J. Math. Soc. Japan
Vol. 19, No. 2, 1967

Horizontal lifts from a manifold to its cotangent bundle

By K. YANO and E. M. PATTERSON

(Received Aug. 10, 1966)

§ 1. Introduction.

The concepts of vertical and complete lifts from a differentiable manifold M of class C^{∞} to its cotangent bundle ${}^{c}T(M)$ were introduced in a recent paper, [4]. Vertical lifts of functions, vector fields, 1-forms and tensor fields of type (1, 1) or (1, 2) were defined. The definitions of complete lifts were restricted to vector fields, tensor fields of type (1, 1) and skew-symmetric tensor fields of type (1, 2). In each case, the complete lift of a tensor field has the same type as the original; however vertical lifts do not have this property. In § 2 of the present paper, we summarise the details of the relevant formulae.

In the present paper we introduce another type of "lift" from M to ${}^{c}T(M)$, which we call the horizontal lift. We apply our definition to vector fields, tensor fields of type (1, 1) and connections in M. As in the previous paper, we obtain from our construction useful information about the relationships between the structures of M and ${}^{c}T(M)$.

The most significant difference between the constructions in the present paper and the earlier constructions is that we now assume that a symmetric affine connection is given in the manifold M. The definition of horizontal lift depends upon this connection, whereas the definitions of vertical and complete lifts were independent of connections.

§ 2. Notations and preliminary results.

Throughout, M denotes a differentiable manifold of class C^{∞} and of dimension n. Its cotangent bundle is denoted by ${}^{c}T(M)$ and $\pi: {}^{c}T(M) \to M$ is the projection mapping. We write U for a coordinate neighbourhood in M and $\pi^{-1}(U)$ for the corresponding coordinate neighbourhood in ${}^{c}T(M)$.

Suffixes A, B, C, D take the values 1 to $2n$. Suffixes $a, b, c, \cdots, h, i, j, \cdots$ take the values 1 to n and $\bar{i} = i+n$, etc. The summation convention for repeated indices is used. Whenever notations such as $(F_{B}{}^{A})$ are used for matrices, the suffix on the left indicates the column and the suffix on the right indicates the row.

We write $\mathcal{T}_s^r(M)$ for the set of tensor fields of class C^∞ and of type (r, s) in M. Vector fields in M are denoted by X, Y, Z, tensor fields of type (1, 1) by F, G, and tensor fields of type (1, 2) by S, T. The Lie product of X and Y is denoted by $[X, Y]$ and the Lie derivative with respect to X by $\mathcal{L}_X$.

If $f \in \mathcal{T}_0^0(M)$, the vertical lift f^V of f is the function in ${}^cT(M)$ defined by

$$f^V = f \circ \pi \tag{2.1}$$

If $X \in \mathcal{T}_0^1(M)$, the vertical lift X^V of X is the function in ${}^cT(M)$ defined by

$$X^V(A, p) = p(X_A) \tag{2.2}$$

where A is a point of M and p is a covariant vector at A, so that (A, p) is a point in ${}^cT(M)$ on the fibre over A. Two vector fields in ${}^cT(M)$ which have the same action on all functions of the form X^V are identically equal (see [4], Proposition 1).

If $\omega \in \mathcal{T}_1^0(M)$, the vertical lift ω^V of ω is a vector field in ${}^cT(M)$ satisfying

$$\omega^V(f^V) = 0 \tag{2.3}$$

and

$$\omega^V(X^V) = (\omega(X))^V. \tag{2.4}$$

If $F \in \mathcal{T}_1^1(M)$, the vertical lift F^V of F is a vector field in ${}^cT(M)$ satisfying

$$F^V(f^V) = 0 \tag{2.5}$$

$$F^V(X^V) = (FX)^V. \tag{2.6}$$

If $X \in \mathcal{T}_0^1(M)$, the complete lift X^C of X is a vector field in ${}^cT(M)$ satisfying

$$X^C(f^V) = (Xf)^V \tag{2.7}$$

$$X^C(Z^V) = [X, Z]^V. \tag{2.8}$$

Suppose that $\hat{S}$, $\tilde{T}$ are tensor fields in ${}^cT(M)$ of type $(0, r)$ or $(1, r)$ which have the same action on vector fields of the form Z^C, where $Z \in \mathcal{T}_0^1(M)$. Then $\hat{S} = \tilde{T}$ (see [4], Proposition 2).

