1})$. 3) The mapping $X_{k-1}\to\Phi(X_{k-1})_x$ gives an isomorphism of $\mathfrak{g}_{k-1}$ onto V_x.

(2) We have easily: $z\sigma\in F_k$ if and only if we have the following equalities:

$$(\sigma X_{-2})_r=\delta_{-2,r}X_{-2} \qquad (-2\leqq r\leqq k-3),$$
$$(\sigma X_p)_r=\delta_{p,r}X_p \qquad (-1\leqq p\leqq k-2,\ -2\leqq r\leqq k-2),$$
$$(\sigma X_{k-1})_r=\delta_{r,k-1}X_{k-1} \qquad (-2\leqq r\leqq k-1)$$

for all $X_p\in\mathfrak{g}_p$ $(-2\leqq p\leqq k-1)$. These equalities clearly mean that σ is of the form $A_k(u, u')B_k(v)$, i.e., $\sigma\in H_k$.

By Lemma 6.3, we see that F_k is a H_k-subbundle of the frame bundle F of P_{k-1}. The projection π_k of F_k onto P_{k-1} is given by the restriction of π to F_k. We denote by ω the basic form of F_k. For any $\sigma\in H_k$, $\bar{R}(\sigma)$ will denote the right translation of F_k induced by σ.

LEMMA 6.4. (1) $\omega_p=\pi_k^*\theta_p^{(k-1)}$ $(-2\leqq p<k-2)$.

(2) *Denote by* $\bar{D}_k$ *the differential system on* F_k (*of codimension* n') *defined by the equaiton* $\omega_{-2}=0$. *Then we have*

$$\omega_{k-2}\,|\,\bar{D}_k=\pi_k^*\theta_{k-2}^{(k-1)}.$$

PROOF. Let Z be any vector in $T_z(F_k)(z\in F_k)$. Then we have $\pi_{k*}Z=z\cdot\omega(Z)=\sum_{p=-2}^{k-1}z\cdot\omega_p(Z)$. Therefore, for each q $(-2\leqq q<k-2)$, we have

$$(\pi_k^* \theta_q^{(k-1)})(Z) = \sum_{p=-2}^{k-2} \theta_q^{(k-1)}(z \cdot \omega_p(Z)) = \sum_{p=-2}^{k-2} \delta_{q,p}\omega_p(Z) = \omega_q(Z) \, ,$$

whence $\pi_k^* \theta_q^{(k-1)} = \omega_q$. We have $\bar{D}_k = \pi_k^* D_{k-1}$ and, for all $Z \in (\bar{D}_k)_z$,

$$(\pi_k^* \theta_{k-2}^{(k-1)})(Z) = \sum_{p=-1}^{k-2} \theta_{k-2}^{(k-1)}(z \cdot \omega_p(Z)) = \sum_{p=-1}^{k-2} \delta_{k-2,p}\omega_p(Z) = \omega_{k-2}(Z) \, ,$$

whence $\pi_k^* \theta_{k-2}^{(k-1)} = \omega_{k-2} | \bar{D}_k$.

LEMMA 6.5.

$$d\omega_p \equiv 0 \ (\mathrm{mod}\ \omega^2) \qquad (-2 \leqq p < k-2) \, ,$$

$$d\omega_{k-2} \equiv 0 \ (\mathrm{mod}\ \omega_{-2}, \omega^2) \, .$$

PROOF. This follows immediately from the existence of a connection in $F_k(P_{k-1}, H_k)$.

Assume for a moment that $k=1$. For any $\sigma \in G_0$ and $X \in \mathfrak{m}_0$, we define an element σX of $\mathfrak{m}_0$ by

$$\sigma X = \sigma X_{-2} + \sigma X_{-1} + \sigma X_0 \sigma^{-1} \, .$$

The mapping $G_0 \times \mathfrak{m}_0 \ni (\sigma, X) \to \sigma X \in \mathfrak{m}_0$ clearly gives a representation of G_0 on $\mathfrak{m}_0$. For any $z \in F_1$ and $\sigma \in G_0$, we define an element $z\sigma$ of F by

$$(z\sigma)X = R(\sigma)_*(z \cdot (\sigma X))$$

for all $X \in \mathfrak{m}_0$.

LEMMA 6.6. *Let $z \in F_1$ and $\sigma, \tau \in G_0$.*

(1) $z\sigma \in F_1$.

(2) $(z\sigma)\tau = z(\sigma\tau)$.

(3) $\pi_1(z\sigma) = \pi_1(z)\sigma$.

(4) *Denote by $E(\sigma)$ the transformation $F_1 \ni z \to z\sigma \in F_1$. Then,*

$$E(\sigma)^*\omega = \sigma^{-1}\omega \, .$$

PROOF. (1) $\theta_{-2}^{(0)}((z\sigma)X_{-2}) = (R(\sigma)^*\theta_{-2}^{(0)})(z(\sigma X_{-2})) = \sigma^{-1}\theta_{-2}^{(0)}(z(\sigma X_{-2})) = \sigma^{-1}\sigma X_{-2} = X_{-2}$ for all $X_{-2} \in \mathfrak{g}_{-2}$. $\theta_{-2}^{(0)}((z\sigma)X_{-1}) = \sigma^{-1}\theta_{-2}^{(0)}(z(\sigma X_{-1})) = 0$ for all $X_{-1} \in \mathfrak{g}_{-1}$. $\theta_{-1}^{(0)}((z\sigma)X_{-1}) = \sigma^{-1}\theta_{-1}^{(0)}(z(\sigma X_{-1})) = \sigma^{-1}\sigma X_{-1} = X_{-1}$. $(z\sigma)X_0 = R(\sigma)_*(z(\sigma X_0 \sigma^{-1})) = R(\sigma)_*\Phi(\sigma X_0 \sigma^{-1})_{\pi_1(z)} = \Phi(\sigma^{-1}\sigma X_0 \sigma^{-1}\sigma)_{\pi_1(z)\sigma} = \Phi(X_0)_{\pi_1(z)\sigma}$ for all $X_0 \in \mathfrak{g}_0$. Hence we get $z\sigma \in F_1$.

(2) and (3) are clear.

(4) Let $Z \in T_z(F_1)$. $\pi_{1*}E(\sigma)_*Z = (z\sigma) \cdot (E(\sigma)^*\omega)(Z)$.

$$\pi_{1*}E(\sigma)_*Z = R(\sigma)_*\pi_{1*}Z = R(\sigma)_*(z \cdot \omega(Z)) = (z\sigma) \cdot \sigma^{-1}\omega(Z) \, .$$

Hence we get $E(\sigma)^*\omega = \sigma^{-1}\omega$.

LEMMA 6.7. *There is a linear mapping $\bar{\Phi}$ of $\mathfrak{g}_0$ into $\mathfrak{X}(F_1)$ having the following properties: Let $X_0 \in \mathfrak{g}_0$. Then,*

1) $\pi_{1*}\bar{\Phi}(X_0)_z = \Phi(X_0)_{\pi_1(z)}$ *at each* $z \in F_1$;

2) $\omega_p(\bar{\Phi}(X_0))=\delta_{p,0}X_0$ $(p=-2,-1,0)$;

3) $\mathcal{L}_{\bar{\Phi}(X_0)}$[3] $\omega_p+[X_0,\omega_p]=0$ $(p=-2,-1,0)$.

PROOF. For any $X_0\in\mathfrak{g}_0$, let $\bar{\Phi}(X_0)$ denote the vector field on F_1 induced by the one parameter group $E(\operatorname{expt}X_0)$. Then the mapping $\bar{\Phi}:\mathfrak{g}_0\ni X_0\to\bar{\Phi}(X_0)$ is linear by Lemma 6.6, (2). 1) and 3) immediately follow from Lemma 6.6, (3) and (4). We have $\pi_{1*}\bar{\Phi}(X_0)_z=z\cdot\omega(\bar{\Phi}(X_0)_z)$ at each $z\in F_1$ and $\pi_{1*}\bar{\Phi}(X_0)_z=\Phi(X_0)_{\pi_1(z)}=z\cdot X_0$, whence $\omega(\bar{\Phi}(X_0))=X_0$. Therefore we have 2).

Now, assume for a moment that $k>1$. For any $\sigma=\bar{S}_{k-1}(X_{k-1})\in G_{k-1}$ and $Y\in\mathfrak{m}_{k-1}$, we define an element σY of $\mathfrak{m}_{k-1}$ by

$$\begin{aligned}\sigma Y&=S_{k-1}(X_{k-1})\sum_{p=-2}^{k-2}Y_p+Y_{k-1}\\&=Y+[X_{k-1},Y_{-1}]+[X_{k-1},Y_{-2}]\,.\end{aligned}$$

Let $\sigma=\bar{S}_{k-1}(X_{k-1})$, $\tau=\bar{S}_{k-1}(Y_{k-1})\in G_{k-1}$. Then we have: If $k=2$,

$$\sigma(\tau Z)=(\sigma\tau)A_2(0,\gamma(X_1,Y_1))Z$$

for all $Z\in\mathfrak{m}_1$; if $k>2$,

$$\sigma(\tau Z)=(\sigma\tau)Z$$

for all $Z\in\mathfrak{m}_{k-1}$. For any $z\in F_k$ and $\sigma\in G_{k-1}$, we now define an element $z\sigma$ of F by

$$(z\sigma)Y=R(\sigma)_*(z(\sigma Y))$$

for all $Y\in\mathfrak{m}_{k-1}$.

LEMMA 6.8. *Let $z\in F_k$ and $\sigma=\bar{S}_{k-1}(X_{k-1})$, $\tau=\bar{S}_{k-1}(Y_{k-1})\in G_{k-1}$.*

(1) $z\sigma\in F_k$.

(2) *If* $k=2$, $(z\sigma)\tau=(z\cdot(\sigma\tau))\cdot A_2(0,\gamma(X_1,Y_1))$; *if* $k>2$, $(z\sigma)\tau=z(\sigma\tau)$.

(3) $\pi_k(z\sigma)=\pi_k(z)\sigma$.

