

## Isometric immersions of Riemannian manifolds

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### Introduction.

It seems to be natural in differential geometry to conjecture that many theorems proved under the condition that the underlying manifolds are compact can be extended to manifolds with complete Riemannian metric under some suitable additional conditions.

Now, it is clear that for any smooth function  $f$  on a compact manifold there is a point  $p$  such that  $\nabla_i f(p) = 0$  and  $\nabla_i \nabla_j f(p)$  is negative semi-definite. It is also clear that if a smooth function  $f(x)$  on a real line  $R$  has an upper bound, then for any  $\varepsilon > 0$  there is  $x \in R$  such that  $f'(x)$  and  $f''(x)$  are  $< \varepsilon$ . This simple fact, however, can not be extended in general to a complete Riemannian manifold. That is, there is a complete Riemannian manifold  $M$  and a bounded function  $f$  on  $M$  such that  $m(p) = \{X^i X^j \nabla_i \nabla_j f(p); \|X\| = 1\}$  is always larger than  $a > 0$ . This example can easily be constructed on  $R^2$  with metric  $dr^2 + g(r)d\theta^2$  (in the polar coordinate expression). Let  $f(r, \theta) = f(r) = \frac{r^2}{1+r^2}$ . Since

$$\nabla \nabla f (= \nabla_i \nabla_j f dx^i dx^j) = f''(r) dr^2 + \frac{1}{2} f'(r) g'(r) d\theta^2,$$

one can choose a suitable function  $g(r)$  so that it satisfies (a)  $g(r)$  is smooth and  $g(r) = r$  for  $0 \leq r < 1/2$ , (b)  $g(r)$  is a solution of  $g'(r)/g(r) = 2c/f'(r)$  (for example  $g(r) = \exp \int_1^r c(1-r)^2/r dr$  for  $r \geq 1$ ). In this example, one can see easily that the sectional curvature has no lower bound.

In this paper, there will be proved first of all a generalization of this example, that is:

**THEOREM A.** *Let  $M$  be a connected and complete Riemannian manifold whose sectional curvature  $K(X, Y)$  has a lower bound i. e.  $K(X, Y) \geq -K_0$ . If a smooth function  $f$  on  $M$  has an upper bound, then for any  $\varepsilon > 0$ , there is a point  $p \in M$  such that  $\|\text{grad } f(p)\| < \varepsilon$  and  $m(p) = \max \{X^i X^j \nabla_i \nabla_j f(p); \|X\| = 1\} < \varepsilon$ .*

For an application of this theorem, an isometric immersion of  $M$  into the Euclidean  $N$ -space  $R^N$  will be considered. It is clear that if  $M$  is compact,

then for any isometric immersion  $\varphi$  of  $M$  into  $R^N$ , there are a point  $p$  and a unit normal  $\xi$  at  $\varphi(p)$  such that the second fundamental form at  $\varphi(p)$  with respect to  $\xi$  is positive definite.

In our case, we have the following theorem :

**THEOREM B.** *Let  $\varphi$  be an isometric immersion of a connected and complete Riemannian manifold  $M$  into  $R^N$ . Assume that the sectional curvature of  $M$  has a lower bound. If there is a unit vector  $n$  at the origin of  $R^N$  such that*

$$\langle \varphi(p), n \rangle / \|\varphi(p)\| \geq \delta > 0$$

*for all  $p \in M$ , then there exist a point  $p_0$  and a unit normal vector  $\xi$  at  $\varphi(p_0)$  such that the second fundamental form at  $\varphi(p_0)$  with respect to  $\xi$  is positive definite.*

From this theorem, we can see immediately the following :

**COROLLARY C.** *Let  $\varphi$  be an isometric immersion of a connected and complete Riemannian manifold  $M$  into  $R^N$ . If  $\varphi(M)$  is a minimal submanifold in  $R^N$ , then for any unit vector  $n$  at the origin of  $R^N$  and for any positive  $\delta$ , there exists a point  $p$  such that  $\langle \varphi(p), n \rangle / \|\varphi(p)\| < \delta$ .*

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### 1. Geodesic spheres and their second fundamental forms.

Let  $M$  be a connected and complete Riemannian  $C^\infty$ -manifold of dimension  $n$  and  $g(t)$  be a geodesic going from  $g(0) = p$  to  $g(h) = q$ . The parameter  $t$  represents the arc length. Assume that  $g(t)$  is not a conjugate point of  $p$  with respect to the geodesic  $g$  for any  $0 < t \leq h$  and that  $g(t); 0 \leq t \leq h$  does not intersect itself. Let  $S_p(a) = \{X \in T_p(M); \|X\| = a\}$  and  $T_t = dg(t)/dt$ . Then,  $T_0 \in S_p(1)$ .

