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On the extreme values of the roots of matrices

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In this paper we shall investigate some properties concerning the behavior of the eigenvalues and singular values of complex matrices. Let A be an n-square matrix and p be any positive integer. Let the eigenvalues of A and the singular values of A^p be denoted by λ_i and $\alpha_i^{(p)}$ $(1 \leqq i \leqq n)$ respectively, which are so arranged that $|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n|$ and $\alpha_1^{(p)} \geqq \alpha_2^{(p)} \geqq \cdots \geqq \alpha_n^{(p)}$. Then in § 1 we shall prove that $\lim_{p\to\infty} \alpha_i^{(p)\frac{1}{p}} = |\lambda_i|$, $i=1, 2, \cdots, n$. This generalizes a Gautschi's result ([3] p. 138). In § 2 we shall treat non-negative matrices and state some properties which improve some results obtained by Gautschi [3] and Brauer [1].

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Notations and definitions: We consider n-square matrices with complex elements and certain notational conventions will be observed throughout this paper. The (i, j) element of an $n \times n$ matrix A will be denoted, using the corresponding small letters, by a_{ij}. $R_i(A)$ stands for $\sum_{j=1}^{n} |a_{ij}|$ and $C_j(A)$ for $\sum_{i=1}^{n} |a_{ij}|$. We shall put $R(A) = \max_i R_i(A)$ and $C(A) = \max_i C_i(A)$. We adopt the notations ${}^tA$ and A^* for a transposed matrix and a conjugate transposed matrix of A respectively. $\lambda(A)$ signifies any one of the eigenvalues of A. The singular values of A are the square roots of the eigenvalues of $(A^*)A$ or AA^*. The spectral radius of A is $\rho(A) = \max|\lambda(A)|$. We mean by a non-negative (positive) matrix the one whose elements are non-negative (positive) real numbers. $|A|$ denotes the matrix whose (i, j) elements are given by $|a_{ij}|$. By a vector x, we mean one column matrix and the Euclidean length of x, $(x^*x)^{1/2}$, will be denoted by $|x|$.

§ 1. The behavior of the singular values

Let $\| \ \|$ be a matrix norm consistent with a vector norm (cf. [4]). Then $\lim_{p\to\infty} \|A^p\|^{\frac{1}{p}} = \rho(A)$ is well known as a special case in the theory of Banach algebra (e. g. [5]) and was also proved by Gautschi [3] in different formulations, but we give here an elementary proof for the sake of completeness.

LEMMA. *Let $\| \ \|$ be a matrix norm consistent with a vector norm, then for every positive integer p, we have*

$$\rho(A) \leqq \| A^p \|^{\frac{1}{p}} \leqq \| A \| \quad \text{and} \quad \rho(A) = \lim_{p\to\infty} \| A^p \|^{\frac{1}{p}} .$$

If p_i $(i=1, 2, \cdots)$ is a strictly increasing sequence of positive integers such that p_i is divisible by p_{i-1} $(i=2, 3, \cdots)$, then the sequence $\{\| A^{p_i} \|^{\frac{1}{p_i}}\}(i=1, 2, 3, \cdots)$ is monotone decreasing and converges towards $\rho(A)$ as $i \to \infty$.

PROOF. Let x be a non-trivial eigenvector corresponding to an eigenvalue λ of A, then $A^p x = \lambda^p x$ for any positive integer p. Hence $|\lambda|^p |x| \leqq \| A^p \| |x|$ and $|x| > 0$. From this it follows that $|\lambda| \leqq \| A^p \|^{\frac{1}{p}} \leqq \| A \|$. Thus the sequence $\{\| A^p \|^{\frac{1}{p}}\}$ $(p=1, 2, \cdots)$ is contained in the bounded closed interval $[\rho(A), \|A\|]$, and has at least one limit point in it. Let α be any limit point of this sequence, then $\rho(A) \leqq \alpha \leqq \| A \|$. Now suppose that $\rho(A) < \alpha$, then there exist a subsequence $\{\|A^{p_i}\|^{\frac{1}{p_i}}\}(i=1, 2, \cdots, 1 \leqq p_1 < p_2 < \cdots)$ such that $\|A^{p_i}\|^{\frac{1}{p_i}} \to \alpha$ as $i \to \infty$, and a positive number α' such that $\rho(A) < \alpha' < \alpha$. Then $\rho\left(\frac{A}{\alpha'}\right) = \frac{1}{\alpha'} \rho(A) < 1$, therefore $\left(\frac{A}{\alpha'}\right)^{p_i} \to 0$ as $i \to \infty$. Since a matrix norm is continuous with respect to the elements, we get $\left\|\left(\frac{A}{\alpha'}\right)^{p_i}\right\| \to 0$ as $i \to \infty$. Hence if we take a positive constant ε, there is an integer $N(\varepsilon)$ such that $\left\|\left(\frac{A}{\alpha'}\right)^{p_i}\right\| < \varepsilon$ for every $i > N(\varepsilon)$, and we have

$$1 < \frac{\alpha}{\alpha'} = \lim_{i\to\infty} \left\|\left(\frac{A}{\alpha'}\right)^{p_i}\right\|^{\frac{1}{p_i}} \leqq \lim_{i\to\infty} \varepsilon^{\frac{1}{p_i}} = 1 .$$