(4) *Denote by $E(\sigma)$ the transformation $F_k\ni z\to z\sigma\in F_k$. Then,*

$$\sigma(E(\sigma)^*\omega)=\omega\,.$$

PROOF. (2)-(4) are just analogous to (2)-(4) in Lemma 6.6. We shall prove (1). Let $Y_p\in\mathfrak{g}_p$ $(-2\leqq p\leqq k-1)$. Then we have:

$$\begin{aligned}\theta_q^{(k-1)}((z\sigma)Y_{-2})&=\theta_q^{(k-1)}(zY_{-2}+z[X_{k-1},Y_{-2}])\\&=\delta_{q,-2}Y_{-2}\qquad(-2\leqq q\leqq k-4)\,,\\\theta_{k-3}^{(k-1)}((z\sigma)Y_{-2})&=(\theta_{k-3}^{(k-1)}-[X_{k-1},\theta_{-2}^{(k-1)}])(zY_{-2}+z[X_{k-1},Y_{-2}])\\&=-[X_{k-1},Y_{-2}]+[X_{k-1},Y_{-2}]=0\,,\end{aligned}$$

3) In general let X be a vector field on a manifold M. $\mathcal{L}_X$ will denote the Lie derivation with respect to X.

$$\theta_q^{(k-1)}((z\sigma)Y_{-1}) = \theta_q^{(k-1)}(zY_{-1}+z[X_{k-1}, Y_{-1}])$$
$$= \delta_{q,-1}Y_{-1} \qquad (-2 \leqq q \leqq k-4),$$
$$\theta_{k-3}^{(k-1)}((z\sigma)Y_{-1}) = (\theta_{k-3}^{(k-1)} - [X_{k-1}, \theta_{-2}^{(k-1)}])(zY_{-1}+z[X_{k-1}, Y_{-1}])$$
$$= \delta_{k-3,-1}Y_{-1},$$
$$\theta_{k-2}^{(k-1)}((z\sigma)Y_{-1}) = (\theta_{k-2}^{(k-1)} - [X_{k-1}, \theta_{-1}^{(k-1)} \mid D_{k-1}])(zY_{-1}+z[X_{k-1}, Y_{-1}])$$
$$= -[X_{k-1}, Y_{-1}]+[X_{k-1}, Y_{-1}] = 0,$$
$$\theta_q^{(k-1)}((z\sigma)Y_p) = \theta_q^{(k-1)}(zY_p)$$
$$= \delta_{q,p}Y_p \qquad (0 \leqq p \leqq k-2,\ -2 \leqq q \leqq k-2),$$
$$(z\sigma)Y_{k-1} = R(\sigma)_*(zY_{k-1}) = R(\sigma)_*\Phi(Y_{k-1})_{\pi_k(z)} = \Phi(Y_{k-1})_{\pi_k(z)\sigma}.$$

This last equality is the case, because $G_{k-1}(k>1)$ is an abelian group. These equalities clearly mean that $z\sigma \in F_k$.

LEMMA 6.9. *There is a linear mapping $\bar{\Phi}$ of $\mathfrak{g}_{k-1}$ into $\mathfrak{X}(F_k)$ having the following properties: Let $X_{k-1} \in \mathfrak{g}_{k-1}$. Then,*

1) $\pi_{k*}\bar{\Phi}(X_{k-1}) = \Phi(X_{k-1})_{\pi_k(z)}$ *at each* $z \in F_k$;

2) $\omega_p(\bar{\Phi}(X_{k-1})) = \delta_{p,k-1}X_{k-1}$ $(-2 \leqq p \leqq k-1)$;

3) $\mathcal{L}_{\bar{\Phi}(X_{k-1})}\omega_p = 0$ $(-2 \leqq p \leqq k-4$ *or* $p = k-1)$,
$\mathcal{L}_{\bar{\Phi}(X_{k-1})}\omega_{k-3} + [X_{k-1}, \omega_{-2}] = 0,$
$\mathcal{L}_{\bar{\Phi}(X_{k-1})}\omega_{k-2} + [X_{k-1}, \omega_{-1}] = 0.$

PROOF. This follows from Lemma 6.8. In the case $k>2$, the situation is just analogous to that in Lemma 6.7. We have only to define $\bar{\Phi}(X_{k-1})$ to be the vector field on F_k induced by the one parameter group $E(\exp t\bar{s}_{k-1}(X_{k-1})) = E(\bar{S}_{k-1}(tX_{k-1}))$.

The case $k=2$. We set $\sigma_t = \exp t\bar{s}_1(X_1) = \bar{S}_1(tX_1)$. Then we see that $E(\sigma_t)$ is a one parameter family of transformations of F_2 and that $E(\sigma_0)$ is equal to the identity transformation of F_2. This being said, we define $\bar{\Phi}(X_1)$ to be the vector field on F_2 induced by the family $E(\sigma_t)$, i.e.,

$$\bar{\Phi}(X_1)_z = \frac{\partial z\sigma_t}{\partial t}\Big|_{t=0}$$

at each $z \in F_2$. Let us show that the mapping $\bar{\Phi}: \mathfrak{g}_1 \ni X_1 \to \bar{\Phi}(X_1) \in \mathfrak{X}(F_2)$ is linear. We set $\sigma_t = \exp t\bar{s}_1(X_1)$ and $\tau_t = \exp t\bar{s}_1(Y_1)$. We have $\sigma_t\tau_t = \exp t\bar{s}_1(X_1 + Y_1)$, and, by Lemma 6.8, (2), we have

$$E(\tau_t)E(\sigma_t) = \bar{R}(A_2(0, t^2\gamma(X_1, Y_1))E(\sigma_t\tau_t).$$

Therefore we have $\bar{\Phi}(X_1)+\bar{\Phi}(Y_1) = \bar{\Phi}(X_1+Y_1)$. Moreover we have clearly $\bar{\Phi}(\lambda X_1) = \lambda\bar{\Phi}(X_1)$ for all $\lambda \in \boldsymbol{R}$, proving our assertion. Now, 1) and 2) follow from Lemma 6.8, (3). To prove 3), we use Lemma 6.8, (4) and the equality

$$\left.\frac{\partial E(\sigma_t)^*\omega}{\partial t}\right|_{t=0}=\mathcal{L}_{\bar{\Phi}(X_1)}\omega\,.$$

Let us return to the general case. Hereafter we assume that $(P_{k-1}, \theta^{(k-1)})$ satisfies condition C_{k-1}. By Lemma 6.4, and condition C_{k-1}, we have easily

LEMMA 6.10.

$$d\omega_{-2}+\frac{1}{2}[\omega_{-1}, \omega_{-1}]\equiv 0 \pmod{\omega_{-2}},$$

$$d\omega_{k-3}+[\omega_{k-2}, \omega_{-1}]\equiv 0 \pmod{\omega_{-2}, (\sum_{p=-1}^{k-3}\omega_p)^2} \qquad (k>1).$$

We set as follows:

$$\Omega'_{-2}=d\omega_{-2}+\frac{1}{2}[\omega_{-1}, \omega_{-1}]+[\omega_0, \omega_{-2}],$$

$$\Omega'_{k-3}=d\omega_{k-3}+[\omega_{k-2}, \omega_{-1}]+[\omega_{k-1}, \omega_{-2}] \qquad (k>1),$$

$$\Omega_{k-2}=d\omega_{k-2}+[\omega_{k-1}, \omega_{-1}] \qquad (k\geqq 1).$$

LEMMA 6.11. *For any $k\geqq 1$, we have:*

$$\Omega'_{k-3}\equiv 0 \pmod{(\omega_{-2})^2, (\sum_{p=-1}^{k-3}\omega_p)^2, \omega_{-2}\cdot\sum_{p=-1}^{k-2}\omega_p},$$

$$\Omega_{k-2}\equiv 0 \pmod{\omega_{-2}, (\sum_{p=-1}^{k-2}\omega_p)^2}.$$

PROOF. By Lemma 6.5, we have

$$\Omega'_{k-3}\equiv 0 \pmod{(\omega_{-2})^2, (\sum_{p=-1}^{k-1}\omega_p)^2, \omega_{-2}\cdot\sum_{p=-1}^{k-1}\omega_p}, \tag{6.1}$$

$$\Omega_{k-2}\equiv 0 \pmod{\omega_{-2}, (\sum_{p=-1}^{k-1}\omega_p)^2}. \tag{6.2}$$

By Lemma 6.10, we have

$$\Omega'_{k-3}\equiv 0 \pmod{\omega_{-2}, (\sum_{p=-1}^{k-3}\omega_p)^2}. \tag{6.3}$$

It follows from (6.1) and (6.3) that

$$\Omega'_{k-3}\equiv 0 \pmod{(\omega_{-2})^2, (\sum_{p=-1}^{k-3}\omega_p)^2, \omega_{-2}\cdot\sum_{p=-1}^{k-1}\omega_p}. \tag{6.4}$$

By Lemmas 6.7 and 6.9, we have

$$\omega_p(\bar{\Phi}(X_{k-1}))=\delta_{p,k-1}X_{k-1} \qquad (-2\leqq p\leqq k-1). \tag{6.5}$$

Let Z by any vector field on F_k. By (6.5), we have

$$\begin{aligned} d\omega_p(\bar{\Phi}(X_{k-1}), Z) &= \bar{\Phi}(X_{k-1})\omega_p(Z)-Z\omega_p(\bar{\Phi}(X_{k-1}))-\omega_p([\bar{\Phi}(X_{k-1}), Z]) \\ &= \bar{\Phi}(X_{k-1})\omega_p(Z)-\omega_p([\bar{\Phi}(X_{k-1}), Z]) \\ &= (\mathcal{L}_{\bar{\Phi}(X_{k-1})}\omega_p)(Z)\,. \end{aligned}$$

Therefore by Lemmas 6.7 and 6.9, we have

$$d\omega_{k-3}(\bar{\Phi}(X_{k-1}), Z)+[X_{k-1}, \omega_{-2}(Z)]=0,$$
$$d\omega_{k-3}(\bar{\Phi}(X_{k-1}), Z)+[X_{k-1}, \omega_{-1}(Z)]=0.$$

Hence by (6.5), we get

$$\Omega'_{k-3}(\bar{\Phi}(X_{k-1}), Z)=d\omega_{k-3}(\bar{\Phi}(X_{k-1}), Z)+[X_{k-1}, \omega_{-2}(Z)]=0, \tag{6.6}$$

$$\Omega_{k-2}(\bar{\Phi}(X_{k-1}), Z)=d\omega_{k-2}(\bar{\Phi}(X_{k-1}), Z)+[X_{k-1}, \omega_{-1}(Z)]=0. \tag{6.7}$$

From (6.4), (6.5) and (6.6), we get the first equality in Lemma 6.11 and, from (6.2), (6.5) and (6.7), we get the second equality in Lemma 6.11.

