J. Math. Soc. Japan
Vol. 18, No. 3, 1966

A characterisation of exponential distribution semi-groups

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(Received March 30, 1966)

§ 1. Introduction.

The notion of the (exponential) distribution semi-group of operators in a Banach space was defined by Lions [1]. He characterized the infinitesimal generator of an exponential distribution semi-group by proving generalized Hille-Yosida theorem (cf. also Foias [2], Yoshinaga [3], [4] and Peetre [5]).

In this paper we shall show another characterisation of exponential distribution semi-group. By virtue of this characterisation, we shall define and characterize holomorphic exponential distribution semi-groups. Finally we shall prove a regularity property of holomorphic distribution semi-groups.

The author wishes to express his gratitude to Professor K. Yosida who kindly gave him many advices in the preparation of this paper.

§ 2. Summary for Lions' results.

We use the following notations: t represents a real variable; $\mathcal{D}_0$ is the space of C_0^∞ functions which vanish in $t<0$; $\mathcal{S}$ is the space of rapidly decreasing C^∞ functions; $\mathcal{E}'$ is the space of distributions with compact support. Let E be a Banach space. If x is an element of E, $\|x\|$ is the norm of x. $L(E, E)$ is the Banach space of bounded linear operators in E. δ_τ is the Dirac distribution concentrated at $t=\tau$.

DEFINITION 1. A distribution semi-group (D. S. G. in short) G is an $L(E, E)$-valued distribution such that

(i) the support of G is contained in $[0, \infty)$,

(ii) $G(\varphi * \phi)=G(\varphi)G(\phi)$, for any φ and ϕ in $\mathcal{D}_0$,

(iii) if $\varphi \in \mathcal{D}_0$ and $x \in E$, and if $y=G(\varphi)x$, the distribution Gy defined by $Gy(\varphi)=G(\varphi)y$ is almost everywhere equal to a function $u(t)$ which is continuous for $t \geqq 0$, $u(+0)=y$ and $u(t)=0$ for $t<0$,

(iv) the set $\mathcal{R}=\left\{\sum_{i=1}^{m} G(\varphi_i)x_i \mid \varphi_i \in \mathcal{D}_0,\ x_i \in E\right\}$ is dense in E,

(v) if $x \in E$, $G(\varphi)x=0$ for any $\varphi \in \mathcal{D}_0$, then $x=0$.

DEFINITION 2. A D. S. G. G is called an exponential distribution semi-group (E. D. S. G. in short) if there is a ξ_0 such that $e^{-\xi_0 t}G$ is a tempered distribution.

DEFINITION 3. (Cf. Yoshinaga [3].) If G is a D. S. G. and S in $\mathcal{E}'$, then we define $G(S)$ on $\mathfrak{R}$ by the formula

$$G(S)\sum_{i=1}^{m} G(\varphi_i)x_i=\sum_{i=1}^{m} G(S*\varphi_i)x_i \qquad \text{for any } x_i\in E,\ \varphi_i\in \mathcal{D}_0\,.$$

Then $G(S)$ is densely defined and closable. $\overline{G(S)}$ represents the minimal closed extension of $G(S)$.

DEFINITION 4. $A=\overline{G(-\delta_0')}$ is called the infinitesimal generator of G.

The following theorem is due to Lions.

THEOREM 1. *A closed linear operator A in E generates an E. D. S. G., if and only if the following conditions hold:*

(i) *the domain D_A of A is dense in E;*

(ii) *there exists a real constant ξ_0 such that, for any $p=\xi+i\eta$ with $\xi>\xi_0$*

$$(pI-A)^{-1}\in L(E,E)\,,$$

(1) (iii)
$$\|(pI-A)^{-1}\|\leqq C(1+|p|)^N$$

for any $p=\xi+i\eta$ with $\xi>\xi_0$, where C and N are independent of p.

§ 3. A characterisation of E. D. S. G.

Let Γ_1 be a parametrix for the operator $\dfrac{d}{dx}$ such that there is an $\omega_1\in\mathcal{D}_0$ satisfying

$$-\delta_0'*\Gamma_1=\delta_0+\omega_1\,. \tag{2}$$

We may assume $\Gamma_1\in L^1(R^1)$.