Since  $g(t)$  is not a conjugate point for any  $0 < t \leq h$  and  $g(t), 0 \leq t \leq h$  does not self-intersect, there exists a neighborhood  $V$  of  $T_0$  in  $S_p(1)$  such that  $\text{Exp}_p : (0, h + \varepsilon) \times V \rightarrow M$  is a diffeomorphism for sufficiently small  $\varepsilon > 0$ , where  $\text{Exp}_p(t, X) = \text{Exp}_p tX$ .

Putting  $S_p(a) = \text{Exp}_p\{S_p(a) \cap (0, h + \varepsilon) \times V\}$ ,  $S_p(a)$  is an  $(n-1)$ -dimensional submanifold of  $M$  for every  $0 < a \leq h$ . Let  $W = \text{Exp}_p\{(0, h + \varepsilon) \times V\}$ .

The following lemma is well-known.

**LEMMA 1.** *Let  $x \in S_p(a)$ ,  $0 < a \leq h$ . For any curve  $C$  joining  $p$  and  $x$  in  $W$ , the length of  $C$  is no less than  $a$ .*

The following lemma is an immediate consequence of Lemma 1.

**LEMMA 2.** *For any points  $x \in S_p(a)$  and  $y \in S_p(h)$  and for any arc  $C$  joining  $x$  and  $y$  in  $W$ , the length of the arc  $C$  is no less than  $h - a$ .*

Let  $X_t$  be a unit vector field that is displaced parallel along  $g$  and is tan-

gent to  $S_p(a)$  and  $S_p(h)$  at  $g(a)$  and  $g(h)$  respectively. We consider the variation of the geodesic  $g$  by the vector field  $X$  in the same way as in § 1 of [1].

The following Lemma is clear from the Lemma in § 1 of [1] and Lemma 2 above.

LEMMA 3. *The first and second variations of arc length are*

$$L'_X(0) = 0, L''_X(0) = H_q(X, X) - H_{g(a)}(X, X) - \int_a^h K(T, X) dt \geq 0,$$

where  $K(T, X)$  is the Riemannian sectional curvature corresponding to the 2-plane  $T \wedge X$  and  $H_{g(a)}(X, X)$  is the second fundamental form for  $S_p(a)$  at  $g(a)$  corresponding to the unit normal  $T$ , evaluated at the tangent vector  $X$ .

Let  $ds^2 = dt^2 + g_{\alpha\beta} d\theta^\alpha d\theta^\beta$  be the metric of  $M$  in the polar coordinate expression with radius  $t$  and with center  $p$ . Since  $(t, \theta^1, \dots, \theta^{n-1})$  can be considered as a coordinate of  $W$ , one can see by an elementary calculation that

$$\frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t}(q) d\theta^\alpha(X) d\theta^\beta(X) = -H_q(X, X),$$

for every  $X \in T_q(S_p(h))$ . Since  $H_q(X, Y) = H_{\alpha\beta} d\theta^\alpha(X) d\theta^\beta(Y)$ ,  $H(X, Y)$  can be considered as a 2-tensor field around  $q$  in  $M$ . That is, for any  $X, Y \in T_q(M)$ ,  $H_q(X, Y)$  is defined by  $H_{\alpha\beta}(q) d\theta^\alpha(X) d\theta^\beta(Y)$ .

On the other hand,

$$(\nabla\nabla t)(x) = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t}(x) d\theta^\alpha d\theta^\beta.$$

Thus,

$$(\nabla\nabla t^2)(q)(X, X) (= X^i X^j (\nabla_i \nabla_j t^2)(q)) = 2\{dt(X)\}^2 - 2hH_q(X, X),$$

for every  $X \in T_q(M)$ .

LEMMA 4. *Notations being as above, if the sectional curvature  $K(X, Y)$  of  $M$  has a lower bound, i. e.  $K(X, Y) \geq -K_0, K_0 > 0$ , then*

$$(\nabla\nabla t^2)(q)(X, X) \leq 2\{1 + h[K_0(h-a) - H_{g(a)}(X, X)]\},$$

where  $X = X_t$  is the vector field that is displaced parallel along the geodesic  $g$ .

This Lemma is an immediate result of Lemma 3.

