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Solvable groups with isomorphic group algebras

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1. Several authors have studied the interesting problem whether non-isomorphic groups can have isomorphic integral group algebras. J. A. Cohn and D. Livingstone [1], D. B. Coleman [2] and D. S. Passman [3] gave partial answers in the case of nilpotent groups. Their results seem to be based mainly on the fact that the center of a group is determined by its group algebra.

In this paper we intend to show that the derived groups of a group are determined by its group algebra. Thus for solvable groups we can prove

MAIN THEOREM. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be finite groups with isomorphic group algebras over the ring of all integers in a finite algebraic number field. If $\mathfrak{G}$ is solvable, then $\mathfrak{H}$ is solvable with the same length of the derived series as that of $\mathfrak{G}$ and the factor groups of their derived series are isomorphic.*

Throughout this paper R denotes the ring of all integers in a finite algebraic number field and Z the ring of rational integers. For an arbitray finite group $\mathfrak{G}$ the group algebra $R(\mathfrak{G})$ (resp. $Z(\mathfrak{G})$) is the algebra over R (resp. Z) with a free basis multiplicatively isomorphic with $\mathfrak{G}$. We shall often identify the elements of this basis with the elements of $\mathfrak{G}$.

2. Let $\mathfrak{G}$ be a finite group. Then the group algebra $R(\mathfrak{G})$ is an augmented algebra with the unit augmentation $\eta_{\mathfrak{G}}$ and the augmentation ideal $I(\mathfrak{G})$ is a two-sided ideal in $R(\mathfrak{G})$ with R-free basis $g-1$, $g \neq 1$, $g \in \mathfrak{G}$, where 1 denotes the identity element of $\mathfrak{G}$.

LEMMA 1. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be finite groups. If $\varphi\,; R(\mathfrak{G}) \cong R(\mathfrak{H})$ is an isomorphism as algebras, then there exists a group $\mathfrak{G}'$ consisting of unit elements of finite order in $R(\mathfrak{G})$ such that $\mathfrak{G}' \cong \mathfrak{G}$, $R(\mathfrak{G}') = R(\mathfrak{G})$ and $\eta_{\mathfrak{G}'} = \eta_{\mathfrak{H}} \circ \varphi$.*

PROOF. For each $g \in \mathfrak{G}$, $\eta_{\mathfrak{H}} \circ \varphi(g)$ is a unit of finite order in R. Then we see easily that $\mathfrak{G}' = \{g' = (\eta_{\mathfrak{H}} \circ \varphi(g))^{-1} g \in R(\mathfrak{G})\,; g \in \mathfrak{G}\}$ is the desired group.

In view of this lemma, we shall always assume implicitly that an isomorphism $\varphi\,; R(\mathfrak{G}) \cong R(\mathfrak{H})$ of group algebras is compatible with augmentation maps; i. e., $\eta_{\mathfrak{G}} = \eta_{\mathfrak{H}} \circ \varphi$.

PROPOSITION 1. *Any algebra isomorphism $\varphi\,; R(\mathfrak{G}) \cong R(\mathfrak{H})$ induces the ring isomorphism $I(\mathfrak{G}) \cong I(\mathfrak{H})$.*

PROOF. This is immediate since $I(\mathfrak{G})$ and $I(\mathfrak{H})$ are the kernels of $\eta_{\mathfrak{G}}$ and $\eta_{\mathfrak{H}}$, respectively, and φ is compatible with $\eta_{\mathfrak{G}}$ and $\eta_{\mathfrak{H}}$.

PROPOSITION 2. *Any algebra isomorphism φ; $R(\mathfrak{G})\cong R(\mathfrak{H})$ implies the group isomorphism $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]\cong\mathfrak{H}/[\mathfrak{H},\mathfrak{H}]$, where $[\mathfrak{G},\mathfrak{G}]$ denotes the commutator of $\mathfrak{G}$.*

PROOF. The augmentation ideal $I_Z(\mathfrak{G})$ of $Z(\mathfrak{G})$ is a Z-submodule of $I(\mathfrak{G})$ and satisfies that $R\otimes_Z I_Z(\mathfrak{G})\cong I(\mathfrak{G})$. Since R is free over Z, we have

$$I(\mathfrak{G})/I^2(\mathfrak{G})\cong R\otimes_Z(I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G}))\cong(I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G}))^{(k)}$$

as abelian groups, where the last term is the direct sum of k ($=Z$-rank of R) copies of $I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G})$. Similarly,

$$I(\mathfrak{H})/I^2(\mathfrak{H})\cong(I_Z(\mathfrak{H})/I_Z^2(\mathfrak{H}))^{(k)}.$$