From Lemma 6.11, we know that there are a unique mapping $T'(z\rightarrow T'_z)$ of F_k to $\mathfrak{T}^{-2}_{k-3}$ and a unique mapping $T(z\rightarrow T_z)$ of F_k to $\mathfrak{T}^{-1}_{k-2}$ respectively as follows:

$$\Omega'_{k-3}+T'(\omega_{-2}, \sum_{p=-1}^{k-2}\omega_p)\equiv 0 \pmod{(\omega_{-2})^2, (\sum_{p=-1}^{k-3}\omega_p)^2},$$

$$\Omega_{k-2}+\frac{1}{2}T(\sum_{p=-1}^{k-2}\omega_p, \sum_{p=-1}^{k-2}\omega_p)\equiv 0 \pmod{\omega_{-2}}.$$

LEMMA 6.12. *Let $\sigma=A_k(u, u')B_k(v)\in H_k$ and $z\in F_k$. Then we have:*

$$(T_{z\sigma}, T'_{z\sigma})=(T_z, T'_z)+\partial(u, u').$$

PROOF. We have $\bar{R}(\sigma)^*\omega=\sigma^{-1}\omega$ and $\sigma^{-1}=A_k(-u, -u')\,B_k(u\circ u'-v)$. Therefore we have

$$\bar{R}(\sigma)^*\omega_p=\omega_p \qquad (-2\leqq p<k-2),$$
$$\bar{R}(\sigma)^*\omega_{k-2}=\omega_{k-2}-u'(\omega_{-2}),$$
$$\bar{R}(\sigma)^*\omega_{k-1}\equiv\omega_{k-1}-u(\sum_{p=-1}^{k-2}\omega_p) \pmod{\omega_{-2}}.$$

Moreover we have

$$d\omega_{-2}+\frac{1}{2}[\omega_{-1}, \omega_{-1}]\equiv 0 \pmod{\omega_{-2}}.$$

We shall prove Lemma 6.12 only for the case where $k=1$. The case where $k>1$ can be similarly dealt with.

$$\bar{R}(\sigma)^*(\Omega'_{-2}+T'(\omega_{-2}, \omega_{-1}))\equiv 0 \pmod{(\omega_{-2})^2}.$$

$$d\omega_{-2}+\frac{1}{2}[\omega_{-1}-u'(\omega_{-2}), \omega_{-1}-u'(\omega_{-2})]+[\omega_0-u(\omega_{-1}), \omega_{-2}]$$
$$+(\bar{R}(\sigma)^*T')(\omega_{-2}, \omega_{-1}-u'(\omega_{-2}))\equiv 0 \pmod{(\omega_{-2})^2}.$$

$$d\omega_{-2}+\frac{1}{2}[\omega_{-1}, \omega_{-1}]+[\omega_0, \omega_{-2}]+(\bar{R}(\sigma)^*T')(\omega_{-2}, \omega_{-1})$$
$$-[u'(\omega_{-2}), \omega_{-1}]-[u(\omega_{-1}), \omega_{-2}]\equiv 0 \pmod{(\omega_{-2})^2}.$$

$$-T'(\omega_{-2}, \omega_{-1})+(\bar{R}(\sigma)^*T')(\omega_{-2}, \omega_{-1})-[u'(\omega_{-2}), \omega_{-1}]$$
$$-[u(\omega_{-1}), \omega_{-2}]\equiv 0 \pmod{(\omega_{-2})^2}.$$

" $\equiv 0$ " in this last equality clearly reduces to " $=0$ ". Therefore we have

$$(\bar{R}(\sigma)^*T')(X_{-2}, Y_{-1})=T'(X_{-2}, Y_{-1})+[u'(X_{-2}), Y_{-1}]-[u(Y_{-1}), X_{-2}] \tag{6.8}$$

for all $X_{-2}\in \mathfrak{g}_{-2}$ and $Y_{-1}\in \mathfrak{g}_{-1}$.

$$\bar{R}(\sigma)^*(\Omega_{-1}+\frac{1}{2}T(\omega_{-1}, \omega_{-1}))\equiv 0 \pmod{\omega_{-2}}.$$

$$d(\omega_{-1}-u'(\omega_{-2}))+[\omega_0-u(\omega_{-1}), \omega_{-1}-u'(\omega_{-2})]$$
$$+\frac{1}{2}(\bar{R}(\sigma)^*T)(\omega_{-1}-u'(\omega_{-2}), \omega_{-1}-u'(\omega_{-2}))\equiv 0 \pmod{\omega_{-2}}.$$

$$d\omega_{-1}+[\omega_0, \omega_{-1}]+\frac{1}{2}(\bar{R}(\sigma)^*T)(\omega_{-1}, \omega_{-1})$$
$$-[u(\omega_{-1}), \omega_{-1}]-u'(d\omega_{-2})\equiv 0 \pmod{\omega_{-2}}.$$

$$-\frac{1}{2}T(\omega_{-1}, \omega_{-1})+\frac{1}{2}(\bar{R}(\sigma)^*T)(\omega_{-1}, \omega_{-1})-[u(\omega_{-1}), \omega_{-1}]$$
$$+\frac{1}{2}u'([\omega_{-1}, \omega_{-1}])\equiv 0 \pmod{\omega_{-2}}.$$

" $\equiv 0$ " in this last equality clearly reduces to " $=0$ ". Therefore we have

$$(\bar{R}(\sigma)^*T)(X_{-1}, Y_{-1})=T(X_{-1}, Y_{-1})+[u(X_{-1}), Y_{-1}]$$
$$-[u(Y_{-1}), X_{-1}]-u'([X_{-1}, Y_{-1}]) \tag{6.9}$$

for all $X_{-1}, Y_{-1}\in \mathfrak{g}_{-1}$. From (6.8) and (6.9), we get

$$(\bar{R}(\sigma)^*T, \bar{R}(\sigma)^*T')=(T, T')+\partial(u, u').$$

Now consider the linear mapping ∂ of $\mathcal{L}_{k-1}$ to $\mathcal{T}_{k-2}$. For each $k\geqq 1$, we choose, once for all, a complementary subspace $\mathcal{T}^{\#}_{k-2}$ of $\partial\mathcal{L}_{k-1}$ in $\mathcal{T}_{k-2}$.

We denote by $P^{\#}_k$ the subset of F_k consisting of all the elements z such that $(T_z, T'_z)\in \mathcal{T}^{\#}_{k-2}$. We shall show that $P^{\#}_k$ is a $G^{\#}_k$-subbundle of $F_k(P_{k-1}, H_k)$. First we have $\pi_k(P^{\#}_k)=P_{k-1}$. Indeed, take any $z\in F_k$. There is a $(u, u')\in\mathcal{L}_{k-1}$ such that $(T_z, T'_z)+\partial(u, u')\in \mathcal{T}^{\#}_{k-2}$. If we set $\sigma=A_k(u, u')$, we have $(T_{z\sigma}, T'_{z\sigma})=(T_z, T'_z)+\partial(u, u')\in \mathcal{T}^{\#}_{k-2}$ (Lemma 6.12). This means $z\sigma\in P^{\#}_k$. Therefore we must have $\pi_k(P^{\#}_k)=P_{k-1}$. Now let $z\in P^{\#}_k$ and $\sigma=A_k(u, u')B_k(v)\in H_k$. We have $(T_{z\sigma}, T'_{z\sigma})=(T_z, T'_z)+\partial(u, u')$ by Lemma 6.12. Since $(T_z, T'_z)\in \mathcal{T}^{\#}_{k-2}$, we have: $z\sigma\in P^{\#}_k$, i.e., $(T_{z\sigma}, T'_{z\sigma})\in \mathcal{T}^{\#}_{k-2}$ if and only if $\partial(u, u')=0$. Therefore by Lemma 6.2, we have: $z\sigma\in P^{\#}_k$ if and only if $\sigma\in G^{\#}_k$, which completes our assertion.

We denote by $\omega^{(k)}$ the basic form of $P_k^\#$. Since $\omega^{(k)}$ is the restriction of ω to $P_k^\#$, we see from Lemmas 6.10 and 6.11 that the $G_k^\#$-structure $(P_k^\#, \omega^{(k)})$ satisfies condition $C_k^\#$. Let α_k be the projection of $P_k^\#$ onto P_{k-1}, which is just the restriction of π_k to $P_k^\#$. Then by Lemma 6.4, (1), we have $\omega_p^{(k)} = \alpha_k^*\theta_p^{(k-1)}$ $(-2 \leqq p < k-2)$. Moreover let $D_k^\#$ denote the differential system on $P_k^\#$ defined by the equation $\omega_{-2}^{(k)} = 0$. Then we have $T_z(P_k^\#) \cap (\bar{D}_k)_z = (D_k^\#)_z$ at each $z \in P_k^\#$, $D_k^\# = \alpha_k^* D_{k-1}$ and hence $\omega_{k-2}^{(k)} | D_k^\# = \alpha_k^* \theta_{k-2}^{(k-1)}$ by Lemma 6.4, (2).

We have thus shown that to every pseudo-G_{k-1}-structure $(P_{k-1}, \theta^{(k-1)})$ on a manifold M_{k-2} satisfying condition C_{k-1}, there is associated, in a canonical manner, a $G_k^\#$-structure $(P_k^\#, \omega^{(k)})$ on P_{k-1} satisfying condition $C_k^\#$ and the condition in Th. 4.2. We must show that the assignment $(P_{k-1}, \theta^{(k-1)}) \to (P_k^\#, \omega^{(k)})$ is compatible with the respective isomorphisms. Let $(P_{k-1}, \theta^{(k-1)})$ (resp. $(P'_{k-1}, \theta'^{(k-1)})$) be a pseudo-$G_{k-1}$-structure on a manifold M_{k-2} (resp. M'_{k-2}) and let $(P_k^\#, \omega^{(k)})$ (resp. $(P_k'^\#, \omega'^{(k)})$) be the corresponding $G_k^\#$-structure on P_{k-1} (resp. P'_{k-1}). Let (F_k, ω) (resp. (F'_k, ω')) be the corresponding H_k-structure on P_{k-1} (resp. P'_{k-1}). We shall write as A' the quantity in $(P'_{k-1}, \theta'^{(k-1)})$ or $(P_k'^\#, \omega'^{(k)})$ or (F'_k, ω') which corresponds to a quantity A in $(P_{k-1}, \theta^{(k-1)})$ or $(P_k^\#, \omega^{(k)})$ or (F_k, ω).

Let φ_{k-1} be an isomorphism of $(P_{k-1}, \theta^{(k-1)})$ onto $(P'_{k-1}, \theta'^{(k-1)})$. From the definition of (F_k, ω) and of (F'_k, ω'), we see that φ_{k-1} yields an isomorphism φ of (F_k, ω) onto (F'_k, ω'). We have clearly $((T')_{\varphi(z)}, (T')'_{\varphi(z)}) = (T_z, T'_z)$ for all $z \in F_k$. Therefore φ induces an isomorphism $\varphi_k^\#$ of $(P_k^\#, \omega^{(k)})$ onto $(P_k'^\#, \omega'^{(k)})$.