## 2. Bounded functions of $M$ .

Let  $M$  be a connected and complete Riemannian manifold of dimension  $n$  and  $f$  a  $C^2$ -function on  $M$  with an upper bound. Let  $b = \sup f$ . For a fixed point  $p$ , we assume without loss of generality that  $f(p) = 0$ .

Considering the graph  $\Gamma = \{(f(x), x); x \in M\}$ ,  $\Gamma$  is a closed submanifold of  $R \times M$ , where  $R$  is the real line with natural Riemannian metric and  $R \times M$  is

the product manifold with product metric.

Though many relations exist between  $M$  and  $R \times M$ , we shall use only the following in the remainder of this paper ;

LEMMA 5. *Let  $\pi$  be the projection from  $R \times M$  onto  $M$ . (i) If  $g'$  is a geodesic in  $R \times M$ , then so is  $g = \pi g'$  in  $M$ . (ii) Let  $y, x$  be two points of a geodesic  $g'$  in  $R \times M$ .  $x$  is a conjugate point of  $y$  with respect to  $g'$  if and only if  $\pi(x)$  is a conjugate point of  $\pi(y)$  with respect to the geodesic  $\pi g'$ . (iii) Let  $(k, y)$  and  $(a, x)$  be two points of  $R \times M$  and  $g'$  a geodesic segment from  $(k, y)$  to  $(a, x)$ ; then  $L(g')^2 = L(\pi g')^2 + (k - a)^2$ , where  $L(g')$  and  $L(\pi g')$  are length of  $g'$  and  $\pi g'$  respectively.*

PROOF. Let  $(l, x^1, \dots, x^n)$  be a local coordinate of  $R \times M$  which is a product of local coordinates of  $R$  and  $M$ . Thus, the equations of a geodesic are

$$\frac{d^2 l}{ds'^2} = 0 \dots\dots\dots (1)$$

$$\frac{d^2 x^i}{ds'^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds'} \frac{dx^k}{ds'} = 0 \dots\dots\dots (2)$$

(1 ≤ i ≤ n)

where  $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$  is the Christoffel's symbol of  $M$  and  $s'$  is the arc length. Thus, (i) follows from the equality (2).

Let  $s', s$  and  $l$  be arc length from  $(k, y)$ ,  $y$  and  $k$  along  $g', \pi g'$  and  $R$  respectively. Then,  $l = As'$  and  $s = as'$  from the equalities (1) and (2). It follows that  $s'^2 = l^2 + s^2$  because of the fact that  $A^2 + a^2 = 1$  (obtained from  $ds'^2 = ds^2 + dl^2$ ).

As for (ii), one can see this easily by the Jacobi equation.

For each positive integer  $k$  and a fixed point  $p$  in  $M$ , consider a point  $\hat{p}_k = (kb, p)$  and a geodesic segment  $\hat{g}_k$  from  $\hat{p}_k$  to  $\Gamma$  which attains the distance between  $\hat{p}_k$  and  $\Gamma$ . Let  $(f(q_k), q_k)$  be another end point of  $\hat{g}_k$ .

LEMMA 6. *Notations being as above, there is no conjugate point of  $\hat{p}_k$  on  $\hat{g}_k$  with respect to  $\hat{g}_k$ .*

PROOF. Let  $\hat{g}_k(t)$  be the geodesic parametrized by the arc length and set  $\hat{g}_k(0) = \hat{p}_k, \hat{g}_k(l'_k) = (f(q_k), q_k), \hat{g}_k([0, l'_k]) = \hat{g}_k$ . It is easy to show that  $\hat{g}_k(t); 0 < t < l'_k$  is not a conjugate point of  $\hat{p}_k$  with respect to  $\hat{g}_k$ .

Assume  $\hat{g}_k(l'_k)$  is a conjugate point of  $\hat{p}_k$  with respect to  $\hat{g}_k$ . Take a point  $\hat{g}_k(l'_k + \epsilon)$  on the line obtained by extension of  $\hat{g}_k$ . Since  $\Gamma$  is a differentiable submanifold of  $R \times M$ , there exists a sufficiently small  $\epsilon > 0$  such that the  $\epsilon$ -sphere of center  $\hat{g}_k(l'_k + \epsilon)$  is in contact with  $\Gamma$  at one point  $\hat{g}_k(l'_k)$ . Since  $\hat{g}_k(l'_k)$  is a conjugate point, there is a geodesic segment  $\hat{g}'$  from  $\hat{p}_k$  to  $\hat{g}_k(l'_k + \epsilon)$  such that  $L(\hat{g}_k) \leq l'_k + \epsilon$ . Let  $\hat{q}'$  be a point of intersection of  $\Gamma$  and  $\hat{g}'$ . Since

