J. Math. Soc. Japan
Vol. 19, No. 1, 1967

On meromorphisms and congruence relations

By Junji HASHIMOTO

(Received June 1, 1966)

1. Introduction

By a *meromorphism* between two algebraic systems admitting the same operations, we mean a many-many correspondence of elements which preserves all algebraic combinations. In the present paper the correspondence of elements under the meromorphism φ shall be written $a \rightarrow b(\varphi)$ or $a\varphi b$. A meromorphism φ is called a *class-meromorphism* if and only if $a\varphi b$, $a'\varphi b$ and $a'\varphi b'$ imply $a\varphi b'$. In Shoda's theory on abstract algebraic systems the following condition is often assumed:

Every meromorphism between two homomorphic images of an algebraic system A is a class-meromorphism.

In a previous paper [4] we have shown that the above condition is equivalent to the condition

(α) *Every meromorphism of A onto itself is a class-meromorphism.*

A meromorphism φ of an algebraic system A onto itself may be considered a relation between elements of A. If φ is reflexive, we shall call φ a *quasi-congruence*. In the paper cited above it has been shown also that a quasi-congruence φ on A is a class-meromorphism if and only if it is a congruence relation on A. Let φ and ψ be two quasi-congruences on A. We shall write $a\varphi\psi b$ to mean $a\varphi c$ and $c\psi b$ for some $c \in A$, and $a\bar{\varphi}b$ to mean $b\varphi a$. Quasi-congruences φ and ψ are called *permutable* if $\varphi\psi = \psi\varphi$. The symmetricity and transitivity of a quasi-congruence φ are written $\bar{\varphi} \leqq \varphi$ and $\varphi^2 \leqq \varphi$ respectively. In the present paper we shall discuss the following conditions on an algebraic system A:

(β) *Every quasi-congruence on A is a congruence.*

(γ) *Every quasi-congruence on A is symmetric.*

(δ) *Every quasi-congruence on A is transitive.*

(ε) *All quasi-congruences on A are permutable.*

(ζ) *All congruences on A are permutable.*

About those conditions it is easy to see that the following implications hold.

$$\begin{tikzcd}[column sep=small] & & (\gamma) \arrow[dr] & & \\ (\alpha) \arrow[r] & (\beta) \arrow[ur] \arrow[dr] & & (\varepsilon) \arrow[r] & (\zeta) \\ & & (\delta) \arrow[ur] & & \end{tikzcd}$$

Our object in the present paper is to clarify what algebraic systems satisfy each of those conditions. For this purpose we shall first rewrite the conditions into more explicit expressions. Using those expressions, we intend to deal with the problem on multiplicative systems. We have succeeded to solve the problem for commutative semigroups and obtained some results on general semigroups and quasigroups. Our main results are the following:

THEOREM 3.2. *Congruences on a commutative semigroup S are permutable if and only if ideals of S are totally ordered under set-inclusion and non-nilpotent elements of S form a group.*

THEOREM 3.3. *Let S be a commutative semigroup containing three or more elements. Then the following assertions on S are equivalent:*

(1) *Every quasi-congruence on S is transitive.*

(2) *Quasi-congruences on S are permutable.*

(3) *S is a group.*

THEOREM 4.2. *Let S be a completely simple semigroup without zero. Then congruences on S are permutable if and only if S is a rectangular band of groups of either one of the types 1×1, 1×2, 2×1, 2×2*[1].

THEOREM 4.3. *Let S be a periodic semigroup containing three or more elements. Then the following assertions on S are equivalent:*

(1) *Every quasi-congruence on S is transitive.*

(2) *Quasi-congruences on S are permutable.*

(3) *S is a rectangular band of groups of either one of the types 1×1, 1×2, 2×1, 2×2.*

THEOREM 4.4. *On a periodic semigroup S the following assertions are equivalent:*

(1) *Every meromorphism of S onto itself is a class-meromorphism.*

(2) *Every quasi-congruence on S is symmetric.*

(3) *S is a group.*

THEOREM 5.1. *Every meromorphism of a periodic quasigroup G onto itself is a class-meromorphism.*

2. General discussions on quasi-congruences and congruences

Defining $\varphi \leqq \psi$ to mean that $a\varphi b$ implies $a\psi b$, the set $Q(A)$ ($\Theta(A)$) of quasi-congruences (congruences) on an algebraic system A forms a complete lattice.

1) We shall define rectangular bands of groups in § 4.