But Proposition 1 implies the isomorphism $I(\mathfrak{G})/I^2(\mathfrak{G})\cong I(\mathfrak{H})/I^2(\mathfrak{H})$, which gives the isomorphism of abelian groups

$$(I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G}))^{(k)}\cong(I_Z(\mathfrak{H})/I_Z^2(\mathfrak{H}))^{(k)}.$$

Hence by the fundamental theorem of abelian groups we obtain that $I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G})\cong I_Z(\mathfrak{H})/I_Z^2(\mathfrak{H})$. On the other hand the natural map $g-1\to g$, $g\in\mathfrak{G}$, induces the isomorphism $I_Z(\mathfrak{G})/I_Z^2(\mathfrak{G})\cong\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$. This shows that $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]\cong\mathfrak{H}/[\mathfrak{H},\mathfrak{H}]$, and completes the proof

COROLLARY 1. ([1], [2] *and* [3]). *If $\mathfrak{G}$ is abelian, then $R(\mathfrak{G})\cong R(\mathfrak{H})$ if and only if $\mathfrak{G}\cong\mathfrak{H}$.*

REMARK. Coleman proved that if Q is the rational number field, then $Q(\mathfrak{G})\cong Q(\mathfrak{H})$ implies $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]\cong\mathfrak{H}/[\mathfrak{H},\mathfrak{H}]$ ([2], Th. 1.1, Cor. 1.1). Then, in case $R=Z$, Proposition 2 follows directly from this result since $Z(\mathfrak{G})\cong Z(\mathfrak{H})$ yields $Q(\mathfrak{G})\cong Q(\mathfrak{H})$.

3. Let $\mathfrak{G}$ and $\mathfrak{H}$ be finite groups with an isomorphism $\varphi: R(\mathfrak{G})\cong R(\mathfrak{H})$ of group algebras. For each normal subgroup $\mathfrak{g}$ of $\mathfrak{G}$ and the natural homomorphism $\rho_{\mathfrak{g}}$ of $R(\mathfrak{G})$ onto $R(\mathfrak{G}/\mathfrak{g})$, the normal subgroup $\Phi(\mathfrak{g})$ of $\mathfrak{H}$ is defined (see [1] and [3]) as follows:

$$\Phi(\mathfrak{g})=\{h\in\mathfrak{H};\ \rho_{\mathfrak{g}}\circ\varphi^{-1}(h)=1\}.$$

We see easily

LEMMA 2. ([1] *and* [3]). *Φ is an isomorphism of the lattice of normal subgroups of $\mathfrak{G}$ onto the lattice of normal subgroups of $\mathfrak{H}$, and φ induces an isomorphism $\bar{\varphi}$; $R(\mathfrak{G}/\mathfrak{g})\cong R(\mathfrak{H}/\Phi(\mathfrak{g}))$ such that $\bar{\varphi}\circ\rho_{\mathfrak{g}}=\rho_{\Phi(\mathfrak{g})}\circ\varphi$. In particular* $(\mathfrak{g}:1)=(\Phi(\mathfrak{g}):1)$.

PROPOSITION 3. *Let $\mathfrak{g}$ be any normal subgroup of $\mathfrak{G}$ and set $\mathfrak{h}=\Phi(\mathfrak{g})$. Then $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\cong\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$.*

PROOF. The exact sequence $0\to I(\mathfrak{G})\xrightarrow{\iota}R(\mathfrak{G})\xrightarrow{\eta_{\mathfrak{G}}}R\to 0$ of $Z(\mathfrak{g})$-modules, where $\mathfrak{g}$ acts on R trivially and ι is the inclusion, gives rise to the exact sequence of homology groups of $\mathfrak{g}$

$$0 \longrightarrow H_1(\mathfrak{g}, R) \longrightarrow H_0(\mathfrak{g}, I(\mathfrak{G})) \xrightarrow{\bar{\iota}} H_0(\mathfrak{g}, R(\mathfrak{G})) \xrightarrow{\bar{\eta}_{\mathfrak{G}}} H_0(\mathfrak{g}, R) \longrightarrow 0$$