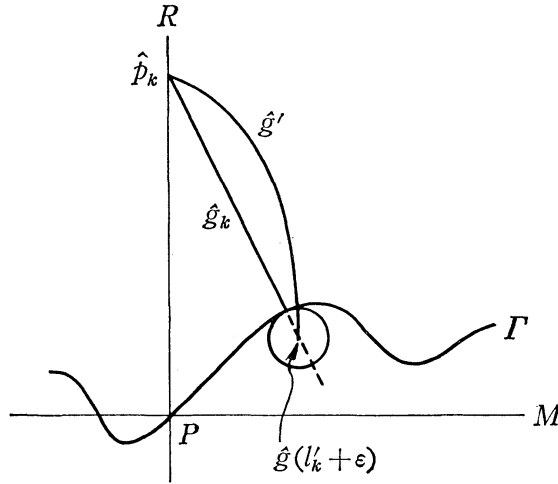


Fig. 1.

$\text{dist.}(\hat{q}', \hat{g}_k(l'_k + \varepsilon)) > \varepsilon$  we have  $\text{dist.}(\hat{p}_k, \hat{q}') < l'_k$ . This is a contradiction.

Now, notations being as above,  $\pi(\hat{g}_k(t))$  is not a conjugate point of  $p$  with respect to  $g_k = \pi\hat{g}_k$  for any  $0 < t \leq l'_k$ , and  $g_k$  does not intersect itself. Let  $T_k$  be the unit tangent vector of  $g_k$  at  $p$ . There is a neighborhood  $W_k$  of  $T_k$  such that  $\text{Exp}_p$  is a diffeomorphism from  $(0, h_k + \delta_k) \times W_k$  into  $M$ , where  $h_k = L(\pi\hat{g}_k)$ . Set  $\phi(x) = \text{Exp}_p^{-1}(x)$  for  $x \in \text{Exp}_p\{(0, h_k + \delta_k) \times W_k\}$  and define the function  $F_k(x)$  by

$$F_k(x) = kb - \sqrt{l_k^2 - \|\phi(x)\|^2}.$$

There is no difficulty in verifying the following:

LEMMA 7. (i)  $f(x) \leq F_k(x)$  and  $f(q_k) = F_k(q_k)$  for every  $k \geq 1$ .

$$(ii) \quad (\nabla\nabla F_k)(q_k) = \frac{1}{2} \frac{(\nabla\nabla \|\phi\|^2)(q_k)}{kb - f(q_k)} + \frac{1}{4} \frac{(\nabla \|\phi\|^2)(q_k)(\nabla \|\phi\|^2)(q_k)}{(kb - f(q_k))^3}.$$

PROOF. From (iii) of Lemma 5, we see that  $l_k^2 \leq (kb - f(x))^2 + \|\phi(x)\|^2$ . Thus,  $f(x) \leq kb - \sqrt{l_k^2 - \|\phi(x)\|^2}$ . (i) follows from this. (ii) is a direct calculation.

Assume  $K(X, Y) \geq -K_0$ ; then, combining with Lemma 4, we see that

$$\begin{aligned} (\nabla\nabla f)(q_k)(X, X) &\leq (\nabla\nabla F_k)(q_k)(X, X) \\ &\leq \frac{1}{kb - f(q_k)} \{1 + h_k[K_0(h_k - a) - H_{g_k(a)}(X, X)]\} \\ &\quad + \frac{1}{(k-1)^3 b^3} h_k^2 \|X\|^2, \quad k \geq 2. \end{aligned}$$

Since all  $g_k$  are geodesic segments from  $p$  to  $q_k$  in  $M$ , all  $g_k(a)$  are points of  $\tilde{\mathcal{S}}_p(a) = \{\text{Exp}_p X; \|X\| = a\}$ . For sufficiently small  $a$ ,  $\tilde{\mathcal{S}}_p(a)$  is a differentiable

submanifold of  $M$ . Let  $m = \min \{H_q(X, X); q \in \tilde{S}_p(a), \|X\| = 1\}$ . Then,

$$(\nabla\nabla f)(q_k)(X, X) \leq \frac{1}{kb-f(q_k)} \{K_0 h_k^2 - (aK_0 + m)h_k + 1\} + \frac{h_k^2}{(k-1)^3 b^3} \dots \quad (3)$$

for all  $\|X\| = 1$ .