If $S \in \mathcal{T}_2^1(M)$, the vertical lift S^V of S is a tensor field of type (1, 1) in ${}^cT(M)$, satisfying

$$S^V(\omega^V) = 0 \tag{2.9}$$

$$S^V(X^C) = (S_X)^V \tag{2.10}$$

where S_X is the tensor field of type (1, 1) in M given by

$$S_X(Y) = S(X, Y).$$

The vertical lifts S^V, T^V of the tensor fields S, T of type (1, 2) in M satisfy

$$S^V T^V = 0 \tag{2.11}$$

and if $F \in \mathcal{T}^1_1(M)$, then

$$S^V F^V = 0. \tag{2.12}$$

If $F \in \mathcal{T}^1_1(M)$, the complete lift F^C of F is a tensor field of type (1, 1) in ${}^cT(M)$, satisfying

$$F^C(\omega^V) = (\omega F)^V \tag{2.13}$$

$$F^C(X^C) = (FX)^C + (\mathcal{L}_X F)^V, \tag{2.14}$$

where ωF is defined by $(\omega F)(Y) = \omega(FY)$ for all $Y \in \mathcal{T}^1_0(M)$. If $S \in \mathcal{T}^1_2(M)$ and $G \in \mathcal{T}^1_1(M)$, then

$$F^C S^V = (SF)^V, \tag{2.15}$$

$$F^C G^V = (GF)^V. \tag{2.16}$$

Also

$$S^V F^C = (SF)^V$$

if and only if

$$S(X, FY) = S(FX, Y)$$

for all $X, Y \in \mathcal{T}^1_0(M)$.

§ 3. The horizontal lift of a vector field.

Let ∇ be a symmetric affine connection in M. Suppose that A is a point of M and that U, U^* are coordinate neighbourhoods containing A. Any point in the fibre over A is of the form (A, p), where p is a covariant vector at A. Let ∇ have components Γ^h_{ji} and Γ^{*h}_{ji} relative to U, U^* respectively at A and let p have components p_i and p^*_i relative to U, U^*. We write

$$\Gamma_{ji} = p_a \Gamma^a_{ji}, \qquad \Gamma^*_{ji} = p^*_a \Gamma^{*a}_{ji}. \tag{3.1}$$

Let X be a vector field in M. Then the vector at (A, p) in ${}^cT(M)$ whose components $\tilde{X}^A$ relative to $\pi^{-1}(U)$ are given by

$$\tilde{X}^h = X^h, \qquad \tilde{X}^{\bar{h}} = \Gamma_{hb} X^b$$

has components $\tilde{X}^{*A}$ relative to $\pi^{-1}(U^*)$, where

$$\tilde{X}^{*h} = X^{*h}, \qquad \tilde{X}^{*\bar{h}} = \Gamma^*_{hb} X^{*b}.$$

This can be proved by a straightforward calculation using the laws of transformation of the components of X, p and ∇ at A.

We denote by X^H the vector field in ${}^cT(M)$ obtained in this way and call X^H the *horizontal lift of* X *to* ${}^cT(M)$.

The vector field X^H satisfies

$$X^H f^V = (Xf)^V \tag{3.2}$$

$$X^H Z^V = (\nabla_X Z)^V, \tag{3.3}$$

where $f \in \mathfrak{T}_0^0(M)$ and $Z \in \mathfrak{T}_0^1(M)$. By Proposition 1 of the previous paper [4], X^H is completely determined by (3.3).

We now prove certain propositions for horizontal lifts, most of which are analogous to those established in the previous paper [4] for complete lifts. We begin with a relationship between the horizontal and complete lifts of a vector field.

PROPOSITION 1. *If* $X \in \mathfrak{T}_0^1(M)$, *then*

$$X^H = X^C + (\nabla X)^V .$$

PROOF. Suppose that $Z \in \mathfrak{T}_0^1(M)$. By (2.8)

$$X^C Z^V = [X, Z]^V$$

and by (2.6)

$$(\nabla X)^V Z^V = ((\nabla X)Z)^V = (\nabla_Z X)^V .$$

Hence

$$(X^C + (\nabla X)^V)Z^V = ([X, Z] + \nabla_Z X)^V = (\nabla_X Z)^V ,$$

so that, by (3.3), the actions of X^H and $X^C + (\nabla X)^V$ on Z^V are the same. The required result follows from Proposition 1 of the previous paper [4].