Let P be a set of ordered pairs (a, b) of elements of A. We denote by $\theta(P)$ ($\theta^*(P)$) the least of quasi-congruences (congruences) φ satisfying $a\varphi b$ for all pairs $(a, b)\in P$. Put $D=\{(a, a); a\in A\}$, $P'=\{(b, a); (a, b)\in P\}$, $P_0=P\cup D$ and $P^*=P\cup D\cup P'$. Then we obtain immediately

LEMMA 2.1. *$u\rightarrow v(\theta(P))$ holds if and only if there exists a polynomial $p(x_1, \cdots, x_\nu)$ (an algebraic combination of $x_1, \cdots, x_\nu$) such that*

$$u=p(a_1, \cdots, a_\nu),\ v=p(b_1, \cdots, b_\nu) \text{ for some } (a_i, b_i)\in P_0,$$

LEMMA 2.2. *$u\equiv v(\theta^*(P))$ holds if and only if there exist a sequence of polynomials $p_i(x_{i1}, \cdots, x_{i\nu_i})$ and elements $u_i, v_i\in A$ such that*

$$u=u_1,\ v=v_n,\ v_i=u_{i+1},\ u_i=p_i(a_{i1}, \cdots, a_{i\nu_i}),\ v_i=p_i(b_{i1}, \cdots, b_{i\nu_i})$$

for some $(a_{ij}, b_{ij})\in P^*$.

In a previous paper [3] we have shown that the condition (ζ) can be written in the following expression.

LEMMA 2.3. *All congruences on an algebraic system A are permutable if and only if, given $a, b, c\in A$, an element u exists such that $a\equiv u(\theta^*(c, b))$ and $u\equiv b(\theta^*(a, c))$.*

Combining Lemma 2.2 and Lemma 2.3, we have

THEOREM 2.1. *All congruences on an algebraic system A are permutable if and only if, given $a, b, c\in A$, two sequences of polynomials $p_i(x, y, z_1, \cdots, z_{\mu_i})$ and $q_j(x, y, z_1, \cdots, z_{\nu_j})$ exist such that*

$$\begin{aligned} a=p_1(b, c, s_{11}, \cdots), \cdots, p_i(c, b, s_{i1}, \cdots)&=p_{i+1}(b, c, s_{i+1,1}, \cdots), \\ \cdots, p_m(c, b, s_{m1}, \cdots)&=q_1(a, c, t_{11}, \cdots), \\ \cdots, q_j(c, a, t_{j1}, \cdots)&=q_{j+1}(a, c, t_{j+1,1}, \cdots), \\ \cdots, q_n(c, a, t_{n1}, \cdots)&=b. \end{aligned}$$

In the same way as above we can show the conditions (γ)—(ε) is rewritten as follows:

THEOREM 2.2. *Every quasi-congruence on an algebraic system A is symmetric if and only if, given $a, b\in A$, $a\rightarrow b(\theta(b, a))$ holds; namely there exists a polynomial $p(x, y_1, \cdots, y_n)$ such that $a=p(b, t_1, \cdots, t_n)$ and $b=p(a, t_1, \cdots, t_n)$.*

THEOREM 2.3. *Every quasi-congruence on an algebraic system A is transitive if and only if, given $a, b, c\in A$, $a\rightarrow b(\theta(a, c)\cup\theta(c, b))$ holds; namely there exists a polynomial $p(x, y, z_1, \cdots, z_n)$ such that $a=p(a, c, t_1, \cdots, t_n)$ and $b=p(c, b, t_1, \cdots, t_n)$.*

THEOREM 2.4. *All quasi-congruences on an algebraic system A are permutable if and only if, given $a, b, c\in A$, an element u exists such that $a\rightarrow u$ $(\theta(c, b))$ and $u\rightarrow b(\theta(a, c))$; namely there exist polynomials $p(x, y_1, \cdots, y_m)$ and $q(x, y_1, \cdots, y_n)$ such that $a=p(c, s_1, \cdots, s_m)$, $p(b, s_1, \cdots, s_m)=q(a, t_1, \cdots, t_n)$ and*

$q(c, t_1, \cdots, t_n)=b$.

By the way we should remark that in the propositions mentioned above a polynomial $p(\cdots, x, \cdots)$ may not necessarily contain x.

Now in the present paper we intend to deal with multiplicative systems. A zero element of a multiplicative system M is an element 0 satisfying $a0=0a=0$ for all $a \in M$. Concerning such an element we first show

LEMMA 2.4. *Let M be a multiplicative system containing three or more elements. If all quasi-congruences on M are permutable, then M does not contain* 0.

PROOF. Given $a, b, c \in M$, we can find polynomials $p(x, y_1, \cdots, y_m)$ and $q(x, y_1, \cdots, y_n)$ such that $a=p(c, s_1, \cdots, s_m)$, $u=p(b, s_1, \cdots, s_m)=q(a, t_1, \cdots, t_n)$ and $q(c, t_1, \cdots, t_n)=b$. If $c=0$ and $p(x, y_1, \cdots, y_m)$ contains x, then $p(c, s_1, \cdots, s_m)=0$. So if $a \neq 0$, p cannot contain x and hence $a=p(s_1, \cdots, s_m)=u$. Similarly if $b \neq 0$, we can infer $b=u$. Therefore if M contains 0, then M cannot contain two elements other then 0.