LEMMA 8. *Notations being as above, if  $h_k$  is bounded for  $k \rightarrow \infty$ , then there exists a point  $q_\infty$  in  $M$  such that  $(\text{grad } f)(q_\infty) = 0$  and  $(\nabla\nabla f)(q_\infty)$  is negative semi-definite.*

PROOF. Since  $\text{grad } f(q_k) = \text{grad } F_k(q_k)$ , we see that

$$\|\text{grad } f(q_k)\| = \frac{h_k}{kb-f(q_k)} \cdot \dots \quad (4)$$

Since  $\{h_k\}$  is bounded, there exists a subsequence  $\{q_{k'}\}$  which converges to  $q_\infty$ . Thus, from the equality (4) and the inequality (3), we have that  $(\text{grad } f)(q_\infty) = 0$  and  $(\nabla\nabla f)(q_\infty)$  is negative semi-definite.

Thus, in the remainder of this section, assume that  $h_k \rightarrow \infty$  for  $k \rightarrow \infty$ .

Since  $K_0 > 0$ , we see that  $K_0 h_k^2 - (aK_0 + m)h_k + 1 \geq 0$  for sufficiently large  $k$ . It follows

$$(\nabla\nabla f)(q_k)(X, X) \leq \frac{1}{(k-1)b} \{K_0 h_k^2 - (aK_0 + m)h_k + 1\} + \frac{h_k^2}{(k-1)^3 b^3} \dots \quad (3)'$$

By the assumption  $f(p) = 0$  and the definition of  $l'_k$  we see that  $l'_k \leq kb$  and thus we can show easily that  $f(q_k) \geq 0$  for all  $k \geq 1$ .

Since  $l'_k \leq \text{dist}(\hat{p}_k, (f(q_j), q_j))$ , one obtains

$$(kb-f(q_k))^2 + h_k^2 \leq (kb-f(q_j))^2 + h_j^2.$$

Thus,  $h_k^2 - h_j^2 \leq 2kb(f(q_k) - f(q_j)) - (f(q_k)^2 - f(q_j)^2)$ .

Since  $h_k \rightarrow \infty$ , for  $k \rightarrow \infty$ , there exists  $k'$  for every  $k$  such that  $k' > k$ ,  $h_{k'} \geq h_k$  and  $h_{k''} < h_k$  for every  $k'', k < k'' < k'$ . If  $f(q_k) > f(q_{k'})$ , then by the above inequality, we see that  $h_k > h_{k'}$ . Thus, one can choose a subsequence  $\{k_i\}$  such that (i)  $k_i < k_{i+1}$ , (ii)  $h_{k_{i+1}} \geq h_{k_i}$ , (iii)  $h_{k'} < h_{k_i}$  for any  $k_i < k' < k_{i+1}$  and (iv)  $f(q_{k_{i+1}}) \geq f(q_{k_i})$ .

Since

$$(k_{i+1}b - f(q_{k_{i+1}}))^2 + h_{k_{i+1}}^2 \leq (k_{i+1}b - f(q_{k_i}))^2 + h_{k_i}^2$$

we see easily that

$$h_{k_{i+1}}^2 - h_{k_i}^2 \leq 2k_{i+1}b(f(q_{k_{i+1}}) - f(q_{k_i})).$$

It follows that

$$\sum_{i=1}^{\infty} \frac{h_{k_{i+1}}^2 - h_{k_i}^2}{k_{i+1}} \leq 2b \sum_{i=1}^{\infty} (f(q_{k_{i+1}}) - f(q_{k_i})) \leq 2b^2.$$

Thus,

$$\sum_{i=1}^{\infty} \frac{1}{k_{i+1}} (h_{k_{i+1}}^2 - h_{k_i}^2)$$

is absolutely convergent.

LEMMA 9. Notations being as above,  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} h_k^2$  is absolutely convergent.

PROOF. Since  $h_{k'}^2 \leq h_{k_i}^2$  for any  $k_i \leq k' < k_{i+1}$ , one obtains that

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{1}{k_{i+1}} (h_{k_{i+1}}^2 - h_{k_i}^2) - \frac{1}{k_1} h_{k_1}^2 &= \sum_{i=1}^{\infty} \left( \frac{1}{k_i} - \frac{1}{k_{i+1}} \right) h_{k_i}^2 \\ &= \sum_{i=1}^{\infty} \left( \sum_{k=k_i}^{k_{i+1}-1} \frac{1}{k(k+1)} \right) h_{k_i}^2 \geq \sum_{k=1}^{\infty} \frac{1}{k(k+1)} h_k^2. \end{aligned}$$

It follows that  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} h_k^2$  is absolutely convergent.