PROPOSITION 2. *Let* $\tilde{S}$, $\tilde{T}$ *be tensor fields in* ${}^cT(M)$ *of type* $(0, r)$ *or* $(1, r)$ (*where* r *is a positive integer*) *such that*

$$\tilde{S}(\tilde{X}_{(1)}, \cdots, \tilde{X}_{(r)}) = \tilde{T}(\tilde{X}_{(1)}, \cdots, \tilde{X}_{(r)})$$

for all vector fields $\tilde{X}_{(s)}$ $(s = 1, \cdots, r)$ *which are of the form* ω^V *or* Z^H, *where* $\omega \in \mathfrak{T}_1^0(M)$ *and* $Z \in \mathfrak{T}_0^1(M)$. *Then* $\tilde{S} = \tilde{T}$.

PROOF. In the coordinate neighbourhood $\pi^{-1}(U)$, the vector fields $E_{(A)}$ whose components are given by

$$E_{(A)}{}^B = \delta_A^B$$

span the module of vector fields in ${}^cT(M)$. Hence any tensor field in ${}^cT(M)$ of type $(0, r)$ or $(1, r)$ is determined in $\pi^{-1}(U)$ by its action on the vector fields $E_{(A)}$.

Let $\omega_{(1)}, \cdots, \omega_{(n)}$ be the 1-forms in M given in U by

$$\omega_{(i)h} = \delta_{ih} .$$

Then

$$E_{(\bar{i})} = \omega_{(i)}^V .$$

Let $X_{(1)}, \cdots, X_{(n)}$ be the vector fields in M given in U by

$$X_{(i)}^h = \delta_i^h .$$

Then $X_{(i)}^H$ is expressible in the form

$$E_{(i)} + f_j E_{(\bar{j})} .$$

It follows that $\omega^V_{(1)}, \cdots, \omega^V_{(n)}, X^H_{(1)}, \cdots, X^H_{(n)}$ also span the module of vector fields in ${}^cT(M)$. Hence any tensor field of type $(0, r)$ or $(1, r)$ is determined in $\pi^{-1}(U)$ by its action on $\omega^V_{(1)}, \cdots, \omega^V_{(n)}, X^H_{(1)}, \cdots, X^H_{(n)}$.

PROPOSITION 3. *If* $X \in \mathcal{T}^1_0(M)$ *and* $\omega \in \mathcal{T}^0_1(M)$, *then*

$$[X^H, \omega^V] = (\nabla_X \omega)^V .$$

PROOF. If $Z \in \mathcal{T}^1_0(M)$, then, by (2.4), (3.2) and (2.3),

$$\begin{aligned}[X^H, \omega^V]Z^V &= X^H \omega^V Z^V - \omega^V X^H Z^V \\ &= X^H(\omega(Z))^V - \omega^V(\nabla_X Z)^V \\ &= \{X(\omega(Z))\}^V - (\omega(\nabla_X Z))^V \\ &= \{\nabla_X(\omega(Z)) - \omega(\nabla_X Z)\}^V \\ &= \{(\nabla_X \omega)(Z)\}^V .\end{aligned}$$

But, by (2.4),

$$(\nabla_X \omega)^V Z^V = \{(\nabla_X \omega)(Z)\}^V .$$

PROPOSITION 4. *If* $X \in \mathcal{T}^1_0(M)$ *and* $F \in \mathcal{T}^1_1(M)$, *then*

$$[X^H, F^V] = (\nabla_X F)^V .$$

This can be proved by a similar argument.

PRPPOSITION 5. *If* $X, Y \in \mathcal{T}^1_0(M)$, *then*

$$[X^H, Y^H] = [X, Y]^H + (K(X, Y))^V ,$$

where K *is the curvature tensor in* M.

PROOF. If $Z \in \mathcal{T}^1_0(M)$, then

$$\begin{aligned}[X^H, Y^H]Z^V &= X^H Y^H Z^V - Y^H X^H Z^V \\ &= X^H(\nabla_Y Z)^V - Y^H(\nabla_X Z)^V \\ &= (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z)^V \\ &= (\nabla_{[X,Y]} Z + K(X, Y)Z)^V \\ &= [X, Y]^H Z^V + \{K(X, Y)\}^V Z^V .\end{aligned}$$

(see [**1**], p. 133).