3. Meromorphisms and congruences on commutative semigroups

In § 3 and § 4 we deal with semigroups, i. e. associative multiplicative systems. The basic concepts and notations on semigroups may be refered to the books of Bruck [1] and of Clifford and Preston [2]. An *ideal* of a semigroup S is a subset J of S satisfying $JS \subseteqq J$ and $SJ \subseteqq J$, and a *principal ideal* generated by a is an ideal $J(a)$ such that $J(a)=SaS \cup aS \cup Sa \cup a$. If S contains no proper ideal, then S is called *simple*.

Now let all congruences on a semigroup S be permutable. Then, given $x, y, z \in S$, we can find an element $u=u(x, y, z)=u(y, x, z) \in S$ such that $x \equiv u$ $(\theta^*(y, z))$ and $y \equiv u(\theta^*(x, z))$. About such an element u we first show

LEMMA 3.1. *$a \notin J(b)$ and $a \notin J(c)$ imply $u(a, b, c)=u(b, a, c)=a$.*

PROOF. $u=u(a, b, c)=u(b, a, c)$ satisfies $a \equiv u(\theta^*(b, c))$. Then by Lemma 2.2 we can find polynomials $p_i(x, y, z_{i1}, \cdots)$ such that

$$a=a_0=p_1(b, c, s_{11}, \cdots),\ a_1=p_1(c, b, s_{11}, \cdots)$$
$$=p_2(b, c, s_{21}, \cdots), \cdots, a_n=p_n(c, b, s_{n1}, \cdots)=u\,.$$

If p_i contains x, then $a_{i-1}=p_i(b, c, s_{i1}, \cdots) \in J(b)$, and if p_i contains y, then $a_{i-1}=p_i(b, c, s_{i1}, \cdots) \in J(c)$. Hence if $a_{i-1} \notin J(b)$ and $a_{i-1} \notin J(c)$, then p_i cannot contain both x and y, and it follows that $a_i=p_i(c, b, s_{i1}, \cdots)=p_i(b, c, s_{i1}, \cdots)=a_{i-1}$. Accordingly $a_0 \notin J(b)$ and $a_0 \notin J(c)$ imply $a_0=a_1=\cdots=a_n$.

THEOREM 3.1. *If all congruences on a semigroup S are permutable, then all ideals of S are totally ordered under set-inclusion.*

PROOF. Let J_1 and J_2 be two ideals and suppose that $J_1-J_2 \ni a$ and J_2-J_1

$\ni b$. Put $ab=c$. Then $J(c)\subseteqq J(a)\subseteqq J_1$, $J(c)\subseteqq J(b)\subseteqq J_2$ and hence $a\notin J(b)$, $a\notin J(c)$, $b\notin J(a)$, $b\notin J(c)$. By Lemma 3.1 we get $u(a, b, c)=a$ and $u(b, a, c)=b$; consequently $a=b$, that is a contradiction.

Next we shall consider commutative semigroups. If $x\equiv u(\theta^*(y, z))$ holds in a commutative semigroup S, then by Lemma 2.2 we get

$$x=y^{\mu_1}z^{\nu_1}t_1^{\delta_1},\ z^{\mu_1}y^{\nu_1}t_1^{\delta_1}=y^{\mu_2}z^{\nu_2}t_2^{\delta_2},\ \cdots,\ z^{\mu_n}y^{\nu_n}t^{\delta_n}=u\,,$$

where μ_i, ν_i, δ_i are non-negative integers such that $\mu_i+\nu_i+\delta_i\geqq 1$, provided y^0, z^0, t_i^0 are delated when they occur. It follows that

$$xy^{\nu_1}z^{\mu_1}=y^{\mu_1+\nu_1}z^{\mu_1+\nu_1}t^{\delta_1}=y^{\mu_1+\mu_2}z^{\nu_1+\nu_2}t_2^{\delta_2}$$

and

$$xy^{\nu_1+\cdots+\nu_n}z^{\mu_1+\cdots+\mu_n}=y^{\mu_1+\cdots+\mu_n+\nu_n}z^{\nu_1+\cdots\nu_n+\mu_n}t_n^{\delta_n}=uy^{\mu_1+\cdots+\mu_n}z^{\nu_1+\cdots+\nu_n}\,;$$

namely $xy^{\nu}z^{\mu}=uy^{\mu}z^{\nu}$ for some non-negative integers μ, ν.

Now let S be a commutative semigroup with or without zero 0, and all congruences on S be permutable. Then $a\notin J(b)$ and $a\notin J(c)$ imply $u(a, b, c)=u(b, a, c)=a$ by Lemma 3.1 and $ba^{\nu}c^{\mu}=ua^{\mu}c^{\nu}=a^{\mu+1}c^{\nu}$, since $b\equiv u(b, a, c)$ $(\theta^*(a, c))$. So we see that S satisfies the following lemmas.