LEMMA 10. For any  $k > 1$ , there is  $k' > k$  such that  $h_{k'}^2 \leq \frac{k'+1}{\log k'}$ .

PROOF. Assume there is  $k_0$  such that  $h_k^2 > \frac{k+1}{\log k}$  for any  $k \geq k_0$ . Then,  $\sum_{k=1}^{\infty} \frac{h_k^2}{k(k+1)} > \sum_{k=k_0}^{\infty} \frac{1}{k \log k} = \infty$ .

PROOF OF THEOREM A. It is easy to show that  $\|\text{grad } f(q_k)\| \leq \frac{h_k}{(k-1)b}$ . From inequality (3)',

$$(\nabla \nabla f)(q_k)(X, X) \leq \frac{1}{(k-1)b} \{K_0 h_k^2 - (aK_0 + m)h_k + 1\} + \frac{h_k^2}{(k-1)^3 b^3},$$

for sufficiently large  $k$ .

From Lemma 10 above, there exists a subsequence  $\{k_j\}$  such that  $h_{k_j}^2 \leq (k_j+1)/\log k_j$ . It follows that

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{h_{k_j}}{(k_j-1)b} &\leq \lim_{j \rightarrow \infty} \frac{k_j+1}{(k_j-1)b \log k_j} = 0 \\ \overline{\lim}_{j \rightarrow \infty} (\nabla \nabla f)(q_{k_j})(X, X) &\leq \lim_{j \rightarrow \infty} \frac{(k_j+1)K_0}{(k_j-1)b \log k_j} + \lim_{j \rightarrow \infty} \frac{k_j+1}{(k_j-1)^3 b^3 \log k_j} = 0. \end{aligned}$$

This completes the proof of Theorem A.

From this proof, one can see more precisely the following:

THEOREM A'. Let  $f$  be a  $C^2$ -function on a complete and connected Riemannian manifold having an upper bound. Assume that the sectional curvature has a lower bound. Then, for an arbitrarily fixed point  $p$  and for any  $\varepsilon > 0$ , there exists a point  $q$  depending on  $p$  such that (i)  $\|\text{grad } f(q)\| < \varepsilon$ , (ii)  $(\nabla \nabla f)(q)(X, X) < \varepsilon$  for  $\|X\|=1$  and (iii)  $f(q) \geq f(p)$ .

**3. Isometric immersions of a complete and connected Riemannian manifold.**

As an application of Theorem A or A', one considers an isometric immersion  $\varphi$  of a complete and connected Riemannian manifold  $M$  into Euclidean space  $R^N$  with natural Euclidean metric. In this section, Theorem B will be proved.

Assume there exists a unit vector  $n$  at the origin of  $R^N$  such that  $\langle \varphi(x), n \rangle / \|\varphi(x)\| \geq a'$  for a fixed  $a' > 0$ . Let  $R^{N-1}$  be the subspace of  $R^N$  which is orthogonal to  $n$ . Set  $p = p(x) = \varphi(x)$  and denote by  $p'$  the  $R^{N-1}$ -component of  $p$ .

Assume without loss of generality that  $\langle p, n \rangle^2 - a'^2 \langle p', p' \rangle \geq 1$  for every  $p = p(x)$ ,  $x \in M$ . For a positive  $a$ ,  $a' > a > 0$ , set

$$f_a(x) = -\langle p(x), n \rangle + \sqrt{a^2 \langle p'(x), p'(x) \rangle + 1}.$$

The meaning of this function is clear, if one changes the equality of the definition to  $(\langle p, n \rangle + f_a(x))^2 - a^2 \langle p', p' \rangle = 1$ . Let  $p_0 = p(x_0)$  for some fixed point  $x_0 \in M$ . It is easy to see that  $\{p(x); f_a(x) \geq f_a(x_0)\}$  is contained in a compact