PROPOSITION 6. *If* $X, Y \in \mathcal{T}^1_0(M)$, *then*

$$[X^C, Y^H] = [X, Y]^H + (\mathcal{L}_X \Gamma)_Y{}^V$$

where the term involving the Lie derivative of the connection is given by

$$(\mathcal{L}_X \Gamma)_Y = \nabla_Y \nabla X + K(X, Y) .$$

PROOF. By Propositions 1, 4 and 5

$$\begin{aligned}[X^C, Y^H] &= [X^H-(\nabla X)^V, Y^H] \\ &= [X^H, Y^H]+[Y^H, (\nabla X)^V] \\ &= [X, Y]^H+\{K(X, Y)+\nabla_Y(\nabla X)\}^V \\ &= [X, Y]^H+\{(\mathcal{L}_X\Gamma)_Y\}^V .\end{aligned}$$

PROPOSITION 7. *If* $F\in\mathcal{T}^1_1(M)$ *and* $X\in\mathcal{T}^1_0(M)$, *then*

$$F^CX^H=(FX)^H+[\nabla F]_X{}^V$$

where $[\nabla F]_X\in\mathcal{T}^1_1(M)$ *is given by*

$$[\nabla F]_XY=(\nabla_XF)Y-(\nabla_YF)X .$$

PROOF. By Proposition 1 and equations (2.14) and (2.16),

$$\begin{aligned}F^CX^H &= F^C(X^C+(\nabla X)^V) \\ &= (FX)^C+(\mathcal{L}_XF)^V+((\nabla X)F)^V \\ &= (FX)^H-(\nabla(FX))^V+(\mathcal{L}_XF)^V+((\nabla X)F)^V \\ &= (FX)^H-(\nabla(FX))^V+\{\nabla_XF-(\nabla X)F+F(\nabla X)\}^V+((\nabla X)F)^V \\ &= (FX)^H+\{\nabla_XF+F(\nabla X)-\nabla(FX)\}^V .\end{aligned}$$

But, if $Y\in\mathcal{T}^1_0(M)$, we have

$$\begin{aligned}\{\nabla_XF+F(\nabla X)-\nabla(FX)\}Y &= (\nabla_XF)Y+F(\nabla_YX)-\nabla_Y(FX) \\ &= (\nabla_XF)Y-(\nabla_YF)X .\end{aligned}$$

PROPOSITION 8. *If* $S\in\mathcal{T}^1_2(M)$, *then*

$$S^VX^H=(S_X)^V .$$

PROOF. By (2.10), (2.12) and Proposition 1,

$$\begin{aligned}S^VX^H &= S^V(X^C+(\nabla X)^V) \\ &= (S_X)^V .\end{aligned}$$

§4. The horizontal lift of a tensor field of type (1, 1).

Suppose now that $F\in\mathcal{T}^1_1(M)$; we shall define the horizontal lift of F. Let U, U^* be coordinate neighbourhoods containing the point A of M and let Γ_{ji}, Γ^*_{ji} be defined as at the beginning of § 3 (see (3.1)). The tensor field of type (1, 1) at the point (A, p) in ${}^cT(M)$ whose components $\tilde{F}_B{}^A$ relative to $\pi^{-1}(U)$ are given by

$$\tilde{F}_i{}^h=F_i{}^h , \qquad \tilde{F}_{\bar{i}}{}^h=0 ,$$

$$\tilde{F}_i{}^{\bar{h}}=-\Gamma_{ia}F_h{}^a+\Gamma_{ha}F_i{}^a , \qquad \tilde{F}_{\bar{i}}{}^{\bar{h}}=F_h{}^i$$

has components $\tilde{F}^{*A}_{B}$ relative to $\pi^{-1}(U^*)$, where

$$\tilde{F}^{*h}_{i} = F^{*h}_{i}\,, \qquad \tilde{F}^{*h}_{\bar{i}} = 0\,,$$

$$\tilde{F}^{*\bar{h}}_{i} = -\Gamma^{*}_{ia}F^{*a}_{h} + \Gamma^{*}_{ha}F^{*a}_{i}\,, \qquad \tilde{F}^{*\bar{h}}_{\bar{i}} = F^{*i}_{h}\,.$$

We denote this tensor field by F^H and call it the *horizontal lift of F*. Since $F^H \in \mathcal{T}^1_1({}^cT(M))$, it is completely determined by its action on vector fields of the form ω^V and X^H, where $\omega \in \mathcal{T}^0_1(M)$ and $X \in \mathcal{T}^1_0(M)$. We have

$$F^H \omega^V = (\omega F)^V\,, \tag{4.1}$$

$$F^H X^H = (FX)^H\,. \tag{4.2}$$

Also, if $G \in \mathcal{T}^1_1(M)$, then

$$F^H G^V = (GF)^V \tag{4.3}$$

and the action of F^H on the complete lift X^C is given by

$$F^H X^C = (FX)^H - \{(\nabla X)F\}^V \tag{4.4}$$

PROROSITION 9. *If $F \in \mathcal{T}^1_1(M)$, then*

$$F^C = F^H + [\nabla F]^V$$

where $[\nabla F] \in \mathcal{T}^1_2(M)$ is given by

$$[\nabla F]Y = (\nabla F)Y - (\nabla_Y F)\,.$$

PROOF. By Proposition 2 it is sufficient to show that the actions of $F^C - F^H$ and $[\nabla F]^V$ on ω^V and X^H are the same. By (2.13) and (4.1),

$$(F^C - F^H)\omega^V = 0$$

and by (2.9)

$$[\nabla F]^V \omega^V = 0\,.$$

By Proposition 7,

$$(F^C - F^H)X^H = (FX)^H + [\nabla F]_X{}^V - (FX)^H$$
$$= [\nabla F]_X{}^V\,.$$