This is a contradiction and we have $\rho(A) = \alpha$, which implies $\lim_{p\to\infty} \| A^p \|^{\frac{1}{p}} = \rho(A)$. Further, if $p_{i+1} = m p_i$ where m is a positive integer, we have

$$\| A^{p_{i+1}} \|^{\frac{1}{p_{i+1}}} = \| A^{m p_i} \|^{\frac{1}{m p_i}} \leqq \| A^{p_i} \|^{\frac{1}{p_i}} .$$

This completes the proof.

COROLLARY. *Let $A=(a_{ij})$ and $B=(b_{ij})$ be two matrices such that $|a_{ij}| \leqq b_{ij}$ $(i, j=1, 2, \cdots, n)$, then $\rho(A) \leqq \rho(|A|) \leqq \rho(B)$.*

This is known as a part of the Perron-Frobenius theorem (cf. [7]), but using Lemma we can prove easily as follows: Let $\| A \| = \sum_{i,j} |a_{ij}|$, then $\| A^p \| = \| |A^p| \| \leqq \| |A|^p \| \leqq \|B^p\|$, *i. e.*, $\|A^p\|^{\frac{1}{p}} \leqq \| |A|^p \|^{\frac{1}{p}} \leqq \|B^p\|^{\frac{1}{p}}$, hence making $p \to \infty$ we obtain $\rho(A) \leqq \rho(|A|) \leqq \rho(B)$.

REMARK. Lemma holds for any matrix function ϕ satisfying the following conditions:

(I) $\phi(A)\geqq 0$, and $\phi(A)=0$ if and only if $A=0$

(II) $\phi(\alpha A)=|\alpha|\phi(A)$ for any complex number α

(III) $\phi(AB)\leqq\phi(A)\phi(B)$

(IV) if $\lim_{p\to\infty}A_p=0$ (considering an $n\times m$ matrix A_p as a point of the nm dimensional complex affine space), then $\lim_{p\to\infty}\phi(A_p)=0$.

Now we prove the following:

THEOREM 1. *Let A be a matrix of order n and p be any positive integer. Let the eigenvalues of A and the singular values of A^p be denoted by λ_i and $\alpha_i^{(p)}$ $(1\leqq i\leqq n)$ respectively, which are so arranged that*

$$|\lambda_1|\geqq|\lambda_2|\geqq\cdots\geqq|\lambda_n|,\quad \alpha_1^{(p)}\geqq\alpha_2^{(p)}\geqq\cdots\geqq\alpha_n^{(p)}.$$

Then we have

$$\lim_{p\to\infty}\alpha_i^{(p)\frac{1}{p}}=|\lambda_i|,\quad i=1,2,\cdots,n.$$

PROOF. As is well known, we have

$$\rho(A)^p=\rho(A^p)\leqq\rho\{(A^p)^*A^p\}^{\frac{1}{2}}\leqq N(A^p) \tag{1}$$

where $N(A)$ stands for the Euclidian norm, i.e., $N(A^p)=\sqrt{\sum_{i,j=1}^{n}|a_{ij}^{(p)}|^2}$ for $A^p=(a_{ij}^{(p)})$. Hence we have from (1)

$$\rho(A)\leqq\rho\{(A^p)^*A^p\}^{\frac{1}{2p}}\leqq N(A^p)^{\frac{1}{p}}\to\rho(A)\quad(p\to\infty)$$

i.e., $\lim_{p\to\infty}\alpha_1^{(p)\frac{1}{p}}=|\lambda_1|$. Applying this to the k-th compound matrix $C_k(A^p)$ of A^p (cf. [**6**]), we get

$$\lim_{p\to\infty}\prod_{i=1}^{k}\alpha_i^{(p)\frac{1}{p}}=\prod_{i=1}^{k}|\lambda_i| \tag{2}$$

since $\rho[\{C_k(A^p)\}^*C_k(A^p)]=\rho[C_k\{(A^p)^*A^p\}]=(\prod_{i=1}^{k}\alpha_i^{(p)})^2$. If $\lambda_1=0$, the assertion is trivial, so, without loss of generality, we may assume that

$$|\lambda_1|\geqq\cdots\geqq|\lambda_k|>0=|\lambda_{k+1}|=\cdots=|\lambda_n|. \tag{3}$$