$(H_1(\mathfrak{g}, R(\mathfrak{G}))=0$ since $R(\mathfrak{G})$ is free over $Z(\mathfrak{g}))$. But $H_0(\mathfrak{g}, R)=R$, $H_0(\mathfrak{g}, R(\mathfrak{G})) = R(\mathfrak{G}/\mathfrak{g})$ and $\bar{\eta}_{\mathfrak{G}}=\eta_{\mathfrak{G}/\mathfrak{g}}$. Then the image of $\bar{\iota}$ is precisely the augmentation ideal $I(\mathfrak{G}/\mathfrak{g})$, and this is free over Z. Therefore $H_0(\mathfrak{g}, I(\mathfrak{G}))$ is isomorphic to the direct sum of $H_1(\mathfrak{g}, R)$ and $I(\mathfrak{G}/\mathfrak{g})$, so that the isomorphic image in $H_0(\mathfrak{g}, I(\mathfrak{G}))$ of $H_1(\mathfrak{g}, R)$ is the torsion part of $H_0(\mathfrak{g}, I(\mathfrak{G}))$. Similarly $H_1(\mathfrak{h}, R)$ is isomorphic to the torsion part of $H_0(\mathfrak{h}, I(\mathfrak{H}))$. On the other hand, the commutativity (Lemma 2) of the following diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(\mathfrak{G})I(\mathfrak{g}) & \longrightarrow & R(\mathfrak{G}) & \xrightarrow{\rho_{\mathfrak{g}}} & R(\mathfrak{G}/\mathfrak{g}) & \longrightarrow & 0 \\ & & & & \Big\downarrow\!\Big\uparrow \varphi & & \Big\downarrow\!\Big\uparrow \bar{\varphi} & & \\ 0 & \longrightarrow & R(\mathfrak{H})I(\mathfrak{h}) & \longrightarrow & R(\mathfrak{H}) & \xrightarrow{\rho_{\mathfrak{h}}} & R(\mathfrak{H}/\mathfrak{h}) & \longrightarrow & 0 \end{array}$$

shows that φ induces the isomorphism $R(\mathfrak{G})I(\mathfrak{g}) \cong R(\mathfrak{H})I(\mathfrak{h})$. Since $\varphi(I(\mathfrak{G}))= I(\mathfrak{H})$ by Proposition 1, it follows that

$$\begin{aligned}\varphi(I(\mathfrak{G})I(\mathfrak{g})) &= \varphi(I(\mathfrak{G}) \cdot R(\mathfrak{G})I(\mathfrak{g})) = \varphi(I(\mathfrak{G})) \cdot \varphi(R(\mathfrak{G})I(\mathfrak{g})) \\ &= I(\mathfrak{H}) \cdot R(\mathfrak{H})I(\mathfrak{h}) = I(\mathfrak{H})I(\mathfrak{h}) .\end{aligned}$$

Thus we have

$$H_0(\mathfrak{g}, I(\mathfrak{G})) = I(\mathfrak{G})/I(\mathfrak{G})I(\mathfrak{g}) \cong I(\mathfrak{H})/I(\mathfrak{H})I(\mathfrak{h}) = H_0(\mathfrak{h}, I(\mathfrak{H}))\,,$$

so that

$$R \otimes_Z (I_Z(\mathfrak{g})/I_Z^2(\mathfrak{g})) = H_1(\mathfrak{g}, R) \cong H_1(\mathfrak{h}, R) = R \otimes_Z (I_Z(\mathfrak{h})/I_Z^2(\mathfrak{h}))$$

since $H_1(\mathfrak{g}, R)$ (resp. $H_1(\mathfrak{h}, R)$) is isomorphic to the torsion part of $H_0(\mathfrak{g}, I(\mathfrak{G}))$ (resp. $H_0(\mathfrak{h}, I(\mathfrak{H}))$). Consequently, the same method as in the proof of Proposition 2 gives an isomorphism $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$, which completes the proof.

COROLLARY 2. *Φ gives a one-to-one correspondence between the set of normal abelian subgroups of $\mathfrak{G}$ and that of $\mathfrak{H}$ and corresponding groups are isomorphic.*

REMARK. This corollary is a generalization to arbitrary groups of the result by Passman [3] in the case of nilpotent groups.

PROPOSITION 4. *If $\mathfrak{G}$ is metabelian* (resp. *metacyclic*), *then $\mathfrak{H}$ is metabelian* (resp. *metacyclic*).

PROOF. Let $\mathfrak{g}$ be the normal abelian subgronp of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{g}$ is abelian and set $\Phi(\mathfrak{g})=\mathfrak{h}$. Then $\mathfrak{g}\cong\mathfrak{h}$ by Corollary 2 and $R(\mathfrak{G}/\mathfrak{g})\cong R(\mathfrak{H}/\mathfrak{h})$ by Lemma 2. Since $\mathfrak{G}/\mathfrak{g}$ is also abelian, Corollary 1 implies that $\mathfrak{G}/\mathfrak{g}\cong\mathfrak{H}/\mathfrak{h}$, which shows that $\mathfrak{H}$ is metabelian. In particular, if $\mathfrak{G}$ is metacyclic, then $\mathfrak{H}$ is metacyclic.