LEMMA 3.2. *$a\notin J(b)$ and $a\notin J(c)$ imply $ba^{\nu}c^{\mu}=a^{\mu+1}c^{\nu}$ for some non-negative integers μ, ν.*

LEMMA 3.3. *Given $a\in S$, a positive integer α exists such that $a^{\alpha}\in a^{\alpha}S$.*

PROOF. Assume that $a\notin aS$. Then $a\notin J(a^2)=a^2S\cup a^2$ and $a\notin J(a^3)$; hence we can infer $a^2a^{\nu}(a^3)^{\mu}=a^{\mu+1}(a^3)^{\nu}$ by Lemma 3.2. Put

$$\mathrm{Min}\,(2+\nu+3\mu,\ \mu+1+3\nu)=\alpha,\ \mathrm{Max}\,(2+\nu+3\mu,\ \mu+1+3\nu)=\alpha+\beta\,.$$

Since $\mu\geqq 0$, $\nu\geqq 0$ and $(2+\nu+3\mu)-(\mu+1+3\nu)=2(\mu-\nu)+1\neq 0$, we get $\alpha\geqq 1$, $\beta\geqq 1$ and $a^{\alpha}=a^{\alpha}a^{\beta}\in a^{\alpha}S$.

LEMMA 3.4. *If $a=ac$, then cS contains an idempotent e satisfying $ae=a$.*

PROOF. If $c=a$, then $c^2=c$. If $c\in aS$, then $c=ax$ satisfies $c^2=cax=ax=c$. In either case we may put $e=c$. If $c\in c^2S$ and $c=c^2x$, then, putting $e=cx\in cS$, we get $e^2=c^2xx=cx=e$ and $ae=accx=ac=a$. It remains only the case that $c\notin J(a)$ and $c\notin J(c^2)$. Then by Lemma 3.2 we get $ac^{\nu}c^{2\mu}=c^{\mu+1}c^{2\nu}$. Since $ac^{\lambda}=a$, we have $a=c^{\alpha}$ $(\alpha=\mu+1+2\nu)$ and $a^2=ac^{\alpha}=a$; thus $e=a$ satisfies the conclusion of the lemma.

LEMMA 3.5. *If e is idempotent, then either $e=0$ or $e=1$ (the identity).*

PROOF. Assume that $e\neq 0$. We shall show $eS=S$. Suppose $a\notin eS$ and put $b=ea$. Then since $a\notin J(e)$ and $a\notin J(b)$, we get $u(e, a, b)=a$, and $e\equiv u(e, a, b)$ $(\theta^*(a, b))$ implies

$$e=a^{\mu_1}b^{\nu_1}t_1^{\delta_1},\ b^{\mu_1}a^{\nu_1}t_1^{\delta_1}=a^{\mu_2}b^{\nu_2}t_2^{\delta_2},\ \cdots,\ b^{\mu_n}a^{\nu_n}t_n^{\delta_n}=a\,.$$

Put $et_i^{\delta_i}=s_i$. Since $ea^{\lambda}=b^{\lambda}$, we have

$$b^{\mu_i+\nu_i}s_i=e(b^{\mu_i}a^{\nu_i}t_i^{\delta_i})=e(a^{\mu_{i+1}}b^{\nu_{i+1}}t_{i+1}^{\delta_{i+1}})=b^{\mu_{i+1}+\nu_{i+1}}s_{i+1}$$

and $e=e^2=b^{\mu_1+\nu_1}s_1=b^{\mu_2+\nu_2}s_2=\cdots=b^{\mu_n+\nu_n}s_n=ea$. Since $e\neq 0$, eS contains $c=ec\neq e$. Then $a\notin J(c)$ and $a\notin J(e)$ imply $ca^\nu e^\mu=a^{\mu+1}e^\nu$. Since $ea=e$ and $ec=c$, we obtain $c=e(ca^\nu e^\mu)=e(a^{\mu+1}e^\nu)=e$, that is a contradiction. Thus $eS=S$ and so $e=1$.

LEMMA 3.6. *If an element $a\in S$ is not nilpotent, then S contains the identity* 1 *and inverse a^{-1} of $a: aa^{-1}=1$.*

PROOF. There exists a positive integer α such that $a^\alpha\in a^\alpha S$; hence we can find an idempotent e satisfying $a^\alpha=a^\alpha e$ by Lemma 3.4. If $a^\alpha\neq 0$, then $e\neq 0$ and hence $e=1$. Suppose that $1\notin aS$. Then we see $1\notin J(a)$, $1\notin J(a^2)$ and $a1^\nu a^{2\mu}=1^{\mu+1}a^{2\nu}$ by Lemma 3.2. Put

$$\operatorname{Min}(2\mu+1,2\nu)=\alpha \quad\text{and}\quad \operatorname{Max}(2\mu+1,2\nu)=\alpha+\beta\,.$$

Then $\beta\geqq 1$ and $a^\alpha=a^\alpha a^\beta$, provided $a^0=1$. It follows from Lemma 3.4 that $a^\beta S$ contains an idempotent e satisfying $a^\alpha=a^\alpha e$. Since a is not nilpotent, $e=1$ and $aS\supseteqq a^\beta S\ni 1$.