Conversely, let $\varphi_k^\#$ be an isomorphism of $(P_k^\#, \omega^{(k)})$ onto $(P_k'^\#, \omega'^{(k)})$. Let φ_{k-1} denote the diffeomorphism of P_{k-1} onto P'_{k-1} induced by $\varphi_k^\#$. Since $\varphi_{k-1} \circ \alpha_k = \alpha'_k \circ \varphi_k^\#$, we have $\varphi_{k-1}^* \theta_p'^{(k-1)} = \theta_p^{(k-1)}$ $(-2 \leqq p \leqq k-2)$. Now, take any $z \in P_k^\#$ and set $z' = \varphi_k^\#(z)$, $x = \alpha_k(z)$ and $x' = \alpha'_k(z') = \varphi_{k-1}(x)$. Then we have $\Phi'(X_{k-1})_{x'} = z' \cdot X_{k-1} = \varphi_{k-1*}(z \cdot X_{k-1}) = \varphi_{k-1*}\Phi(X_{k-1})_x$. This means that $r'(X) = \varphi_{k-1*} r(X)$ for all $X \in \mathfrak{g}_{k-1}$. Since the Lie group G_{k-1} is connected, it follows that $\varphi_{k-1}(x\sigma) = \varphi_{k-1}(x)\sigma$ for all $x \in P_{k-1}$ and $\sigma \in G_{k-1}$. We have thereby proved φ_{k-1} to be an isomorphism of $(P_{k-1}, \theta^{(k-1)})$ onto $(P'_{k-1}, \theta'^{(k-1)})$.

§ 7. Applications to the geometry of differential systems.

Let D (resp. D') be a differential system on a manifold M (resp. M'). A diffeomorphism f of M onto M' is called an isomorphism of (M, D) onto (M', D') if we have $f_* D_x = D'_{f(x)}$ at each $x \in M$.

Let (n, n') be a pair of integers with $n \geqq 0$, $n' \geqq 0$ and $n + n' > 0$. We set $m = n + n'$. We consider the vector space $\mathfrak{m} = \boldsymbol{R}^m$, the space of m real variables, and denote by $e'_1, \cdots, e'_{n'}, e_1, \cdots, e_n$ the natural base of it. Let $\mathfrak{g}_{-2}$ (resp. $\mathfrak{g}_{-1}$) be the subspace of $\mathfrak{m}$ spanned by $(e'_\alpha)_{1 \leqq \alpha \leqq n'}$ (resp. $(e_i)_{1 \leqq i \leqq n}$). We have $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$

(direct sum). Let H be the closed subgroup of $GL(\mathfrak{m})=GL(m, \boldsymbol{R})$ consisting of all the elements σ which leave $\mathfrak{g}_{-1}$ invariant. For each $\sigma\in H$, σ_{-2} and σ_{-1} will denote the linear automorphisms of $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ respectively defined by $\sigma_{-2}X_{-2}\equiv\sigma X_{-2}$ (mod $\mathfrak{g}_{-1}$) for all $X_{-2}\in\mathfrak{g}_{-2}$ and by $\sigma_{-1}X_{-1}=\sigma X_{-1}$ for all $X_{-1}\in\mathfrak{g}_{-1}$. Let S be the closed subgroup of H consisting of all the elements σ which leave $\mathfrak{g}_{-2}$ invariant, and let N_0 be the kernel of the homomorphism $\sigma\rightarrow(\sigma_{-2}, \sigma_{-1})$ of H onto $S=GL(\mathfrak{g}_{-2})\times GL(\mathfrak{g}_{-1})=GL(n', \boldsymbol{R})\times GL(n, \boldsymbol{R})$. We have $H=N_0S$.

As is easily observed, to every n-dimensional differential system D on an m-dimensional manifold M, there is associated a H-structure (F, ω) on M, unique up to equivalence, such that the differential system π^*D on F is defined by the equation $\omega_{-2}=0$, π being the projection of F onto M and ω_{-2} being the $\mathfrak{g}_{-2}$-component of ω, and vice versa. The assignment $(M, D)\rightarrow(F, \omega)$ is clearly compatible with the respective isomorphisms.

We shall now define the "torsion" T of any H-structure (F, ω).

We denote by $\mathfrak{M}$ the vector space consisting of all the linear mappings L of the second exterior product $\Lambda^2\mathfrak{g}_{-1}$ of $\mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$. The group $GL(\mathfrak{g}_{-1})$ linearly acts on $\Lambda^2\mathfrak{g}_{-1}$ by the rule: $\sigma(X_{-1}\wedge Y_{-1})=\sigma X_{-1}\wedge\sigma Y_{-1}$ for all $\sigma\in GL(\mathfrak{g}_{-1})$ and $X_{-1}, Y_{-1}\in\mathfrak{g}_{-1}$. Let us now make the group H operate on $\mathfrak{M}$ as follows: Let $L\in\mathfrak{M}$ aud $\sigma\in H$. Then $L^\sigma\in\mathfrak{M}$ is defined to be

$$L^\sigma(B)=(\sigma_{-2})^{-1}L(\sigma_{-1}B)$$

for all $B\in\Lambda^2\mathfrak{g}_{-1}$. We have $(L^\sigma)^\tau=L^{\sigma\tau}$. We shall denote by $(\mathfrak{M}, H)$ this transformation group H on $\mathfrak{M}$.

LEMMA 7.1. *Let (F, ω) be a H-structure on a manifold M. There is a unique mapping $T(z\rightarrow T_z)$ of F to $\mathfrak{M}$ such that*

$$d\omega_{-2}+\frac{1}{2}T(\omega_{-1}\wedge\omega_{-1})\equiv 0 \pmod{\omega_{-2}},$$

where ω_p $(p=-2, -1)$ denotes the $\mathfrak{g}_p$-component of ω.

PROOF. Since $F(M, H)$ admits a connection, we have

$$d\omega_{-2}\equiv 0 \pmod{\omega_{-2}, (\omega_{-1})^2}.$$

Lemma 7.1 follows immediately from this equality.

LEMMA 7.2. *The notations being as above, we have*

$$T_{z\sigma}=(T_z)^\sigma$$

for all $z\in F$ and $\sigma\in H$.

PROOF. For each $\sigma\in H$, let $R(\sigma)$ denote the right translation of F induced by σ. We have $R(\sigma)^*\omega=\sigma^{-1}\omega$, from which follows that $R(\sigma)^*\omega_{-2}=(\sigma_{-2})^{-1}\omega_{-2}$ and $R(\sigma)^*\omega_{-1}\equiv(\sigma_{-1})^{-1}\omega_{-1}$ (mod ω_{-2}). Lemma 7.2 follows easily from these equalities.

Given a H-structure (F, ω) on a manifold M, we shall denote by $T(F, \omega)$

the subset of $\mathfrak{M}$ consisting of all the elements $T_z (z \in F)$. By Lemma 7.2, we see that $T(F, \omega)$ is composed of several orbits of $(\mathfrak{M}, H)$. If the pseudo-group Γ of all the local automorphisms of (F, ω) is transitive on M, then it is clear that $T(F, \omega)$ forms a single orbit.

DEFINITION 7.1. Let $\mathfrak{N}$ be an orbit of $(\mathfrak{M}, H)$. We say that a H-structure (F, ω) on a manifold M or the corresponding differential system D on M is of type $\mathfrak{N}$ if $T(F, \omega)$ coincides with the given $\mathfrak{N}$.

DEFINITION 7.2. An orbit $\mathfrak{N}$ of $(\mathfrak{M}, H)$ is called $\mathfrak{g}_{-2}$-maximal if some and hence any $L \in \mathfrak{N}$ maps $\Lambda^2 \mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$.

Let L be any element of $\mathfrak{M}$. We denote by $G_0^\#(L)$ the isotropy group of $(\mathfrak{M}, H)$ at the point L, and set $G_0(L) = G_0^\#(L) \cap S$. Then we have $G_0^\#(L) = N_0 \cdot G_0(L)$ and $\sigma L(X_{-1} \wedge Y_{-1}) = L(\sigma X_{-1} \wedge \sigma Y_{-1})$ for all $\sigma \in G_0(L)$ and $X_{-1}, Y_{-1} \in \mathfrak{g}_{-1}$. Let $\mathfrak{g}_0(L)$ be the Lie algebra of $G_0(L)$. Let us now make the direct sum $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0(L)$ a graded Lie algebra by defining a bracket operation $[\ ,\ ]$ as follows: $[\mathfrak{g}_{-2}, \mathfrak{g}_{-2} + \mathfrak{g}_{-1}] = \{0\}$, $[X_0, Y] = X_0 Y$ for all $X_0 \in \mathfrak{g}_0(L)$, $Y \in \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$, $[X_{-1}, Y_{-1}] = L(X_{-1} \wedge Y_{-1})$ for all $X_{-1}, Y_{-1} \in \mathfrak{g}_{-1}$, and $[X_0, Y_0] = X_0 Y_0 - Y_0 X_0$ for all $X_0, Y_0 \in \mathfrak{g}_0(L)$. Now, assume that L maps $\Lambda^2 \mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$. Then we know[4)] that the graded Lie algebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0(L)$, thus defined, satisfies condition (2.1) in § 2, and that the groups N_0, $G_0^\#(L)$ and $G_0(L)$ are just associated[5)] with this graded Lie algebra. We denote by $\mathfrak{g}(L) = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \sum_{p=0}^{\infty} \mathfrak{g}_p(L)$ the prolongation of the graded Lie algebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0(L)$. Finally we remark the following points: The mapping $\sigma \to \sigma_{-1}$ gives an isomorphism of $G_0(L)$ onto a subgroup of $GL(\mathfrak{g}_{-1})$, and this subgroup consists of all the elements $\tau \in GL(\mathfrak{g}_{-1})$ such that $\tau L^{-1}(0) = L^{-1}(0)$.

Let $\mathfrak{N}$ be a $\mathfrak{g}_{-2}$-maximal orbit of $(\mathfrak{M}, H)$ and let (F, ω) be a H-structure of type $\mathfrak{N}$ on a manifold M. We fix an element L of $\mathfrak{N}$ and use the notations as above. T being as in Lemma 7.1, let $P_0^\#(L)$ denote the subset of F consisting of all the elements z such that $T_z = L$. Since $T(F, \omega) = \mathfrak{N}$, we see from Lemma 7.2 that $P_0^\#(L)$ is a $G_0^\#(L)$-subbundle of F. We denote by $\omega^{(0)}(L)$ the basic form of $P_0^\#(L)$, which is the restriction of ω to $P_0^\#(L)$. Then we know from Lemma 7.1 that the $G_0^\#(L)$-structure $(P_0^\#(L), \omega^{(0)}(L))$ satisfies condition $C_0^\#$. It is clear that the assignment $(F, \omega) \to (P_0^\#(L), \omega^{(0)}(L))$ is compatible with the

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4) Let σ be any element of H. Then the group $G_0(L^\sigma)$ (resp. the graded Lie algebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0(L^\sigma)$) is canonically isomorphic with the group $G_0(L)$ (resp. the graded Lie algebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0(L)$).