But

$$[\nabla F]^V X^H = [\nabla F]^V (X^C + (\nabla X)^V)$$
$$= [\nabla F]_X{}^V\,.$$

PROPOSITION 10. *If $F \in \mathcal{T}^1_1(M)$ and $S \in \mathcal{T}^1_2(M)$, then*

$$F^H S^V = (SF)^V\,.$$

PROOF. By Proposition 9 and equations (2.11) and (2.15),

$$F^H S^V = F^C S^V - [\nabla F]^V S^V$$
$$= (SF)^V\,.$$

PROPOSITION 11. *Suppose that* $F \in \mathcal{T}_1^1(M)$ *and* $S \in \mathcal{T}_2^1(M)$. *Then*

$$S^V F^H = (SF)^V$$

if and only if

$$S(X, FY) = S(FX, Y)$$

for all $X, Y \in \mathcal{T}_0^1(M)$.

PROOF. By Proposition 9 and equation (2.11),

$$S^V F^H = S^V F^C$$

so that the proposition follows at once from the result stated at the end of § 2.

PROPOSITION 12. *If* $F, G \in \mathcal{T}_1^1(M)$, *then*

$$F^H G^H + G^H F^H = (FG + GF)^H .$$

PROOF. If $\omega \in \mathcal{T}_1^0(M)$, then, by (4.1)

$$F^H G^H \omega^V = F^H (\omega G)^V = (\omega GF)^V = (GF)^H \omega^V$$

so that

$$(F^H G^H + G^H F^H)\omega^V = (FG + GF)^H \omega^V .$$

If $X \in \mathcal{T}_0^1(M)$, then, by (4.2)

$$F^H G^H X^H = F^H (GX)^H = (FGX)^H = (FG)^H X^H .$$

Therefore

$$(F^H G^H + G^H F^H) X^H = (FG + GF)^H X^H .$$

The required result now follows from Proposition 2.

§ 5. Almost complex structures in the cotangent bundle.

We now show how horizontal lifts can be used to obtain almost complex and similar structures on ${}^cT(M)$.

THEOREM 1. *Let* F *be an almost complex structure on* M. *Then* F^H *is an almost complex structure on* ${}^cT(M)$.

PROOF. Since F is an almost complex structure,

$$F^2 = -I$$

where I is the unit tensor of type (1, 1).

By Propostion 12,

$$(F^H)^2 = (F^2)^H$$

and hence we have

$$(F^H)^2 = (-I)^H .$$

Since the horizontal lift of the unit tensor in M is clearly the unit tensor in ${}^cT(M)$, it follows that F^H is an almost complex structure on ${}^cT(M)$.

THEOREM 2. *Let* F *be an* f*-structure on* M, (*see* [2]), *so that*

$$F^3+F=0.$$

Then F^H satisfies

$$(F^H)^3+F^H=0.$$

PROOF. From Proposition 12, we have

$$(F^H)^2=(F^2)^H.$$

Again applying Proposition 12, but this time with G replaced by F^2, we have

$$(F^H)(F^2)^H+(F^2)^H F^H=(2F^3)^H.$$

Hence

$$(F^H)^3=(F^3)^H.$$

It follows that if $F^3+F=0$, then $(F^H)^3+F^H=0$.

An almost complex structure F on a manifold is integrable if and only if the Nijenhuis tensor of F is zero. We now consider the Nijenhuis tensor of F^H in order to determine the circumstances in which F^H is integrable. We first prove the following proposition, in which F is any tensor field of type (1, 1) in M.

PROPOSITION 13. *Suppose that $F\in\mathcal{T}_1^1(M)$ and that $\tilde{N}$ is the Nijenhuis tensor of F^H. Then if $\phi, \omega\in\mathcal{T}_1^0(M)$ and $X, Y\in\mathcal{T}_0^1(M)$, we have*

$$\tilde{N}(\phi^V, \omega^V)=0$$

$$\tilde{N}(X^H, \omega^V)=\{\omega(\nabla_{FX}F)-\omega(\nabla_X F)F\}^V$$

$$\tilde{N}(X^H, Y^H)=\{N(X, Y)\}^H$$
$$+\{K(FX, FY)-K(FX, Y)F-K(X, FY)F+K(X, Y)F^2\}^V$$

where N is the Nijenhuis tensor of F.