LEMMA 3.7. *If G is the set of non-nilpotent elements of S, then G is a group.*

PROOF. Let a and b be any elements of G. Then S contains 1 and a^{-1}. Since $(a^{-1})^\alpha(ab)^\alpha=b^\alpha\neq 0$, $(ab)^\alpha\neq 0$ and $ab\in G$. Since $(a^{-1})^\alpha a^\alpha=1\neq 0$, $(a^{-1})^\alpha\neq 0$ and $a^{-1}\in G$.

In summary we have

THEOREM 3.2. *Congruences on a commutative semigroup S are permutable if and only if* (i) *ideals of S are totally ordered under set-inclusion, and* (ii) *non-nilpotent elements of S form a group.*

PROOF. The necessity of the conditions (i), (ii) has been proved above. Now suppose that S satisfies (i) and (ii). If $a\neq b$ and $a\notin bS$, then $J(a)\nsubseteq J(b)$, whence we get $b\in J(b)\subseteqq J(a)$ by the condition (i). So it follows that (i)′ two distinct elements a, b of S satisfy $a\in bS$ or $b\in aS$. Let S contain non-nilpotent elements and e be the identity of the group G formed by them. If a is nilpotent, then ax is also nilpotent. Hence $a\notin G$ implies $e\notin aS$ and $a\in eS$ by (i)′; accordingly e becomes the identity of S. Now let θ and φ be any congruences on S and assume $a\theta c\varphi b$ for three distinct elements a, b, c. We may assume $b\in aS$ by (i)′.

Case I: $a\in cS$. Then $b\in aS\subseteqq cS$, and $a=cx$ and $b=cy$ imply $a=cx\varphi bx=cxy=ay\theta cy=b$.

Case II: $a\notin cS$. Then we may write $c=au\in aS$ by (i)′. Since $u\in G$ implies $a=auu^{-1}=cu^{-1}\in cS$, $u\notin G$ and $u^\alpha=0$ for some integer α. It follows that $a\theta au$, $au\theta au^2, \cdots, au^{\alpha-1}\theta au^\alpha=0$; hence $a\theta 0$ and $b=av\theta 0$. Thus we have $a\varphi a\theta b$. So θ and φ are permutable.

In the above theorem S may contain no non-nilpotent element. For instance a nilpotent cyclic semigroup $Z(a, n)=\{a, a^2, \cdots, a^n=0\}$ satisfies the condition.

On the other hand if two distinct elements a, b of S satisfy $J(a)=J(b)$, then $a=bx$, $b=ay$, $a=axy$ and we can show $x, y \in G$. Indeed, if $x \notin G$, then $z=xy$ is nilpotent and $a=az=az^{\alpha}=0$ for some α. Conversely it is easy to see that $x \in G$ implies $J(ax)=J(a)$. So if we classify the elements by $x \equiv y$ to mean $J(x)=J(y)$, choose arbitrarily a representative a_i from each nil class $\{x; J(x)=J(a_i)\}$, $a_iS \neq S$, and set $Z=\{a_i\}$, then S is written either $S=Z$ or $S=(Z\cup 1)G$, where Z is a nil subset forming a totally ordered set under the ordering $a_i \leqq a_j$ to mean $a_i \in J(a_j)$, and G is a group.

Further let S satisfy the ascending condition for ideals. Then we can find a maximal principal nil ideal $J(a)$, provided S contains 0. If b is any nilpotent element, then $J(b) \subseteqq J(a)$ and b is written $b=a^{\alpha}$ or $b=a^{\alpha}x$ for some positive integer α and $x \notin J(a)$; that is, $x \in G$. Hence we can infer

COROLLARY 1. *Let S be a commutative semigroup with zero satisfying the ascending condition for ideals. Then congruences on S are permutable if and only if a nilpotent element $a \in S$ exists and either S is a nilpotent cyclic semigroup Z generated by a or $S=(Z\cup 1)G=\bigcup a^iG \cup G$, where G is a group.*

A simple application of Corollary 1 is the following

EXAMPLE. Let S be the multiplicative semigroup of residue classes of integers modulo m. Then congruences on S are permutable if and only if m is a power of a prime number.

The following corollary is obvious.

COROLLARY 2. *Congruences on a commutative semigroup S without zero are permutable if and only if S is a group.*

In a previous paper [4] we have shown that every quasi-congruence on a group is transitive. Therefore combining Lemma 2.4 and the above corollary, we have

THEOREM 3.3. *Let S be a commutative semigroup containing three or more elements. Then the following assertions on S are equivalent:*

(1) *Every quasi-congruence on S is transitive.*

(2) *Quasi-congruences on S are permutable.*

(3) *S is a group.*

Further we have also proved in the paper cited above

THEOREM 3.4. *The following assertions on a commutative semigroup S are equivalent:*

(1) *Every meromorphism of S onto itself is a class-meromorphism.*

(2) *Every quasi-congruence on S is a congruence.*

(3) *Every quasi-congruence on S is symmetric.*

(4) *S is a periodic group.*

4. Meromorphisms and congruences on general semigroups

About a semigroup on which congruences are permutable we have obtained Theorem 3.1. As for a semigroup on which quasi-congruences are permutable we show the following

