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A problem on the existence of an Einstein metric

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This note is to present a variational problem, to which the problem of the existence of an Einstein metric is reduced. The latter has called certain geometers' attention from various points of view; for instance, if an Einstein metric is proved to exist on a compact simply connected 3-manifold then the Poincaré conjecture for the 3-sphere is answered affirmatively (see e. g. [**1**]). The variational problem arises from the characterization of Einstein metrics in the following theorem.

THEOREM. *Let M be an n-dimensional compact orientable manifold with a fixed volume element Ω (or an $SL(n, \boldsymbol{R})$-structure P). And let $\mathcal{G}$ be the family of all Riemannian metrics with Ω on M (or all $SO(n)$-structures contained in P). Then a Riemannian metric g_0 in $\mathcal{G}$ is Einstein if and only if the function $I=I(g)=\int_M R\Omega$ on $\mathcal{G}$ attains a critical value at g_0, where R is the scalar curvature of g.*

Before the proof some comments will be adequate. An $SL(n, \boldsymbol{R})$-structure is essentially unique on M [**2**]. I attains its critical value at g_0 by definition if one has $D(I(g(t)))=g_0$ at $t=0$, $D=d/dt$, for any differentiable one-parameter family $\{g(t)\}$ with $g(0)=g_0$ of Riemannian metrics in $\mathcal{G}$. The theorem is trivial when $n=2$, since I is then a constant by the Gauss-Bonnet formula.

Now the proof is given by a straightforward tensor calculus. We put $a=Dg(t)$. Such an $a=(a_{ij})$ is characterized as a symmetric covariant tensor field of degree 2 whose "trace" $a_a^a=g^{ij}a_{ij}$ vanishes identically due to the assumption on the volume element. It is easy to obtain $DR_{ij}=\nabla_a D\left\{{a \atop i\ \ j}\right\}$ by noting $D\left\{{a \atop a\ \ j}\right\}=0$, so that we have $DI(g)=D\int_M R\Omega=\int_M (DR)\Omega=\int_M (g^{ab}DR_{ab}+R_{ab}Dg^{ab})\Omega=\int_M (\nabla_a g^{bc}D\left\{{a \atop b\ \ c}\right\}-R_{ab}g^{ab})\Omega=-\int_M R_{ab}a^{ab}\Omega$, since the second integrand in the third integral is the divergence of the vector field $(v^i)=\left(g^{bc}D\left\{{i \atop b\ \ c}\right\}\right)$. Thus $DI(g)$ vanishes if and only if $R_{ab}a^{ab}$ vanishes everywhere on M. This occurs when and only when (R_{ab}) is a scalar multiple of g. The theorem is proved.

It should be noted that in the variation above the volume element is left fixed, which is a condition which was not posed by Hilbert (Nachr. Ges. Wiss. Göttingen, p. 395, 1915) and Einstein in deriving the Einstein field equation (see A. H. Taub, Conversation laws and variational principles in general relativity, Lecture I, Santa Barbara, 1962).

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