PROOF. If $\phi, \omega\in\mathcal{T}_1^0(M)$, then

$$\begin{aligned}\tilde{N}(\phi^V, \omega^V)&=[F^H\phi^V, F^H\omega^V]+(F^H)^2[\phi^V, \omega^V]\\&\quad -F^H[F^H\phi^V, \omega^V]-F^H[\phi^V, F^H\omega^V]\\&=[(\phi F)^V, (\omega F)^V]+(F^H)^2[\phi^V, \omega^V]\\&\quad -F^H[(\phi F)^V, \omega^V]-F^H[\phi^V, (\omega F)^V]\\&=0\end{aligned}$$

by Proposition 4 of the previous paper [**4**].

If $X\in\mathcal{T}_0^1(M)$ and $\omega\in\mathcal{T}_1^0(M)$, then, by (4.1), (4.2) and Proposition 3,

$$\begin{aligned}\tilde{N}(X^H, \omega^V) &= [F^H X^H, F^H \omega^V] + (F^H)^2[X^H, \omega^V] \\ &\quad - F^H[F^H X^H, \omega^V] - F^H[X^H, F^H\omega^V] \\ &= [(FX)^H, (\omega F)^V] + (F^H)^2[X^H, \omega^V] \\ &\quad - F^H[(FX)^H, \omega^V] - F^H[X^H, (\omega F)^V] \\ &= \{\nabla_{FX}(\omega F)\}^V + (F^H)^2(\nabla_X \omega)^V \\ &\quad - F^H(\nabla_{FX}\omega)^V - F^H(\nabla_X(\omega F))^V \\ &= \{\nabla_{FX}(\omega F) + (\nabla_X\omega)F^2 - (\nabla_{FX}\omega)F - (\nabla_X(\omega F))F\}^V \\ &= \{\omega(\nabla_{FX}F) - \omega(\nabla_X F)F\}^V .\end{aligned}$$

If $X, Y \in \mathfrak{T}^1_0(M)$, then by (4.2) and Proposition 5,

$$\begin{aligned}\tilde{N}(X^H, Y^H) &= [F^H X^H, F^H Y^H] + (F^H)^2[X^H, Y^H] \\ &\quad - F^H[F^H X^H, Y^H] - F^H[X^H, F^H Y^H] \\ &= [(FX)^H, (FY)^H] + (F^H)^2[X^H, Y^H] \\ &\quad - F^H[(FX)^H, Y^H] - F^H[X^H, (FY)^H] \\ &= [FX, FY]^H + (F^H)^2[X, Y]^H - F^H[FX, Y]^H - F^H[X, FY]^H \\ &\quad + \{K(FX, FY)\}^V + (F^H)^2\{K(X, Y)\}^V \\ &\quad - F^H\{K(FX, Y)\}^V - F^H\{K(X, FY)\}^V \\ &= [FX, FY]^H + \{F^2[X, Y]\}^H - \{F[FX, Y]\}^H - \{F[X, FY]\}^H \\ &\quad + \{K(FX, FY) + K(X, Y)F^2 - K(FX, Y)F - K(X, FY)F\}^V \\ &= \{N(X, Y)\}^H \\ &\quad + \{K(FX, FY) + K(X, Y)F^2 - K(FX, Y)F - K(X, FY)F\}^V .\end{aligned}$$

THEOREM 3. *Let F be a Kählerian structure in M, with respect to the connection ∇. Then the almost complex structure F^H in ${}^cT(M)$ is integrable.*

PROOF. If F is Kählerian, then

(i) F is a complex structure in M,

(ii) $\nabla F = 0$,

(iii) the curvature tensor of ∇ satisfies

$$K(FX, FY) = K(X, Y) .$$

(see [3], Chapter IV). From (i) it follows that the Nijenhuis tensor of F is zero. From (iii) we get

$$K(FX, Y) = -K(X, FY)$$

since $F^2 = -I$. Hence, again using $F^2 = -I$,

$$K(FX, FY)+K(X, Y)F^2-K(FX, Y)F-K(X, FY)F=0.$$

It follows from Proposition 13 that

$$\tilde{N}(\psi^V, \omega^V)=0, \qquad \tilde{N}(X^H, \omega^V)=0,$$
$$\tilde{N}(X^H, Y^H)=0.$$

Since $\tilde{N}$ is skew-symmetric, we also have

$$\tilde{N}(\omega^V, X^H)=0.$$

Hence, by Proposition 2, $\tilde{N}$ is zero and so F^H is integrable.