THEOREM 4.1. *If quasi-congruences on a semigroup S containing three or more elements are permutable, then S is simple.*

PROOF. Let $J(a)$ be any principal ideal of S and suppose that $J(a) \not\ni e$. Given $b \in S$, we can find polynomials $p(x, y_1, \cdots, y_m)$ and $q(x, y_1, \cdots, y_n)$ such that

$$e = p(a, s_1, \cdots, s_m),\ u = p(b, s_1, \cdots, s_m) = q(e, t_1, \cdots, t_n),\ b = q(a, t_1, \cdots, t_n).$$

If p contains x, then $e = p(a, s_1, \cdots, s_m) \in J(a)$; hence p cannot contain x and we get $e = u = p(s_1, \cdots, s_m)$. On the other hand if q does not contain x, then $b = u = e$; so it follows that $b = q(a, t_1, \cdots, t_n) \in J(a)$ for $b \neq e$; namely $J(a) = S - e$. Further $J(b) \subseteq J(a)$ implies $J(b) \not\ni e$, and $J(b) = S - e$ is deduced similarly as above. Then if q contains y_i such that $t_i \neq e$, we have $e = q(e, t_1, \cdots, t_n) \in J(t_i) = S - e$; hence it follows that $e = q(e, e, \cdots, e) = e^\alpha$. If $\alpha = 1$, then $q(x, y_1, \cdots, y_n) = x$ and $b = a$. Since S contains at least one element b different from e and a, $e^\alpha = e$ must hold for some $\alpha > 1$. If $e^\alpha = e$ and $e^{\alpha-1} \neq e$, then $e = e^\alpha \in J(e^{\alpha-1}) = S - e$; consequently we get $e^2 = e$. We can now assert from the arguments stated above that an element b different from e and a can be written $b = q(a, e)$. Since $ea \in J(a)$, $ea \neq e$. If $ea \neq a$, it follows that $a = q(ea, e)$ is rewritten either $a = eaq'(ea, e)$ or $a = eq'(ea, e)$ and in either case $ea = a$, since $e^2 = e$. So that we see $ea = ae = a$ and $b = q(a, e) = a^\beta$ for every $b \neq e$. Then S is commutative and we infer that S must be a group by Theorem 3.3, contradicting $e \notin J(a)$. Thus $J(a) = S$ for all $a \in S$.

Rees [5] has proved that if a periodic semigroup, that is, a semigroup in which every element has a finite order, is simple, then it is completely simple. A completely simple semigroup without zero is a union of disjoint subgroups $\{H_{\alpha\lambda}; \alpha \in X, \lambda \in Y\}$ satisfying $H_{\alpha\lambda} H_{\beta\mu} = H_{\alpha\mu}$, which we shall call a *rectangular band of groups of type* $|X| \times |Y|$, provided $|X|$ means the cardinal number of X. In such a semigroup S, $R_\alpha = \bigcup_\lambda H_{\alpha\lambda}$ is a minimal right ideal, $L_\lambda = \bigcup_\alpha H_{\alpha\lambda}$ is a minimal left ideal, and $S = \bigcup_\alpha R_\alpha = \bigcup_\lambda L_\lambda$.

LEMMA 4.1. *Let S be a semigroup on which congruences are permutable. Then S cannot be decomposed into the union of three or more disjoint right (left) ideals.*

PROOF. Suppose that $S = R_1 \cup R_2 \cup R_3 \cup \cdots$, where the R_α are right ideals of S and mutually disjoint. If congruences on S are permutable, then, given

$a \in R_1$, $b \in R_2$, $c \in R_3$, we can find sequences of polynomials p_i, q_j such that

$$a = p_1(b, c, s_{11}, \cdots), p_1(c, b, s_{11}, \cdots) = p_2(b, c, s_{21}, \cdots), \cdots, p_m(c, b, s_{m1}, \cdots)$$
$$= u = q_1(a, c, t_{11}, \cdots), q_1(c, a, t_{11}, \cdots) = q_2(a, c, t_{21}, \cdots), \cdots, q_n(c, a, t_{n1}, \cdots) = b .$$

$a_i = p_i(b, c, s_{i1}, \cdots)$ is written into either one of the forms $bp_i'(b, c, s_{i1}, \cdots)$, $cp_i'(b, c, s_{i1}, \cdots)$, $s_{ij}p_i'(b, c, s_{i1}, \cdots)$, and $bp_i' \in R_2$, $cp_i' \in R_3$. Therefore if $a_i \in R_1$, then $a_i = s_{ij}p_i'(b, c, s_{i1}, \cdots)$ with $s_{ij} \in R_1$ and hence $a_{i+1} = s_{ij}p_i'(c, b, s_{i1}, \cdots) \in R_1$. Repeating this, we deduce $u = a_{m+1} \in R_1$ from $a = a_1 \in R_1$. Similarly $b \in R_2$ implies $u \in R_2$, contradicting $u \in R_1$.