Then we have $\alpha_k^{(p)}>0$ for any p. For, suppose that $\alpha_k^{(p)}=0$ for some p, then it follows that $\alpha_k^{(m)}=0$ for $m\geqq p$ since

$$\operatorname{rank}\{(A^{p+1})^*A^{p+1}\}=\operatorname{rank}(A^{p+1})\leqq\operatorname{rank}(A^p)=\operatorname{rank}\{(A^p)^*A^p\}.$$

Hence we have $\prod_{i=1}^{k}\alpha_i^{(m)\frac{1}{m}}=0$ for $m\geqq p$, but $\lim_{m\to\infty}\prod_{i=1}^{k}\alpha_i^{(m)\frac{1}{m}}=\prod_{i=1}^{k}|\lambda_i|>0$ by (2) and (3), which is a contradiction. Therefore we have $\alpha_i^{(p)}>0$ $(1\leqq i\leqq k)$ for any p, and

$$\lim_{p\to\infty} \alpha_j^{(p)\frac{1}{p}} = \lim_{p\to\infty} \left\{ \left(\prod_{i=1}^{j} \alpha_i^{(p)} \right)^{\frac{1}{p}} \Big/ \left(\prod_{i=1}^{j-1} \alpha_i^{(p)} \right)^{\frac{1}{p}} \right\}$$

$$= \left(\prod_{i=1}^{j} |\lambda_i| \right) \Big/ \left(\prod_{i=1}^{j-1} |\lambda_i| \right) = |\lambda_j|, \quad j \leqq k+1 .$$

Since $0 \leqq \alpha_j^{(p)\frac{1}{p}} \leqq \alpha_{k+1}^{(p)\frac{1}{p}} \to |\lambda_{k+1}| = 0$ as $p\to\infty$ for every $j > k+1$, it is clear that

$$\lim_{p\to\infty} \alpha_j^{(p)\frac{1}{p}} = |\lambda_j| \qquad \text{for } j > k+1 .$$

This completes the proof.

COROLLARY. *Let the assumptions and notations be the same as in Theorem 1. Then we have*

$$\prod_{i=1}^{k} \alpha_i^{(1)} \geqq \prod_{i=1}^{k} \alpha_i^{(2)\frac{1}{2}} \geqq \prod_{i=1}^{k} \alpha_i^{(4)\frac{1}{4}} \geqq \cdots \geqq \lim_{p\to\infty} \prod_{i=1}^{k} \alpha_i^{(2^p)2^{-p}} \tag{4}$$

$$= \prod_{i=1}^{k} |\lambda_i|, \qquad k = 1, 2, \cdots, n .$$

PROOF. For any matrix A, B, it is well known that

$$\rho\{(AB)^*(AB)\} \leqq \rho(A^*A) \cdot \rho(B^*B) .$$

Hence we have

$$\rho\{(A^{2^p})^* A^{2^p}\} \leqq [\rho\{(A^{2^{p-1}})^* A^{2^{p-1}}\}]^2, \qquad p = 1, 2, \cdots .$$

Applying this to the k-th compound matrix $C_k\{(A^{2^p})^* A^{2^p}\}$, we get (4) from Theorem 1.

THEOREM 2. *Let A be a matrix, then we have*

$$\lim_{p\to\infty} \rho({}^t|A^p|\,|A^p|)^{\frac{1}{2p}} = \lim_{p\to\infty} \rho(|(A^p)^* A^p|)^{\frac{1}{2p}} = \rho(A) .$$

PROOF. This is an immediate consequence of Lemma and the following inequalities:

$$\rho(A) = \rho(A^p)^{\frac{1}{p}} \leqq \rho(|A^p|)^{\frac{1}{p}} \leqq \rho({}^t|A^p|\,|A^p|)^{\frac{1}{2p}} \leqq N(A^p)^{\frac{1}{p}}$$

and

$$\rho(A) \leqq \rho\{(A^p)^* A^p\}^{\frac{1}{2p}} \leqq \rho\{|(A^p)^* A^p|\}^{\frac{1}{2p}} \leqq N(A^p)^{\frac{1}{p}}$$

where $N(A)$ is the Euclidian norm defined in (1).

§ 2. Non-negative matrices

A square matrix A is called reducible in case there exists a permutation matrix P such that

$${}^tPAP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} and A_{22} are square submatrices; otherwise it is called irreducible.