5) The group $G_0(L)$ is generally not connected. Therefore we should rigorously consider the connected component of the identity of $G_0(L)$ and that of $G^\#_0(L)$ instead of $G_0(L)$ and $G^\#_0(L)$. However since we are exclusively concerned with local problems, this difference does not have any essential influences on our applications of the main theorems. The same remark will hold for the groups $G_0(L)$ and $G^\#_6(L)$ in § 8.

respective isomorphisms.

From Cor. 3 to Ths. 4.1 and 4.2, we get

PROPOSITION 7.1. *Let $\mathfrak{N}$ be a $\mathfrak{g}_{-2}$-maximal orbit of $(\mathfrak{M}, H)$ and let D be an n-dimensional differential system of type $\mathfrak{N}$ on an m-dimensional connected manifold M. Let L be an element of $\mathfrak{N}$. If $\dim \mathfrak{g}(L) < \infty$, then the Lie algebra of all the infinitesimal automorphisms of (M, D) is finite dimensional and of dimension $\leqq \dim \mathfrak{g}(L)$.*

Finally, we shall observe, in several cases, the maximal orbits $\mathfrak{N}$ of the transformation group $(\mathfrak{M}, H)$, the graded Lie algebras $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$ $(L \in \mathfrak{N})$, and their prolongations $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\sum_{p=0}^{\infty} \mathfrak{g}_p(L)$.

(1) The case where $n'=1$ and $n=2k$ $(k \geqq 1)$. In this case, every element L of $\mathfrak{M}$ may be considered as an anti-symmetric bilinear form on $\mathfrak{g}_{-1}$, and vice versa. Let $\mathfrak{N}_0$ denote the subset of $\mathfrak{M}$ consisting of all the elements L which are non-degenerate. Then we see that $\mathfrak{N}_0$ is open and dense in $\mathfrak{M}$ and that it is the unique maximal orbit of $(\mathfrak{M}, H)$. A H-structure (F, ω) of type $\mathfrak{N}_0$ on a manifold M or the corresponding differential system D on M is called a contact structure. Now, define a skew-symmetric matrix $I=(I_{ij})$ of degree $2k$ by $I_{ij}=I_{k+i,k+j}=0$ and $I_{i,k+j}=-I_{k+i,j}=\delta_{ij}$ $(1 \leqq i, j \leqq k)$, and define an element L of $\mathfrak{N}_0$ by $L(e_i \wedge e_j)=I_{ji}e'_i$ for all i, j $(1 \leqq i, j \leqq 2k)$. Then the group $G_0(L)$ consists of all the matrices σ of degree $2k+1$ of the form:

$$\begin{pmatrix} \varepsilon a^2 & 0 \\ 0 & ab \end{pmatrix},$$

where $a>0$, $b \in GL(2k, \boldsymbol{R})$, $\varepsilon^2=1$ and ${}^t bIb=\varepsilon I$. From the above argument, we may say that a contact structure is a $G_0^\#(L)$-structure $(P_0^\#(L), \omega^{(0)}(L))$ on a manifold M satisfying condition $C_0^\#$.

(1′) We use the same notations as in (1). Let G_0 be the subgroup of $G_0(L)$ defined by the equations $\varepsilon=1$ and $a=1$. Hence the group G_0 is isomorphic with the group $Sp\ (k, \boldsymbol{R})$. Setting $G_0^\#=N_0 \cdot G_0$, we say that a $G_0^\#$-structure $(P_0^\#, \omega^{(0)})$ on a manifold M satisfying condition $C_0^\#$, is a strict contact structure. As is well known, the principal fiber bundle $P_0^\#$ naturally reduces to a G_0-subbundle. Let $\mathfrak{g}_0$ be the Lie algebra of G_0 which is isomorphic with the simple Lie algebra $\mathfrak{sp}\ (k, \boldsymbol{R})$. Then the direct sum $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$ forms a subalgebra of $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$. Let $\mathfrak{g}=\sum_{p=-2}^{\infty} \mathfrak{g}_p$ be the prolongation of $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0$. Then we can easily prove: $\mathfrak{g}_p$ $(p \geqq 0)$ is isomorphic with the p-th prolonged space $\mathfrak{sp}\ (k, \boldsymbol{R})^{(p)}$ in the usual sense of the representation of $\mathfrak{sp}(k, \boldsymbol{R})$ on $\boldsymbol{R}^{2k}$, and $[\mathfrak{g}_p, \mathfrak{g}_{-2}]=\{0\}$ for any $p \geqq 0$. It follows that $\mathfrak{g}$ is infinite dimensional. Moreover it follows that the prolongation $\mathfrak{g}(L)=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\sum_{p=0}^{\infty} \mathfrak{g}_p(L)$ of $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$ is also infinite dimensional.

(2) The case where $n'=\frac{1}{2}n(n-1)-1$ and $n=2k$ $(k\geqq 2)$. Let L be the linear mapping of $\Lambda^2\mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$ such that the kernel $L^{-1}(0)$ of L is spanned by the element $\sum_{i,j=1}^{n} I_{ij}e_i\wedge e_j$, where $I=(I_{ij})$ denotes the matrix defined in (1). Let $\mathfrak{N}_0$ be the orbit of $(\mathfrak{M}, H)$ through L. Then we see that $\mathfrak{N}_0$ is open and dense in $\mathfrak{M}$ and hence that it is the unique maximal orbit of $(\mathfrak{M}, H)$. Moreover we see that the group $G_0(L)$ may be identified with the subgroup of $GL(\mathfrak{g}_{-1})=GL(n, \boldsymbol{R})$ consisting of all the elements $b\in GL(n, \boldsymbol{R})$ such that $bI^tb=\rho I$ with some $\rho\in\boldsymbol{R}$. An easy computation shows that $\mathfrak{g}_p(L)=\{0\}$ for all $p\geqq 1$. Hence we have $\dim\mathfrak{g}(L)=\dim\mathfrak{g}_{-2}+\dim\mathfrak{g}_{-1}+\dim\mathfrak{g}_0(L)=\frac{3}{2}(n^2+n)$.

(3) The case where $n'=\frac{1}{2}n(n-1)$ $(n\geqq 3)$. Let $\mathfrak{N}_0$ denote the subset of $\mathfrak{M}$ consisting of all the elements $L\in\mathfrak{M}$ which map $\Lambda^2\mathfrak{g}_{-1}$ isomorphically onto $\mathfrak{g}_{-2}$. Then we see that $\mathfrak{N}_0$ is open and dense in $\mathfrak{M}$ and that $\mathfrak{N}_0$ is the unique maximal orbit of $(\mathfrak{M}, H)$. We take a fixed element L of $\mathfrak{N}_0$ and identify $\mathfrak{g}_{-2}$ with $\Lambda^2\mathfrak{g}_{-1}$ by the isomorphism L. Then we see that the group $G_0(L)$ may be identified with the group $GL(\mathfrak{g}_{-1})$, and we can easily prove: $\mathfrak{g}_1(L)\neq\{0\}$ and $\mathfrak{g}_p(L)=\{0\}$ $(p>2)$. Since both the representations ρ_{-2} and ρ_{-1} on $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are irreducible, we know from Prop. 2.1 that the Lie algebra $\mathfrak{g}(L)$ is simple and $\dim\mathfrak{g}(L)=2(\dim\mathfrak{g}_{-2}+\dim\mathfrak{g}_{-1})+\dim\mathfrak{g}_0(L)=2n^2+n$: More precisely, $\mathfrak{g}(L)$ is isomorphic with the simple Lie algebra $\mathfrak{so}(n, n+1)$, and we have the natural identifications: $\mathfrak{g}_0(L)=\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-1}^*$, $\mathfrak{g}_1(L)=\mathfrak{g}_{-1}^*$ and $\mathfrak{g}_{-2}(L)=\Lambda^2\mathfrak{g}_{-1}^*$, where $\mathfrak{g}_{-1}^*$ denotes the dual space of $\mathfrak{g}_{-1}$.

REMARK. A general study of the equivalence of n-dimensional differential systems D on $(n+n')$-dimensional manifolds M will require graded Lie algebras of the form $\mathfrak{g}=\sum_{p=k}^{\infty}\mathfrak{g}_p$ $(k\leqq -3)$. This will be the case especially in the case where $n'\geqq\frac{1}{2}n(n-1)$. The particular case where $n'=3$ and $n=2$, has thoroughly been investigated by E. Cartan [2]. In his paper, he implicitly utilizes a graded Lie algebra of the form $\mathfrak{g}=\sum_{p=-3}^{3}\mathfrak{g}_p$.

§ 8. Applications to the geometry of real submanifolds in complex manifolds.

In this section, we always assume the differentiability of class C^ω.

DEFINITION 8.1 ([6] and [7]). Let f (resp. f') be an imbedding of a real manifold M(resp. M') to a complex manifold $\tilde{M}$(resp. $\tilde{M}'$). An analytic homeomorphism φ of M onto M' is called an isomorphism of (M, f) onto (M', f') if there is a complex analytic homeomorphism $\tilde{\varphi}$ of a neighborhood of $f(M)$ onto a neighborhood of $f'(M')$ such that $f' \circ \varphi = \tilde{\varphi} \circ f$. Furthermore, we say that (M, f) and (M', f') are equivalent if $M = M'$ and if the identity transformation of M is an isomorphism of (M, f) onto (M', f').

Let V be a complex vector space and let W be a real subspace of V. Then, $W \cap \sqrt{-1}\, W$ (resp. $W + \sqrt{-1}\, W$) is the maximum (resp. the minimum) complex subspace of V contained in W (resp. containing W). We say that W is of type (n, n') if $\dim_C(W \cap \sqrt{-1}\, W) = n$ and $\dim_C(W + \sqrt{-1}\, W) = n + n'$. We have $\dim_R W = 2n + n'$.