In the case that S is completely simple, S is decomposed into $S = \bigcup_\alpha R_\alpha = \bigcup_\lambda L_\lambda$. If congruences on S are permutable, $X = \{\alpha\}$ can contain at most two indices and so does $Y = \{\lambda\}$; hence $|X| \times |Y|$ must be either one of 1×1, 1×2, 2×1, 2×2.

THEOREM 4.2. *Let S be a completely simple semigroup without zero. Then congruences on S are permutable if and only if S is a rectangular band of groups of either one of the types* 1×1, 1×2, 2×1, 2×2.

PROOF. The half of the theorem has been proved above. Now let S be a union of group $\{H_{\alpha\lambda}; \alpha \in X, \lambda \in Y\}$ satisfying $H_{\alpha\lambda}H_{\beta\mu} = H_{\alpha\mu}$ and both X and Y contain at most two indices. Then we shall show that $a\theta c\varphi b$ implies $a\varphi u\theta b$ for any congruences θ, φ. Set $a \in H_{\alpha\lambda}$, $b \in H_{\beta\mu}$, $c \in H_{\gamma\nu}$. Suppose that $\gamma \neq \alpha$ and $\gamma \neq \beta$. Then $\alpha = \beta$ must hold. If e is the identity of the group $H_{\alpha\lambda}$, then e becomes a left identity of $H_{\beta\mu} = H_{\alpha\lambda}H_{\beta\mu}$ and we have $a = ea\theta ec\varphi eb = b$ with $ec \in H_{\alpha\nu}$. So we may assume $\gamma = \alpha$, and it is sufficient to consider the two cases: (1) $\gamma = \alpha$, $\nu = \mu$; (2) $\gamma = \alpha$, $\nu = \lambda$.

Case (1): $\gamma = \alpha$, $\nu = \mu$. Let e be the identity of the group $H_{\gamma\nu} = H_{\alpha\mu}$ and c^{-1} the inverse of c in $H_{\gamma\nu}$. Then e becomes a left identity of $H_{\alpha\lambda}$ and a right identity of $H_{\beta\mu}$. Hence we get $a = ea = cc^{-1}a\varphi bc^{-1}a\theta bc^{-1}c = be = b$.

Case (2): $\gamma = \alpha$, $\nu = \lambda$. Let e and f be the identities of $H_{\alpha\lambda}$ and $H_{\alpha\mu}$ respectively, c^{-1} the inverse of c in $H_{\gamma\nu} = H_{\alpha\lambda}$ and b^{-1} the inverse of eb in $H_{\alpha\mu}$. Then we obtain the following formulas:

$$ef = f, fe = e, bf = b, ea = ae = a, cc^{-1} = c^{-1}c = e, fc^{-1} = c^{-1}, b^{-1}eb = f;$$

and

$$c^{-1} = (b^{-1}eb)c^{-1}\varphi b^{-1}ecc^{-1} = b^{-1}e,$$
$$a = cc^{-1}ac^{-1}c\varphi bc^{-1}a(b^{-1}e)b\theta bc^{-1}c(b^{-1}eb) = bef = b ,$$

completing the proof.

We can further prove that every quasi-congruence on S is transitive if S is a rectangular band of groups of either one of the types 1×1, 1×2, 2×1, but not so does that of type 2×2. On the other hand, if every element of

$H_{\alpha\lambda}$ has a finite order, all semigroups of the above types satisfy the condition (δ). Indeed, let $a\theta c\theta b$ for any quasi-congruence θ. Then using the same notations as the above proof, we can deduce the following implications:

Case (1): $c\theta b$ and $a\theta c$ imply $a=cc^{-1}a\theta bc^{-1}c=b$.

Case (2): Let the order of eb in $H_{\alpha\mu}$ be m and that of c in $H_{\alpha\lambda}$ n. Then $c=ec\theta eb$ implies $c^{mn-1}\theta(eb)^{mn-1}$; that is, $c^{-1}\theta b^{-1}$; hence we see $a=cc^{-1}ac^{-1}ec\theta bc^{-1}cb^{-1}eb=b$.

As shown in Theorem 4.1, if quasi-congruences on a periodic semigroup S are permutable, then S is simple and accordingly completely simple; so we infer

THEOREM 4.3. *The following assertions on a periodic semigroup S containing three or more elements are equivalent:*

(1) *Every quasi-congruence on S is transitive.*

(2) *Quasi-congruences on S are permutable.*

(3) *S is a rectangular band of groups of either one of the types* 1×1, 1×2, 2×1, 2×2.

Next we shall deal with the symmetricity of quasi-congruences.

LEMMA 4.2. *If every quasi-congruence on a semigroup S is symmetric, then S is join-irreducible with respect to right (left) ideals; namely at least one of right ideals R_1, R_2 satisfying $S=R_1\cup R_2$ coincides with S.*

PROOF. Suppose that $a\notin R_1$. Then $a\in R_2$. Since every quasi-congruence on S is symmetric, given $b\in S$, we can find a polynomial p such that $a=p(b, t_1, \cdots, t_n)$, $b=p(a, t_1, \cdots, t_n)$. Then we get either $a=bp'(b, \cdots)$, $b=ap'(a, \cdots)\in R_2$ or $a=t_ip'(b, \cdots)$, $b=t_ip'(a, \cdots)$. In the latter case $a\notin R_1$ implies $t_i\notin R_1$, $t_i\in R_2$ and hence $b\in R_2$. Thus $R_2=S$.