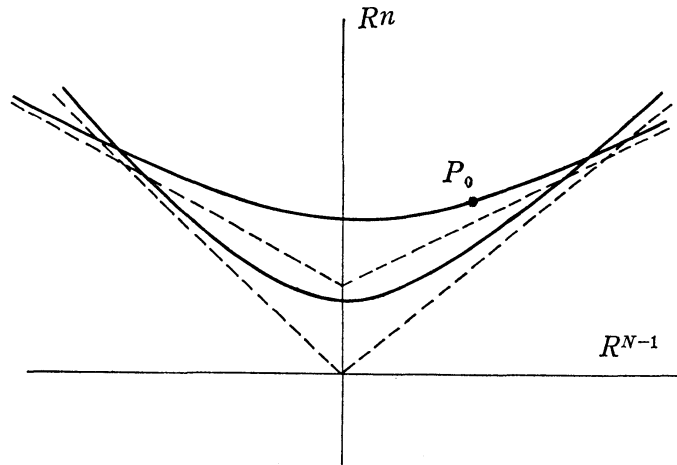


Fig. 2.

subset  $K$  for any  $a$ ,  $0 < a < a' - \delta'$ . Thus, there is a real number  $a$  such that  $a^2 \langle p(x), p(x) \rangle \leq \frac{1}{2}$  for any  $x$  satisfying  $f_a(x) \geq f_a(x_0)$ . Choose such a number  $a$  and fix it throughout the remainder of this paper. Put  $f(x) = f_a(x) - f_a(x_0)$ . Clearly,  $f(x)$  has an upper bound and  $f(x_0) = 0$ .

By the definition of  $f(x)$ , one has directly that



$$\begin{aligned} \nabla_i \nabla_j f &= -\langle p_{i,j}, n \rangle + \frac{a^2(\langle p_{i,j}, p \rangle - \langle p, n \rangle \langle p_{i,j}, n \rangle)}{\sqrt{a^2(\langle p, p \rangle - \langle p, n \rangle^2) + 1}} \\ &+ \frac{a^4}{\sqrt{a^2(\langle p, p \rangle - \langle p, n \rangle^2) + 1}} \left\{ (\langle p_i, p_j \rangle - \langle p_i, n \rangle \langle p_j, n \rangle) (\langle p, p \rangle - \langle p, n \rangle^2 + \frac{1}{a^2}) \right. \\ &\left. - (\langle p_i, p \rangle - \langle p, n \rangle \langle p_i, n \rangle) (\langle p_j, p \rangle - \langle p, n \rangle \langle p_j, n \rangle) \right\} \end{aligned}$$

where  $p_i = \nabla_i p$ ,  $p_{i,j} = \nabla_j \nabla_i p$  for some coordinate neighborhood of  $M$ . Let  $\hat{H}_{ij}$  be the first and second terms of  $\nabla_i \nabla_j f$  and  $R_{ij}$  the third term. Let  $N_\xi$  be the normal vectors of  $M$  in  $R^N$ , where  $\xi = 1, 2, \dots, N-n$ .

Since  $p_{i,j} = \sum_{\xi} H_{ij} N_\xi$ , we see easily that  $\hat{H}(X, X) = \hat{H}_{ij} X^i X^j = \sum_{\xi} H(X, X) \langle N_\xi, \hat{m} \rangle$ , where

$$\hat{m} = \frac{a^2 p + a^2 \langle p, n \rangle n}{\sqrt{a^2 \langle p', p' \rangle + 1}} - n.$$

For  $X$ ,  $\|X\| = 1$ , we have that

$$\begin{aligned} R(X, X) &= \frac{a^4}{\sqrt{a^2 \|p'\|^2 + 1}} (1 - \langle X, n \rangle^2) (\langle p, p \rangle - \langle p, n \rangle^2 + \frac{1}{a^2}) \\ &\quad - (\langle X, p \rangle - \langle p, n \rangle \langle X, n \rangle)^2, \end{aligned}$$

where  $R(X, X) = R_{ij} X^i X^j$ .

By the definition of  $f(x)$  and Theorem A', there exists a sequence  $\{x_n\}$  in  $M$  such that  $f(x_n) \geq 0$  and

$$\lim_{n \rightarrow \infty} \|\text{grad } f(x_n)\| = \lim_{n \rightarrow \infty, \|X\|=1} \max \left\{ -\langle X, n \rangle + \frac{a^2 \langle X, p \rangle - \langle p, n \rangle \langle X, n \rangle}{\sqrt{a^2 \|p'(x_n)\|^2 + 1}} \right\} = 0 \quad \dots (5)$$

$$\lim_{n \rightarrow \infty} (\nabla \nabla f)(x_n)(X, X) = \lim_{n \rightarrow \infty} \sum_{\xi} H_{x_n}(X, X) \langle N_\xi, \hat{m} \rangle + \lim_{n \rightarrow \infty} R_{x_n}(X, X) = 0, \dots (6)$$

for  $X \in T_{x_n}(M)$  and  $\|X\| = 1$ , where the tangent space  $T_{x_n}(M)$  at  $x_n$  is identified with the subspace of  $R^N$  by the immersion  $\varphi$ .