Let f be an imbedding of a real manifold M to a complex manifold $\tilde{M}$. Let x be any point of M. $\tilde{M}$ being a comlpex manifold, the tangent space $T_{f(x)}(\tilde{M})$ to $\tilde{M}$ at $f(x)$ is a complex vector space, and the image $f_* T_x(M)$ of the tangent space $T_x(M)$ to M at x by the differential f_* of f is a real subspace of the complex vector space $T_{f(x)}(\tilde{M})$.

DEFINITION 8.2. The notation being as above, we say that f is of type (n, n') if, at each $x \in M$, $f_* T_x(M)$ is a subspace of type (n, n') of $T_{f(x)}(\tilde{M})$.

An imbedding f of type $(n, 0)$ means that the image $f(M)$ of M by f is an n-dimensional complex submanifold of $\tilde{M}$, and an imbedding f of type $(0, n')$ means that $f(M)$ is a real part of $\tilde{M}$ if $\dim_C \tilde{M} = n'$. Moreover, any imbedding f of a $(2n+1)$-dimensional manifold M to an $(n+1)$-dimensional complex manifold $\tilde{M}$ is of type $(n, 1)$.

REMARK 1. Let f be an imbedding of a real connected manifold M to a complex manifold $\tilde{M}$. Denote by p the maximum of the integers $\dim_C(f_* T_x(M) + \sqrt{-1} f_* T_x(M))$ $(x \in M)$. Set $n = \dim_R M - p$ and $n' = 2p - \dim_R M$ and denote by M_0 the subset of M consisting of all the points x such that $f_* T_x(M)$ is a subspace of type (n, n') of $T_{f(x)}(\tilde{M})$. Then it can be proved that the subset M_0 is open and dense in M. This fact justifies the introduction of the notion of imbeddings of type (n, n').

REMARK 2. Let f be an imbedding of a real manifold M to a complex manifold $\tilde{M}$. Then we can prove the following reduction theorem: If f is of type (n, n'), there is a "unique" $(n+n')$-dimensional complex submanifold $\tilde{M}_0$ of $\tilde{M}$ such that $f(M) \subset \tilde{M}_0$.

REMARK 3. Let f(resp. f') be an imbedding of a real manifold M(resp. M') to a complex manifold $\tilde{M}$(resp. $\tilde{M}'$). We assume that both f and f' are of type (n, n') and that $\dim_C \tilde{M} = \dim_C \tilde{M}' = n + n'$. Then we can prove the following uniqueness theorem: If φ is an isomorphism of (M, f) onto (M', f'), there is a "unique" complex analytic homeomorphism $\tilde{\varphi}$ of a neighborhood of $f(M)$ onto a neighborhood of $f'(M')$ such that $f' \circ \varphi = \tilde{\varphi} \circ f$.

The results in Remarks 2 and 3 can be given reasonable proofs by the use of the E. Cartan's theory of involutive exterior differential systems. From now on, we shall exclusively deal with imbeddings of type (n, n') from $(2n+n')$-

dimensional real manifolds to $(n+n')$-dimensional complex manifolds. By virtue of Remark 2, this does not offer any essential restrictions for our problem.

Let (n, n') be a pair of integers with $n \geqq 0$, $n' \geqq 0$ and $n+n' > 0$. We set $m = 2n+n'$ and $\tilde{m} = n+n'$. Now, consider the complex vector space $\tilde{\mathfrak{m}} = \boldsymbol{C}^{\tilde{m}}$, the space of $\tilde{m}$ complex variables, and denote by $e'_1, \cdots, e'_{n'}, e_1, \cdots, e_n$ the natural base of it. Denote by $\mathfrak{g}_{-2}$ (resp. $\mathfrak{g}_{-1}$) the real subspace (resp. the complex subspace) of $\tilde{\mathfrak{m}}$ spanned by $(e'_\alpha)_{1 \leqq \alpha \leqq n'}$ over $\boldsymbol{R}$ (resp. by $(e_i)_{1 \leqq i \leqq n}$ over $\boldsymbol{C}$), and set $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$. Note that $\mathfrak{m}$ is a real subspace of type (n, n') of $\tilde{\mathfrak{m}}$. We set $\tilde{H} = GL(\tilde{\mathfrak{m}}) = GL(\tilde{m}, \boldsymbol{C})$ and denote by H the subgroup of $\tilde{H}$ consisting of all the elements σ which leave $\mathfrak{m}$ invariant. As is easily observed, the representation of H on $\mathfrak{m}$ is faithful. Therefore we may identify H with a subgroup of $GL(\mathfrak{m}) = GL(m, \boldsymbol{R})$. Let σ be an element of $\tilde{H}$. Then we see that $\sigma \in H$ if and only if σ satisfies the followings: 1) $\sigma\mathfrak{g}_{-1} = \mathfrak{g}_{-1}$, 2) for each $X_{-2} \in \mathfrak{g}_{-2}$, there is a $Y_{-2} \in \mathfrak{g}_{-2}$ such that $\sigma X_{-2} \equiv Y_{-2} \pmod{\mathfrak{g}_{-1}}$. For each $\sigma \in H$, denote by σ_{-1} and σ_{-2} the complex linear automorphism of $\mathfrak{g}_{-1}$ and the real linear automorphism of $\mathfrak{g}_{-2}$ respectively defined by $\sigma_{-1}X_{-1} = \sigma X_{-1}$ for all $X_{-1} \in \mathfrak{g}_{-1}$ and by $\sigma_{-2}X_{-2} \equiv \sigma X_{-2} \pmod{\mathfrak{g}_{-1}}$ for all $X_{-2} \in \mathfrak{g}_{-2}$. Let S denote the closed subgroup of H consisting of all the elements σ which leave $\mathfrak{g}_{-2}$ invariant, and let N_0 denote the kernel of the homomorphism $\sigma \to (\sigma_{-2}, \sigma_{-1})$ of H onto $S = GL(\mathfrak{g}_{-2}) \times GL(\mathfrak{g}_{-1}) = GL(n', \boldsymbol{R}) \times GL(n, \boldsymbol{C})$. We have $H = N_0 \cdot S$.

Since H is identified with a Lie subgroup of $GL(\mathfrak{m})$, we have the notion of H-structures. Let (F, ω) be a H-structure on a manifold M of dimension m. ω being an $\mathfrak{m}$-valued 1-form on F, it may be expressed as

$$\omega = \sum_{\alpha=1}^{n'} \omega'_\alpha e'_\alpha + \sum_{i=1}^{n} \omega_i e_i ,$$

where (ω'_α) are real valued 1-forms on F and (ω_i) are complex valued 1-forms on F. It is clear that the m forms $\omega'_1, \cdots, \omega'_{n'}, \omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent over $\boldsymbol{C}$ at each $z \in F$.

DEFINITION 8.3. The notations being as above, we say that the H-structure (F, ω) is integrable if we have the equalities:

$$d\omega'_\alpha \equiv 0 \pmod{\omega'_1, \cdots, \omega'_{n'}, \omega_1, \cdots, \omega_n},$$

$$d\omega_i \equiv 0 \pmod{\omega'_1, \cdots, \omega'_{n'}, \omega_1, \cdots, \omega_n} \quad (1 \leqq \alpha \leqq n',\ 1 \leqq i \leqq n),$$

where the meaning of mod should be considered with respect to complex valued forms. Furthermore, we say that (F, ω) satisfies condition (C) if we have the first n' equalities.

We shall show that to every imbedding f from an m-dimensional manifold M to an $\tilde{m}$-dimensional complex manifold $\tilde{M}$, there is associated, in a natural way, an integrable H-structure (F, ω) on M.

Let $\tilde{M}$ be an $\tilde{m}$-dimensional complex manifold. We denote by $\mathcal{G}_0$ the manifold of all the real contact elements of type (n, n') to $\tilde{M}$, which is a fiber bundle over $\tilde{M}$.

REMARK 4. Let $\mathcal{G}$ be the manifold of all the m-dimensional real contact elements to $\tilde{M}$. Then it can be shown that $\mathcal{G}_0$ is open and dense in $\mathcal{G}$. This implies that there are many real submanifolds of type (n, n') in $\tilde{M}$.

As is well known, there is associated to $\tilde{M}$ an integrable $\tilde{H}$-structure $(\tilde{F}, \tilde{\omega})$ on M. Let $\tilde{\pi}$ be the projection of $\tilde{F}$ onto $\tilde{M}$. Every element z of $\tilde{F}$ induces a complex isomorphism of $\tilde{\mathfrak{m}}$ onto $T_{\tilde{\pi}(z)}(\tilde{M})$, and we have clearly $z\cdot\mathfrak{m} \in \mathcal{G}_0$ for any $z \in \tilde{F}$. It is easily seen that the mapping $\rho(z \to z\cdot\mathfrak{m})$ maps $\tilde{F}$ onto $\mathcal{G}_0$ and that $\tilde{F}$ becomes a principal fiber bundle over the base space $\mathcal{G}_0$ with structure group H with projection ρ (the action of H on $\tilde{F}$ is given by the restriction to H of the action of $\tilde{H}$ on $\tilde{F}$). $\tilde{\omega}$ being an $\tilde{\mathfrak{m}}$-valued 1-form on $\tilde{F}$, it may be expressed as

$$\tilde{\omega} = \sum_{\alpha=1}^{n'} \tilde{\omega}'_\alpha e'_\alpha + \sum_{i=1}^{n} \tilde{\omega}_i e_i ,$$

where $(\tilde{\omega}'_\alpha)$ and $(\tilde{\omega}_i)$ are complex valued forms on $\tilde{F}$. Since $(\tilde{F}, \tilde{\omega})$ is integrable, we have the equalities:

$$d\tilde{\omega}'_\alpha \equiv 0 \ (\mathrm{mod}\ \tilde{\omega}'_1, \cdots, \tilde{\omega}'_{n'}, \tilde{\omega}_1, \cdots, \tilde{\omega}_n) ,$$

$$d\tilde{\omega}_i \equiv 0 \ (\mathrm{mod}\ \tilde{\omega}'_1, \cdots, \tilde{\omega}'_{n'}, \tilde{\omega}_1, \cdots, \tilde{\omega}_n) \quad (1 \leqq \alpha \leqq n',\ 1 \leqq i \leqq n) .$$

Now, let f be an imbedding of type (n, n') from an m-dimensional real manifold M to $\tilde{M}$. Let f_0 be the mapping of M into $\mathcal{G}_0$ defined by $f_0(x) = f_* T_x(M)$ for any $x \in M$. We denote by F the principal fiber bundle over the base space M with structure group H which is induced from the principal fiber bundle $\tilde{F}(\mathcal{G}_0, H)$ by the mapping f_0. From the definition of F, it follows that there is a unique mapping $\bar{f}_0$ of F to $\tilde{F}$ such that $\rho \circ \bar{f}_0 = f_0 \circ \pi$ and $\bar{f}_0(z\sigma) = \bar{f}_0(z)\sigma$ for any $z \in F$ and $\sigma \in H$, where π denotes the projection of F onto M. We define an $\tilde{\mathfrak{m}}$-valued 1-form ω on F to be $\omega = \bar{f}_0^* \tilde{\omega}$.