Since the union of right ideals is also a right ideal, in the above semigroup S, $S=R_1\cup\cdots\cup R_n$ implies $S=R_i$ for some i.

THEOREM 4.4. *The following assertions on a periodic semigroup S are equivalent:*

(1) *Every meromorphism of S onto itself is a class-meromorphism.*

(2) *Every quasi-congruence on S is a congruence.*

(3) *Every quasi-congruence on S is symmetric.*

(4) *S is a group.*

PROOF. Suppose that a periodic semigroup S satisfies (3). If S contains three or more elements, then from Theorem 4.1 S is simple and hence completely simple. It is easy to see that a two-element semigroup is either a commutative semigroup or a right or left group; and it follows from Theorem 3.4 that the former does not satisfy (3) unless it is a group. Anyhow S is completely simple, and indecomposable with respect to both right and left ideals, as shown in the above lemma; accordingly S is a group. Then it suffices to prove that every meromorphism of a periodic group onto itself is a class-meromorphism. We shall show that on quasigroups below.

5. Meromorphisms of quasigroups

It has been proved by Trevisan [8] that congruences on any finite quasigroup are permutable. We shall show more strongly that any periodic quasigroup (in the sense mentioned below) possesses the property (α).

LEMMA 5.1. *A meromorphism θ of a quasigroup G onto itself is a class-meromorphism if any two of $a\theta b$, $x\theta y$, $ax\theta by$ imply the third.*

PROOF. Suppose that $a\theta b$, $a'\theta b$ and $a'\theta b'$. Choose elements x, y, z and u so that $a=ax$, $b=by$, $a'=az$ and $b'=uy$. Then by the assumption we see the following implications:

$$a\theta b,\ ax\theta by \rightarrow x\theta y;\ a\theta b,\ az\theta by \rightarrow z\theta y;\ z\theta y,\ az\theta uy \rightarrow a\theta u;$$

consequently $a\theta u,\ x\theta y \rightarrow ax\theta uy$; that is, $a\theta b'$.

Now in a quasigroup G we shall denote

$$a^n x = a(a(\cdots(a(ax)))),\quad xa^n = (((((xa)a)\cdots)a)a).$$

By the *order* of an element a we shall mean the maximal cardinal number of the sets $\{a^n x; n=0,1,2,\cdots\}$ and $\{xa^n; n=0,1,2,\cdots\}$ when x runs through G, and if every element of G has a finite order we shall call G *periodic*. Then we can show

THEOREM 5.1. *Every meromorphism of a periodic quasigroup G onto itself is a class-meromorphism.*

PROOF. If the set $\{a^n x; n=0,1,2,\cdots\}$ is finite, then $a^m x = a^n x$ for some integers $m, n (m > n \geqq 0)$, and we get $a^{m-n}x = x$ by successive cancellation. Let θ be a meromorphism of G onto itself, and suppose that $a\theta b$ and $ax\theta by$. Then we have $a^k x\theta b^k y$ for all positive integers k. If $a^m x = x$ and $b^n y = y$, then we see $a^{mn}x = x$, $b^{mn}y = y$ and so $x\theta y$. Thus θ satisfies the condition of Lemma 5.1.

Again the proof of Theorem 4.4. has been completed.

Kobe University

References

[1] R.H. Bruck, A survey of Binary Systems, Ergebnisse der Math., Heft 20, Berlin, 1958.

[2] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups I, Math. Surveys, No. 7, Amer. Math. Soc., Providence, 1961.

[3] J. Hashimoto, Congruence relations and congruence classes in lattices, Osaka Math. J., 15 (1963), 71–86.

[4] J. Hashimoto, On meromorphisms of algebraic systems, Nagoya Math. J., **27** (1966), 559–569.

[5] D. Rees, On semi-groups, Proc. Cambridge Philos. Soc., **36** (1940), 387–400.

[6] K. Shoda, Uber die allgemeinen algebraischen Systeme I, Proc. Imp. Acad. Japan, **17** (1941), 323–327. II, ibid., **18** (1942), 179–184. III, ibid., **18** (1942), 227–232. IV, ibid., **18** (1942), 276–279. V, ibid., **19** (1943), 114–118. VI, ibid., **19** (1943), 259-263. VII, ibid., **19** (1943), 515–517. VIII, ibid., **20** (1944), 584–588.

[7] K. Shoda, General Algebra (Daisugaku Tsuron in Japanese), Tokyo, 1949.

[8] G. Trevisan, A proposita della relazione di congruenza sui quasigrouppi, Rend. Sem. Mat. Univ. Padova, **19** (1950), 367–370.