§6. The horizontal lift of a connection.

In the previous paper, we used the idea of a Riemann extension of a symmetric affine connection in order to define the complete lift ∇^C of a symmetric connection ∇ in M. We now consider other possible connections in ${}^cT(M)$ and select one which we call the horizontal lift of ∇.

In order to construct a connection $\tilde{\nabla}$ in ${}^cT(M)$, we can choose any tensor field $\tilde{T}$ of type (1, 2) in ${}^cT(M)$ and write

$$\tilde{\nabla}=\nabla^C+\tilde{T}.$$

The most interesting connections which we can obtain in this way will be those which, like ∇^C itself, have the property

$$\tilde{\Gamma}^h_{ji}=\Gamma^h_{ji},$$

so that $\tilde{\nabla}$ coincides with ∇ on M. We therefore choose $\tilde{T}$ to be such that

$$\tilde{T}_{ji}{}^h=0.$$

The simplest method of constructing such a tensor field T in ${}^cT(M)$ is to begin with a tensor field T of type (1, 3) in M and to form its vertical lift T^V. We construct T^V in the same way as we constructed vertical lifts of tensor fields of type (1, 1) or (1, 2) in the previous paper [4]. Thus the components of T^V are given by

$$\tilde{T}_{ji}{}^{\bar{h}}=p_a T_{hij}{}^a$$

and the remaining components are zero.

In particular, we can form a connection ∇^H in ${}^cT(M)$ by writing

$$\nabla^H=\nabla^C-K^V \tag{6.1}$$

where K is the curvature tensor of ∇ in M. We call ∇^H the *horizontal lift of* ∇.

It follows quickly from the definition that if the components of ∇ in a coordinate neighbourhood U of M are Γ_{ji}^{h}, then the components $\tilde{\Gamma}_{CB}^{A}$ of ∇^{H} in $\pi^{-1}(U)$ are given by

$$
\begin{aligned}
&\Gamma_{ji}^{h}=\Gamma_{ji}^{h}, \qquad \tilde{\Gamma}_{j\bar{i}}^{h}=0, \qquad \tilde{\Gamma}_{\bar{j}i}^{h}=0, \qquad \tilde{\Gamma}_{\bar{j}\bar{i}}^{h}=0, \\
&\tilde{\Gamma}_{ji}^{\bar{h}}=p_{a}(-\partial_{j}\Gamma_{ih}^{a}+\Gamma_{hb}^{a}\Gamma_{ji}^{b}+\Gamma_{ib}^{a}\Gamma_{hj}^{b}) \\
&\tilde{\Gamma}_{j\bar{i}}^{\bar{h}}=-\Gamma_{jh}^{i}, \qquad \tilde{\Gamma}_{\bar{j}i}^{\bar{h}}=-\Gamma_{hi}^{j}, \qquad \tilde{\Gamma}_{\bar{j}\bar{i}}^{\bar{h}}=0 .
\end{aligned} \tag{6.2}
$$

From these formulae, we can readily deduce that covariant differentiation with respect to the connection ∇^{H} satisfies

$$
\begin{aligned}
&\nabla_{\phi^{V}}^{H}\omega^{V}=0, \qquad \nabla_{\phi^{V}}^{H}Y^{H}=0, \\
&\nabla_{X^{H}}^{H}\omega^{V}=(\nabla_{X}\omega)^{V}, \qquad \nabla_{X^{H}}^{H}Y^{H}=(\nabla_{X}Y)^{H}.
\end{aligned} \tag{6.3}
$$

We can also prove the following result without difficulty.

THEOREM 4. *Let C be an autoparallel curve of ∇^{H} in ${}^{c}T(M)$. Then the projection of C in M is a geodesic in M.*

§ 7. The torsion and curvature tensors of the horizontal lift of a connection in M.

In the next two propositions, we determine the torsion and curvature tensors of ∇^{H}.

PROPOSITION 14. *Let $\tilde{T}$ be the torsion tensor of the horizontal lift ∇^{H}. Then $\tilde{T}$ is the skew-symmetric tensor field determined by*

$$
\begin{gathered}
\tilde{T}(\phi^{V}, \omega^{V})=0, \qquad \tilde{T}(X^{H}, \omega^{V})=0, \\
\tilde{T}(X^{H}, Y^{H})=-(K(X, Y))^{V}
\end{gathered}
$$

where $\phi, \omega \in \mathfrak{T}_{1}^{0}(M)$ and $X, Y \in \mathfrak{T}_{0}^{1}(M)$.