LEMMA 8.1. (1) *ω is an $\mathfrak{m}$-valued 1-form on F.*

(2) *The pair (F, ω) is an integrable H-structure on M.*

PROOF. (1) Let z be any point of F and set $y = \bar{f}_0(z)$. We have $y\cdot\tilde{\omega}(Y) = \tilde{\pi}_* Y$ for any $Y \in T_y(\tilde{F})$. Since we have $\tilde{\pi} \circ \bar{f}_0 = f \circ \pi$, we have, for all $Z \in T_z(F)$,

$$y\cdot\omega(Z) = y\cdot\tilde{\omega}(\bar{f}_{0*}Z) = (\tilde{\pi} \circ \bar{f}_0)_* Z = (f \circ \pi)_* Z .$$

Therefore we have

$$y\cdot\omega(T_z(F)) = f_* T_{\pi(z)}(M) = f_0 \circ \pi(z) = \rho \circ \bar{f}_0(z) = y\cdot\mathfrak{m} ,$$

Hence we get $\omega(T_z(F))=\mathfrak{m}$.

(2) Let $R(\sigma)$ (resp. $\tilde{R}(\sigma)$) denote the right translation of F (resp. $\tilde{F}$) induced by $\sigma\in H$ (resp. $\sigma\in\tilde{H}$). We have $\tilde{R}(\sigma)^*\tilde{\omega}=\sigma^{-1}\tilde{\omega}$ for any $\sigma\in\tilde{H}$. Since $\bar{f}_0(z\sigma)=\bar{f}_0(z)\sigma$ for any $z\in F$ and $\sigma\in H$, we have clearly $R(\sigma)^*\omega=\sigma^{-1}\omega$ for any $\sigma\in H$. Let Z be a tangent vector to F at $z\in F$. Since f is an imbedding, we see from the above argument that $\omega(Z)=0$ if and only if $\pi_*Z=0$, i.e., Z is a vertical vector in $F(M, H)$. We have thereby proved (F, ω) to be a H-structure on M. That (F, ω) is integrable follows from the integrability of $(\tilde{F}, \tilde{\omega})$.

LEMMA 8.2. *The assignment $(M, f)\rightarrow(F, \omega)$ is compatible with the respective isomorphisms.*

This has already been proved in the case where $n'=1$ ([**6**], Th. 1). The proof in the general case is just similar to this case.

REMARK 5. We can prove the following realization theorem for integrable H-structures: Let (F, ω) be any integrable H-structure on a manifold M of dimension m. Then there are an $\tilde{m}$-dimensional complex manifold $\tilde{M}$ and an imbedding f of type (n, n') from M to $\tilde{M}$ such that the given H-structure (F, ω) is equivalent to the H-structure corresponding to (M, f). By Lemma 8.2, we know that (M, f) is uniquely (up to equivalence) determined by (F, ω). Therefore this theorem combined with Lemma 8.2 generalizes a theorem of Whitney-Bruhat [**9**] concerning the complexification of real analytic manifolds. (Note that, in the case where $n=0$, F is nothing but the frame bundle of M and hence it does not give any structure on M.)

We shall now define the "torsion" or the "Levi form" T of any H-structure (F, ω) satisfying condition (C).

Let $\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-1}$ denote the complex tensor product of the complex vector space $\mathfrak{g}_{-1}$ and itself. We set

$$X_{-1}\,\bar{\wedge}\,Y_{-1}=\frac{1}{2}(X_{-1}\otimes\bar{Y}_{-1}-Y_{-1}\otimes\bar{X}_{-1})$$

for all $X_{-1}, Y_{-1}\in\mathfrak{g}_{-1}$, where $\bar{X}_{-1}$ denotes the vector conjugate to X_{-1} with respect to the real form $\boldsymbol{R}^n$ of $\mathfrak{g}_{-1}=\boldsymbol{C}^n$. We denote by $\bar{\wedge}^2\mathfrak{g}_{-1}$ the real subspace of $\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-1}$ spanned by $X_{-1}\bar{\wedge}Y_{-1}(X_{-1}, Y_{-1}\in\mathfrak{g}_{-1})$ over $\boldsymbol{R}$. The group $GL(\mathfrak{g}_{-1})=GL(n, \boldsymbol{C})$ acts[6] linearly on $\bar{\wedge}^2\mathfrak{g}_{-1}$ by the rule: $\sigma(X_{-1}\bar{\wedge}Y_{-1})=(\sigma X_{-1})\bar{\wedge}(\sigma Y_{-1})$ for all $\sigma\in GL(\mathfrak{g}_{-1})$ and $X_{-1}, Y_{-1}\in\mathfrak{g}_{-1}$. We denote by $\mathfrak{M}$ the vector space con-

6) The vector space $\bar{\Lambda}^2\mathfrak{g}_{-1}$ is composed of all the elements $\sum_{i,j=1}^{n} b_{ij}e_i\otimes e_j$, where $b=(b_{ij})$ is a skew-hermitian matrix of degree n. Let $\sigma=(\sigma_{ij})$ be a matrix in $GL(n, \boldsymbol{C})$ and let $b=(b_{ij})$ be a skew-hermitian matrix of degree n. Then we have

$$\sigma(\sum_{i,j} b_{ij}e_i\otimes e_j)=\sum_{i,j,k,l}\sigma_{ik}b_{kl}\bar{\sigma}_{jl}e_i\otimes e_j =\sum_{i,j}(\sigma b^t\bar{\sigma})_{ij}e_i\otimes e_j\,.$$

sisting of all the linear mappings L of $\overline{\wedge}^2\mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$. Let us now make the group H operate on $\mathfrak{M}$ as follows: Let $L\in\mathfrak{M}$ and $\sigma\in H$. Then, $L^\sigma\in\mathfrak{M}$ is defined to be

$$L^\sigma(B)=(\sigma_{-2})^{-1}L(\sigma_{-1}B)$$

for all $B\in\overline{\Lambda}^2\mathfrak{g}_{-1}$. We have $(L^\sigma)^\tau=L^{\sigma\tau}$.

LEMMA 8.3. *Let (F,ω) be a H-structure on a manifold M satisfying condition (C). There is a unique mapping $T(z\to T_z)$ of F to $\mathfrak{M}$ satisfying the equality*

$$d\omega_{-2}+\frac{1}{2}T(\omega_{-1}\overline{\wedge}\omega_{-1})\equiv 0 \pmod{\omega_{-2}},$$

where ω_p $(p=-2,-1)$ denotes the $\mathfrak{g}_p$-component of ω.

PROOF. Uniqueness is clear. Since (F,ω) satisfies condition (C), we can find a unique system $(T^\alpha_{ij})_{1\leqq i,j\leqq n,1\leqq\alpha\leqq n'}$ of complex valued functions on F such that

$$d\omega'_\alpha+\frac{1}{2}\sum_{i,j=1}^{n}T^\alpha_{ij}\omega_i\wedge\bar{\omega}_j\equiv 0 \pmod{\omega'_1,\cdots,\omega'_{n'}} \quad (1\leqq\alpha\leqq n'),$$

$$\bar{T}^\alpha_{ij}+T^\alpha_{ji}=0 \quad (1\leqq i,\ j\leqq n,\ 1\leqq\alpha\leqq n').$$

We have only to define $T_z\in\mathfrak{M}$ to be

$$T_z(X_{-1}\overline{\wedge}Y_{-1})=\frac{1}{2}\sum_{\alpha=1}^{n'}T^\alpha_{ij}(z)(x_i\bar{y}_j-y_i\bar{x}_j)e'_\alpha$$

for all $X_{-1}=\sum_i x_ie_i$, $Y_{-1}=\sum_j y_je_j\in\mathfrak{g}_{-1}$.

LEMMA 8.4. *The notation being as above, we have*

$$T_{z\sigma}=(T_z)^\sigma$$

for all $z\in F$ and $\sigma\in H$.

The proof of this is analogous to that of Lemma 7.2.

Once we have established Lemmas 8.3 and 8.4, we may now proceed just as in §7.

Let (F,ω) be a H-structure on a manifold M satisfying condition (C). We denote by $T(F,\omega)$ the subset of $\mathfrak{M}$ consisting of all the elements $T_z(z\in F)$.

DEFINITION 8.4. Let $\mathfrak{N}$ be an orbit of $(\mathfrak{M},H)$ and let (F,ω) be a H-structure on a manifold M satisfying condition (C). We say that (F,ω) is of type $\mathfrak{N}$ if we have $T(F,\omega)=\mathfrak{N}$. An imbedding f of type (n,n') from an m- dimensional real manifold M to an $\tilde{m}$-dimensional complex manifold $\tilde{M}$ is said to be of type $\mathfrak{N}$ if the corresponding H-structure (F,ω) on M is of type $\mathfrak{N}$.

DEFINITION 8.5. An orbit $\mathfrak{N}$ of $(\mathfrak{M},H)$ is called $\mathfrak{g}_{-2}$-maximal if some and hence any $L\in\mathfrak{M}$ maps $\overline{\Lambda}^2\mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$.

Let L be any element of $\mathfrak{M}$. We define the group $G_0^\#(L)=N_0G_0(L)$ and the graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$ just as in §7. Note that we have

$[X_{-1}, Y_{-1}]=L(X_{-1}\bar{\wedge}Y_{-1})$ for all $X_{-1}, Y_{-1}\in\mathfrak{g}_{-1}$. Under the assumption that L maps $\bar{\Lambda}^2\mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$, we see that the graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$ satisfies condition (2.1) in §2 and that the groups N_0 and $G_0(L)$ are just associated with this graded Lie algebra. We denote by $\mathfrak{g}(L)=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\sum_{p=0}^{\infty}\mathfrak{g}_p(L)$ the prolongation of the graded Lie algebra $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$.

Let $\mathfrak{N}$ be a $\mathfrak{g}_{-2}$-maximal orbit of $(\mathfrak{M}, H)$ and let (F, ω) be a H-structure of type $\mathfrak{N}$ on a manifold M. We fix an element L of $\mathfrak{N}$. Then we define the $G_0^\#(L)$-structure $(P_0^\#(L), \omega^{(0)}(L))$ on M just as in §7. By Lemma 8.3, this $G_0^\#(L)$-structure satisfies condition $C_0^\#$.