Thus, we have only to show that  $R_{x_n}(X, X) \geq \delta' > 0$  for sufficiently large  $n$ .

Let  $\hat{X}$  be the  $R^{N-1}$ -component of  $X$ . Thus,  $X = \lambda n + \hat{X}$ ,  $\lambda = \langle X, n \rangle$  and then  $\langle X, p \rangle - \langle p, n \rangle \langle X, n \rangle = \langle \hat{X}, p \rangle$ . It follows that

$$\begin{aligned} &\frac{1}{\|p\|^2} (1 - \langle X, n \rangle^2) (\langle p, p \rangle - \langle p, n \rangle^2 + \frac{1}{a^2}) - (\langle X, p \rangle - \langle p, n \rangle \langle X, n \rangle)^2 \\ &= (1 - \lambda^2) \left( 1 - \mu^2 + \frac{1}{a^2 \|p\|^2} \right) - \langle \hat{X}, p_0 \rangle^2, \end{aligned}$$

where  $\mu = \langle p, n \rangle / \|p\|$  and  $p_0 = p / \|p\|$ . Since  $\langle \hat{X}, p_0 \rangle \leq \|\hat{X}\| = 1 - \mu^2$ , one obtains that

$$R(X, X) \geq \frac{a^4 \|p\|^2}{\sqrt{a^2 \|p'\|^2 + 1}} (1 - \lambda^2) \left( \frac{1}{a^2 \|p\|^2} - \mu^2 \right),$$

for all  $x \in M$ . Since  $f(x_n) \geq 0$ , we see that  $a^2 \|p(x_n)\|^2 < 1/2$ . It follows that  $1/a^2 \|p\|^2 - \mu^2 \geq \delta > 0$  for all  $x_n$ .

Assume that  $1 - \lambda^2$  is not larger than  $\delta'' > 0$  for every  $x_n$  and for every element of  $T_{x_n}(M)$  satisfying  $\|X\|=1$ . Then, there is a subsequence  $\{x_{n'}\}$  of  $\{x_n\}$  such that, for every  $x_{n'}$ , there exists  $X_{n'} \in T_{x_{n'}}(M)$  and  $1 - \langle X_{n'}, n \rangle^2$  converges to 0 for  $n' \rightarrow \infty$ . Put  $\lambda_{n'} = \langle X_{n'}, n \rangle$ . Without loss of generality, we assume  $\lambda_{n'} \leq 0$ . Since  $X_{n'}$  converges to  $n$ , one obtains that  $\lim \| \text{grad} f(x_{n'}) \| = 1$ , contradicting the equality (5). Thus, there exists an  $\varepsilon > 0$  such that  $1 - \lambda^2 \geq \varepsilon > 0$  for sufficiently large  $n$  and for all  $X \in T_{x_n}(M)$  satisfying  $\|X\|=1$ . It follows that

$$R_{x_n}(X, X) \geq \frac{a^4 \|p_n\|^2 \delta \varepsilon}{\sqrt{a^2 \|p(x_n)\|^2 + 1}} \quad \text{for a sufficiently large } n.$$

Since  $\|p_n\|$  is bounded,  $\frac{a^4 \|p_n\|^2}{\sqrt{a^2 \|p(x_n)\|^2 + 1}} \geq C$  for some positive constant. Thus, we see that

$$\lim_{n \rightarrow \infty} (\nabla \nabla f)(x_n)(X, X) \geq \lim_{n \rightarrow \infty} \sum_{\xi} H_{x_n}(X, X) \langle N, \hat{m} \rangle_{\xi} + C \delta \varepsilon.$$

It follows that  $\sum_{\xi} H_{x_n}(X, X) \langle N, \hat{m} \rangle_{\xi}$  is negative definite for sufficiently large  $n$ , since  $\sum_{\xi} \langle N, \hat{m} \rangle_{\xi} H_{x_n}$  is the second fundamental form with respect to the normal vector  $\hat{N}$  at  $x_n$  with the coefficient  $\langle \hat{N}, N \rangle_{\xi} = \langle N, \hat{m} \rangle_{\xi}$ . This completes the proof of Theorem B.

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### Reference

- [ 1 ] T. Frankel, Manifolds with positive curvature, Pacific J. Math., 11 (1961), 165-174.