PROPOSITION 5. *If $\mathfrak{G}$ is supersolvable, then $\mathfrak{H}$ is also supersolvable.*

PROOF. Let $\mathfrak{G}$ be supersolvable. Then there exists a normal series $\mathfrak{G}=\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = \{1\}$ such that all factors $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ are cyclic. Setting $\Phi(\mathfrak{g}_i)=\mathfrak{h}_i$, we have a normal series of $\mathfrak{H}$:

$$\mathfrak{H}=\mathfrak{h}_0 \supset \mathfrak{h}_1 \supset \cdots \supset \mathfrak{h}_n=\{1\},$$

and

$$R(\mathfrak{G}/\mathfrak{g}_i) \cong R(\mathfrak{H}/\mathfrak{h}_i), \qquad i=1, 2, \cdots, n.$$

Since each $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is a normal cyclic subgroup of $\mathfrak{G}/\mathfrak{g}_i$ and $\Phi(\mathfrak{g}_{i-1}/\mathfrak{g}_i)=\mathfrak{h}_{i-1}/\mathfrak{h}_i$(*), Corollary 2 yields that $\mathfrak{h}_{i-1}/\mathfrak{h}_i$ is isomorphic to $\mathfrak{g}_{i-1}/\mathfrak{g}_i$, so that cyclic. Thus $\mathfrak{H}$ is supersolvable.

LEMMA 3. *Let* $\mathfrak{g}$ *be any normal subgroup of* $\mathfrak{G}$ *and set* $\Phi(\mathfrak{g})=\mathfrak{h}$, *then* $\Phi([\mathfrak{g}, \mathfrak{g}])=[\mathfrak{h}, \mathfrak{h}]$.

PROOF. Set $\Phi([\mathfrak{g}, \mathfrak{g}])=\mathfrak{h}'$. We have seen in Lemma 2 that $R(\mathfrak{G}/[\mathfrak{g}, \mathfrak{g}]) \cong R(\mathfrak{H}/\mathfrak{h}')$. Then $\mathfrak{h}/\mathfrak{h}'=\Phi(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ is isomorphic to the normal abelian subgroup $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{G}/[\mathfrak{g}, \mathfrak{g}]$ (Corollary 2), which shows that $\mathfrak{h}' \supset [\mathfrak{h}, \mathfrak{h}]$ and $(\mathfrak{h}':1)=([\mathfrak{g}, \mathfrak{g}]:1)$. But it follows from Proposition 3 that $([\mathfrak{h}, \mathfrak{h}]:1)=([\mathfrak{g}, \mathfrak{g}]:1)$. Hence $(\mathfrak{h}':1)=([\mathfrak{h}, \mathfrak{h}]:1)$ so that $\mathfrak{h}'=[\mathfrak{h}, \mathfrak{h}]$. This proves the lemma.

MAIN THEOREM. *Let* $\mathfrak{G}$ *and* $\mathfrak{H}$ *be finite groups such that* $R(\mathfrak{G}) \cong R(\mathfrak{H})$. *If* $\mathfrak{G}$ *is solvable with the length n of the derived series, then so is* $\mathfrak{H}$, *and if* $\{\mathfrak{g}_i\}$ *and* $\{\mathfrak{h}_i\}$ *are the derived series of* $\mathfrak{G}$ *and* $\mathfrak{H}$, *respectively, then* $\mathfrak{g}_{i-1}/\mathfrak{g}_i \cong \mathfrak{h}_{i-1}/\mathfrak{h}_i$ *for all i.*

PROOF. Since $\mathfrak{g}_i=[\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}]$, Lemma 3 insures that $\Phi(\mathfrak{g}_i)$ is precisely the commutator $[\Phi(\mathfrak{g}_{i-1}), \Phi(\mathfrak{g}_{i-1})]$. Then the induction process shows that $\Phi(\mathfrak{g}_i)$ is the i-th derived group $\mathfrak{h}_i$ of $\mathfrak{H}$ and $\mathfrak{h}_n=\{1\}$. Furthermore $R(\mathfrak{G}/\mathfrak{g}_i) \cong R(\mathfrak{H}/\mathfrak{h}_i)$, $\Phi(\mathfrak{g}_{i-1}/\mathfrak{g}_i)=\mathfrak{h}_{i-1}/\mathfrak{h}_i$, and $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ is a normal abelian subgroup of $\mathfrak{G}/\mathfrak{g}_i$. Therefore $\mathfrak{g}_{i-1}/\mathfrak{g}_i \cong \mathfrak{h}_{i-1}/\mathfrak{h}_i$ for all i by

COROLLARY 2. *This shows the theorem.*

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References

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(*) This follows directly from the definition of Φ.