Consequently from Cor. 3 to Ths. 4.1 and 4.3, we get

PROPOSITION 8.1. (1) *Let $\mathfrak{N}$ be a $\mathfrak{g}_{-2}$-maximal orbit of $(\mathfrak{M}, H)$ and let (F, ω) be a H-structure of type $\mathfrak{N}$ on a connected manifold M. Let L be an element of $\mathfrak{N}$. If* $\dim\mathfrak{g}(L)<\infty$, *then the Lie algebra af all the infinitesimal automorphisms of (F, ω) is finite dimensional and of dimension* $\leqq \dim\mathfrak{g}(L)$.

(2) *Let f be an imbedding of type (n, n') from an m-dimensional connected manifold M to an $\tilde{m}$-dimensional complex manifold $\tilde{M}$. We assume that f is of type $\mathfrak{N}$ with respect to a $\mathfrak{g}_{-2}$-maximal orbit $\mathfrak{N}$ of $(\mathfrak{M}, H)$. Let L be an element of $\mathfrak{N}$. If* $\dim\mathfrak{g}(L)<\infty$, *then the Lie algebra of all the infinitesimal automorphisms of (M, f) is finite dimensional and of dimension* $\leqq\dim\mathfrak{g}(L)$.

Finally, we shall observe, in several cases, the maximal orbits $\mathfrak{N}$ of the transformation group $(\mathfrak{M}, H)$, the graded Lie algebras $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0(L)$ $(L\in\mathfrak{N})$ and their prolongations $\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\sum_{p=0}^{\infty}\mathfrak{g}_p(L)$.

(1) The case where $n'=1$ and $n\geqq 1$. Let λ be an integer with $0\leqq\lambda\leqq\left[\frac{n}{2}\right]$. We set as follows: $\varepsilon_i=-1$ $(1\leqq i\leqq\lambda)$ and $\varepsilon_i=1$ $(\lambda<i\leqq n)$, and define a skew-hermitian matrix $I_\lambda=(I_{\lambda ij})$ of degree n by $I_{\lambda ij}=\sqrt{-1}\varepsilon_i\delta_{ij}$ $(1\leqq i, j\leqq n)$. By using the matrix I_λ, we now define an element L_λ of $\mathfrak{M}$ by $L_\lambda(X_{-1}\bar{\wedge}Y_{-1})=\frac{1}{2}\sum_{i,j=1}^{n}I_{\lambda ji}(x_i\bar{y}_j-y_i\bar{x}_j)e_1'$ for all $X_{-1}=\sum_i x_ie_i$, $Y_{-1}=\sum_j y_je_j\in\mathfrak{g}_{-1}$. Let $\mathfrak{N}_\lambda$ denote the orbit of $(\mathfrak{M}, H)$ through L_λ. Then we see easily that $\mathfrak{N}_\lambda$ $\left(0\leqq\lambda\leqq\left[\frac{n}{2}\right]\right)$ are open sets of $\mathfrak{M}$ and that the maximal orbits of $(\mathfrak{M}, H)$ are just given by $\mathfrak{N}_\lambda$. The group $G_0(L_\lambda)$ consists of all the matrices σ of degree $n+1$ of the form

$$\begin{pmatrix}\varepsilon a^2 & 0\\ 0 & ab\end{pmatrix},$$

where $a>0$, $b\in GL(n, \boldsymbol{C})$, $\varepsilon^2=1$ and ${}^t\bar{b}I_\lambda b=\varepsilon I_\lambda$. An easy calculation shows: $\mathfrak{g}_1(L_\lambda)\neq\{0\}$ and $\mathfrak{g}_p(L_\lambda)=\{0\}$ $(p>2)$. Since both the representations ρ_{-2} and ρ_{-1} of $\mathfrak{g}_0(L_\lambda)$ on $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are irreducible, we see from Prop. 2.1 that $\mathfrak{g}(L_\lambda)$ is a simple Lie algebra and that $\dim\mathfrak{g}(L_\lambda)=2(\dim\mathfrak{g}_{-2}+\dim_R\mathfrak{g}_{-1})+\dim\mathfrak{g}_0(L_\lambda)=n^2+4n+3$. More precisely, $\mathfrak{g}(L_\lambda)$ is isomorphic with the simple Lie algebra

$\mathfrak{su}(n+1-\lambda, 1+\lambda)$.

By Prop. 8.1, we have the following ([6] and [7]): Let f be an imbedding of a $(2n+1)$-dimensional connected manifold M to an $(n+1)$-dimensional complex manifold $\tilde{M}$. If f is of type $\mathfrak{N}_\lambda$, then the Lie algebra of all the infinitesimal automorphisms of (M, f) is finite dimensional and of dimension $\leqq n^2+4n+3$.

An imbedding f of type $\mathfrak{N}_\lambda$ means that the real hypersurface $f(M)$ of $\tilde{M}$ is non-degenerate of index λ in the sense of [6]. In particular, an imbedding f of type $\mathfrak{N}_0$ means that $f(M)$ is a pseudo-convex hypersurface of $\tilde{M}$.

(2) The case where $n' = n^2-1$ $(n \geqq 2)$. This case is just analogous to case (2) in § 7. Let λ be an integer with $0 \leqq \lambda \leqq \left[\frac{n}{2}\right]$. Let L_λ be the linear mapping of $\bar{\Lambda}^2\mathfrak{g}_{-1}$ onto $\mathfrak{g}_{-2}$ such that the kernel $L_\lambda^{-1}(0)$ of L_λ is spanned by the element $\sum_{i,j=1}^{n} I_{\lambda ij} e_j \otimes e_j$, where $I_\lambda = (I_{\lambda ij})$ denotes the matrix defined in (1). Let $\mathfrak{N}_\lambda$ denote the orbit of $(\mathfrak{M}, H)$ through L_λ. Then we see that $\mathfrak{N}_\lambda$ $\left(0 \leqq \lambda \leqq \left[\frac{n}{2}\right]\right)$ are open sets of $\mathfrak{M}$ and that the maximal orbits of $(\mathfrak{M}, H)$ are just given by $\mathfrak{N}_\lambda$. Moreover we know that the group $G_0(L_\lambda)$ may be identified with the subgroup of $GL(\mathfrak{g}_{-1}) = GL(n, \boldsymbol{C})$ consisting of all the elements $b \in GL(n, \boldsymbol{C})$ such that $bI_\lambda{}^t\bar{b} = \rho I_\lambda$ with some $\rho \in \boldsymbol{R}$. We have, as before, $\mathfrak{g}_p(L_\lambda) = \{0\}$ for all $p \geqq 1$. Hence we have $\dim \mathfrak{g}(L_\lambda) = \dim \mathfrak{g}_{-2} + \dim_R \mathfrak{g}_{-1} + \dim \mathfrak{g}_0(L_\lambda) = 2(n^2+n)$.

(3) The case where $n' = n^2$ $(n \geqq 1)$. This case is just analogous to case (3) in § 7. We denote by $\mathfrak{N}_0$ the subset of $\mathfrak{M}$ consisting of all the elements $L \in \mathfrak{M}$ which map $\bar{\Lambda}^2\mathfrak{g}_{-1}$ isomorphically onto $\mathfrak{g}_{-2}$. Then we see that $\mathfrak{N}_0$ is open and dense in $\mathfrak{M}$ and that $\mathfrak{N}_0$ is the unique maximal orbit of $(\mathfrak{M}, H)$. Let L be a fixed element of $\mathfrak{N}_0$ and identify $\mathfrak{g}_{-2}$ with $\bar{\Lambda}^2\mathfrak{g}_{-1}$ by the isomorphism L. As before the group $G_0(L)$ may be identified with the group $GL(\mathfrak{g}_{-1}) = GL(n, \boldsymbol{C})$, and we have $\mathfrak{g}_1(L) \neq \{0\}$ and $\mathfrak{g}_p(L) = \{0\}$ $(p > 2)$. Since both the representations ρ_{-2} and ρ_{-1} of $\mathfrak{g}_0(L)$ on $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are irreducible, we know from Prop. 2.1 that $\mathfrak{g}(L)$ is a simple Lie algebra and that $\dim \mathfrak{g}(L) = 2(\dim \mathfrak{g}_{-2} + \dim_R \mathfrak{g}_{-1}) + \dim \mathfrak{g}_0(L) = 4(n^2+n)$. More precisely, $\mathfrak{g}(L)$ is isomorphic with the simple Lie algebra $\mathfrak{su}(n, n+1)$, and we have the natural isomorphisms: $\mathfrak{g}_0(L) = \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}^*$, $\mathfrak{g}_1(L) = \mathfrak{g}_{-1}^*$ and $\mathfrak{g}_2(L) = \bar{\Lambda}^2\mathfrak{g}_{-1}^*$, $\mathfrak{g}_{-1}^*$ being the dual space of the complex vector space $\mathfrak{g}_{-1}$.

Remark 6. Let f be an imbedding of type $(1, 2)$ from a 4-dimensional connected manifold M to a 3-dimensional complex manifold $\tilde{M}$. We have proved that the Lie algebra of all the infinitesimal automorphisms of (M, f) is generally finite dimensional and of dimension $\leqq 5$. Note that $n' = 2 > 1 = n^2$. The proof of this fact uses a real graded Lie algebra of the form $\mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0$, where $\mathfrak{g}_{-1}$ is equipped with a structure of 1-dimensional complex vector space and $\dim \mathfrak{g}_p = 1$ ($p = -3, -2, 0$).

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Bibliography

[1] E. Cartan, Les sous-groupes des groupes continue de transformations, Ann. Sci. Ecole Norm. Sup., **25** (1908), 57-194.
[2] E. Cartan, Les systèmes de Pfaff à cinq variables et les équations aux derivées partielles du second ordre, Ann. Sci. Ecole Norm. Sup., **27** (1910), 109-192.
[3] S. Kobayashi, Theory of connections, Ann. Mat. Pura Appl., **43** (1957), 119-194.
[4] T. Ochiai, On the automorphism group of a G-structure, J. Math. Soc. Japan, **18** (1966), 189-193.
[5] I. M. Singer and S. Sternberg, The infininite group of Lie and Cartan, J. Analyse Math., **15** (1965), 1-114.
[6] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan, **14** (1962), 397-429.
[7] N. Tanaka, Graded Lie algebras and geometric structures, Proc. U. S.—Japan Seminar in Diff. Geometry (1965), 147-150.
[8] N. Tanaka, On generalized graded Lie algebras and geometric structures II (in preparation).
[9] H. Whitney and F. Bruhat, Quelques propriétés fondamentales des ensembles analytiques-réels, Comment. Math. Helv., **33** (1959), 132-160.