We have

THEOREM 3. *Let A be a non-negative irreducible matrix of order n, then we have*

$$\rho(A)=\lim_{p\to\infty} R_i(A^p)^{\frac{1}{p}}, \qquad i=1,2,\cdots,n.$$

PROOF. By the Perron-Frobenius theorem on non-negative matrices, there exists a positive eigenvalues λ_A with $\rho(A)=\lambda_A$ and a positive eigenvector x with $x_i>0$ $(i=1,2,\cdots,n)$ corresponding to λ_A. Let $A^p=(a_{ij}^{(p)})$, then, from the relation $A^p x=\lambda_A^p x$, we have $\lambda_A^p x_i=\sum_j a_{ij}^{(p)} x_j$. Put $\max_k x_k=x_\alpha$, $\min_k x_k=x_\beta$ and $\frac{x_\beta}{x_\alpha}=\delta>0$, then

$$\lambda_A x_i^{\frac{1}{p}}=(\sum_j a_{ij}^{(p)} x_j)^{\frac{1}{p}} \leqq (\sum_j a_{ij}^{(p)})^{\frac{1}{p}} x_\alpha^{\frac{1}{p}} \leqq (\max_k \sum_j a_{kj}^{(p)})^{\frac{1}{p}} x_\alpha^{\frac{1}{p}},$$

hence

$$\lambda_A \delta^{\frac{1}{p}} \leqq \lambda_A\Big(\frac{x_i}{x_\alpha}\Big)^{\frac{1}{p}} \leqq R_i(A^p)^{\frac{1}{p}} \leqq R(A^p)^{\frac{1}{p}}, \qquad i=1,2,\cdots,n.$$

Since $R(A)$ is a matrix norm, we have $R(A^p)^{\frac{1}{p}}\to\rho(A)=\lambda_A$ as $p\to\infty$. Therefore we have $R_i(A^p)^{\frac{1}{p}}\to\lambda_A$ $(1\leqq i\leqq n)$ as $p\to\infty$ since $\lambda_A\delta^{\frac{1}{p}}\to\lambda_A$ as $p\to\infty$.

From this proof, we see that Theorem 3 also holds whenever an eigenvalue with the largest non-zero absolute value has a positive eigenvector. By Brauer's theorem [2], power positive matrices have these properties, and so we have the following:

COROLLARY 1. *Let A be a power positive matrix, i.e., a real matrix such that A^k is a positive matrix for some k, then*

$$\rho(A)=\lim_{p\to\infty} R_i(A^p)^{\frac{1}{p}} \quad \textit{and} \quad \rho(A)=\lim_{p\to\infty} C_i(A^p)^{\frac{1}{p}}, \quad i=1,2,\cdots,n.$$

The next corollary is an improvement of the result of Brauer [1].

COROLLARY 2. *Let A be a non-negative matrix and $r(A)=\min_i R_i(A)$. Then we have*

$$r(A)\leqq r(A^2)^{\frac{1}{2}} \leqq r(A^4)^{\frac{1}{4}} \leqq \cdots \leqq \rho(A)=\lim_{p\to\infty} R(A^{2^p})^{2^{-p}}$$

$$\leqq \cdots \leqq R(A^4)^{\frac{1}{4}} \leqq R(A^2)^{\frac{1}{2}} \leqq R(A).$$

Moreover if non-negative matrix A is irreducible, then

$$r(A^{2^p})^{2^{-p}} \to \rho(A) \quad \textit{as} \quad p\to\infty.$$

PROOF. For any non-negative matrix A and B, it is easy to see that $r(AB) \geqq r(A)r(B)$, hence we have $r(A^{2^p})\geqq r(A^{2^{p-1}})^2$, i.e., $r(A^{2^p})^{\frac{1}{2^p}} \geqq r(A^{2^{p-1}})^{\frac{1}{2^{p-1}}}$. Take the greatest non-negative eigenvalue λ of ${}^tA$ and a non-negative vector y corresponding to λ, then from ${}^tAy=\lambda y$, we have $\sum_{i=1}^n R_i(A)y_i=\lambda\sum_{i=1}^n y_i$. Since $y_i\geqq 0$ for every i and $\sum_{i=1}^n y_i>0$, we see that $\lambda\geqq r(A)$. Hence applying this to A^p we get $\rho(A)\geqq r(A^p)^{\frac{1}{p}}$. The assertion $\lim_{p\to\infty} r(A^p)^{\frac{1}{p}}=\rho(A)$ will follow in case of non-negative irreducible matrices from Theorem 3.

THEOREM 4. *Let A be a non-negative matrices, then*

$$\rho(A)=\lim_{p\to\infty}\rho\Big(\frac{A^p+{}^tA^p}{2}\Big)^{\frac{1}{p}}.$$

PROOF. As is easily seen, we have $\rho(A)\leqq\rho\Big(\frac{A^p+{}^tA^p}{2}\Big)^{\frac{1}{p}}\leqq\rho({}^tA^pA^p)^{\frac{1}{2p}}$. Hence Theorem 4 follows from Theorem 1.

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