PROOF. We can prove this result by using (6.1) and the components of ∇^{H} given by (6.2). Alternatively, since

$$
\tilde{T}(\tilde{X}, \tilde{Y})=\nabla_{\tilde{X}}^{H}\tilde{Y}-\nabla_{\tilde{Y}}^{H}\tilde{X}-[\tilde{X}, \tilde{Y}]
$$

we have

$$
\begin{aligned}
\tilde{T}(\phi^{V}, \omega^{V}) &=\nabla_{\phi^{V}}^{H}\omega^{V}-\nabla_{\omega^{V}}^{H}\phi^{V}-[\phi^{V}, \omega^{V}] \\
&=0
\end{aligned}
$$

by (6.3) and Proposition 4 of the previous paper [4];

$$
\begin{aligned}
\tilde{T}(X^{H}, \omega^{V}) &=\nabla_{X^{H}}^{H}\omega^{V}-\nabla_{\omega^{V}}^{H}X^{H}-[X^{H}, \omega^{V}] \\
&=(\nabla_{X}\omega)^{V}-(\nabla_{X}\omega)^{V}=0
\end{aligned}
$$

by (6.3) and Proposition 3;

$$\tilde{T}(X^H, Y^H) = \nabla^H_{X^H} Y^H - \nabla^H_{Y^H} X^H - [X^H, Y^H]$$
$$= (\nabla_X Y)^H - (\nabla_Y X)^H - [X, Y]^H - (K(X, Y))^V$$
$$= -(K(X, Y))^V$$

by (6.3), Proposition 5 and the fact that ∇ is symmetric.

PROPOSITION 15. *Let $\tilde{K}$ be the curvature tensor of ∇^H. Then, if $\phi, \psi, \omega \in \mathcal{T}^0_1(M)$ and $X, Y, Z \in \mathcal{T}^1_0(M)$, we have*

$$\tilde{K}(\phi^V, \psi^V) = 0, \qquad \tilde{K}(X^H, \psi^V) = 0,$$
$$\tilde{K}(X^H, Y^H)\omega^V = -(\omega(K(X, Y)))^V$$
$$\tilde{K}(X^H, Y^H)Z^H = (K(X, Y)Z)^H.$$

PROOF. This can be deduced from equations (6.3) by a routine verification.

In terms of the components of $\tilde{K}$ relative to the coordinate neighbourhood $\pi^{-1}(U)$, the result of Proposition 15 is equivalent to the statement that

$$\tilde{K}_{kji}{}^h = K_{kji}{}^h$$
$$\tilde{K}_{kji}{}^{\bar{h}} = p_a(\Gamma^a_{hb} K_{kji}{}^b + \Gamma^a_{ib} K_{kjh}{}^b)$$
$$\tilde{K}_{kj\bar{i}}{}^{\bar{h}} = -K_{kjh}{}^i$$

and the remaining components of $\tilde{K}$ are zero.

PROPOSITION 16. *The covariant derivative $\nabla^H \tilde{K}$ of the curvature tensor is given by*

$$(\nabla^H \tilde{K})(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$$

if one or more of $\tilde{X}, \tilde{Y}, \tilde{Z}$ is the vertical lift of a 1-form in M, and

$$(\nabla^H \tilde{K})(X^H, Y^H, Z^H)\omega^V = -\{\omega(\nabla_X K(Y, Z))\}^V$$
$$(\nabla^H \tilde{K})(X^H, Y^H, Z^H)W^H = \{(\nabla K)(X, Y, Z)W\}^H.$$

This result can also be proved by means of a routine verification. Similar results for higher order covariant derivatives can also be obtained by this process.

THEOREM 5. *The curvature tensor $\tilde{K}$ of ${}^cT(M)$ with respect to ∇^H is parallel if and only if the curvature tensor K of M with respect to ∇ is parallel.*

PROOF. This follows at once from Proposition 16, since clearly $\nabla^H \tilde{K}$ is zero if and only if ∇K is zero.

Tokyo Institute of Technology
and
University of Aberdeen, Scotland

Bibliography

[1] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Tract No. 5, 1963.

[2] K. Yano, On a structure defined by a tensor field f of type (1, 1) satisfying $f^3+f=0$, Tensor, **14** (1963), 9-19.

[3] K. Yano, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

[4] K. Yano and E. M. Patterson, Vertical and complete lifts from a manifold to its cotangent bundle, J. Math. Soc. Japan, **19** (1967), 91-113.