LEMMA 1. *If G is an E. D. S. G. then for any $\xi>\xi_0$, there exist a constant $C>0$ and an integer $N>0$ such that, for any $\varphi,\psi\in\mathcal{D}_0$,*

$$\|e^{-\xi t}G(\varphi*\psi)\|\leqq C\Big(\sum_{p=0}^{N}\|\varphi^{(p)}\|_{L^1}\Big)\big(\|\Gamma_1*\varphi\|_{L^1}+\|\omega_1*\psi\|_{L^1}\big)\,. \tag{3}$$

PROOF. As $e^{-\frac{\xi+\xi_0}{2}t}G$ is a continuous linear map from $\mathcal{S}$ to $L(E,E)$ and, for any polynomial $Q(t)$, $e^{-\frac{\xi-\xi_0}{2}t}Q(t)$ is bounded in $t>0$,

$$\|e^{-\xi t}G(\varphi*\psi)\|\leqq C\sum_{l=0}^{N-1}\|\delta_0^{(l)}*\varphi*\psi\|_{L^1}\,. \tag{4}$$

From (2) we have

$$\|\delta_0^{(l)}*\varphi*\psi\|_{L^1}\leqq\|\delta_0^{(l+1)}*\varphi\|_{L^1}\cdot\|\psi*\Gamma_1\|_{L^1}+\|\delta^{(l)}*\varphi\|_{L^1}\cdot\|\omega_1*\psi\|_{L^1} \tag{5}$$

so that (3) follows from (4) and (5).

COROLLARY. *If G is an E.D.S.G. for which ξ_0 is negative and φ is a C^N-function with compact support contained in $t>0$, where N is the integer introduced in lemma 1, then the operator $\overline{G(\varphi)}$ is a bounded linear operator on E with the estimate*

$$\|\overline{G(\varphi)}\| \leqq C\cdot(\sum_{l=0}^{N}\|\varphi^{(l)}\|_{L^1}) \tag{6}$$

where C is a constant independent of φ.

PROOF. Let ρ_n be a mollifier, then

$$\|G(\rho_n*\varphi)-G(\rho_m*\varphi)\| \leqq C(\sum_{l=0}^{N}\|\varphi^{(l)}\|_{L^1})(\|\Gamma_1*(\rho_n-\rho_m)\|_{L^1}+\|\omega_1*(\rho_n-\rho_m)\|_{L^1})\,.$$

When n and m tend to infinity, the right-hand side tends to zero. Therefore $\{G(\rho_n*\varphi)\}_n$ converges in norm to $G(\varphi)$. The inequallity

$$\|G(\rho_n*\varphi)\| \leqq C(\sum_{l=0}^{N}\|\varphi^{(l)}\|_{L^1})(\|\Gamma_1*\rho_n\|_{L^1}+\|\omega_1*\rho_n\|_{L^1})$$

gives

$$\|G(\varphi)\| \leqq C(\|\Gamma_1\|_{L^1}+\|\omega_1\|_{L^1})(\sum_{l=0}^{N}\|\varphi^{(l)}\|_{L^1})\,.$$

For any densely defined closed linear operator A, define D_{A^∞} by

$$D_{A^\infty}=\bigcap_{n=1}^{\infty}D_{A^n}\,,$$

where D_{A^n} is the domain of the operator A^n. This becomes a Fréchet space with respect to the system of semi-norms $\|x\|_n=\|A^nx\|$ $(n=0,1,2,\cdots)$. The operator A maps D_{A^∞} into itself continuously. We denote by $L(D_{A^\infty}, D_{A^\infty})$ the space of continuous linear operators from D_{A^∞} into itself.

The following two theorems will characterize the structure of E.D.S.G.

THEOREM 2. *If a closed linear operator A in E generates an E.D.S.G. G, then there exist a positive constant $2p$ and a linear set $F\subset D_{A^\infty}$ such that*

(I) $(2pI-A)^{-1}\in L(E,E)$,

(II) *F is dense in E,*

(III) *F constitutes a closed linear subspace in D_{A^∞},*

(IV) *$(A-2pI)$ generates an equi-continuous semi-group T_t, $t\geqq 0$ of continuous linear operators of F with respect to the topology of F induced from that of D_{A^∞}.*

REMARK. About equi-continuous semi-groups of bounded linear operators in a Fréchet space, see K. Yosida [6].

PROOF OF THEOREM 2. Let F be the closure of $\mathfrak{R}$ (given in (iv)) in D_{A^∞}, then (I) is satisfied for $2p>\xi_0$ Considering $e^{-2pt}G$ instead of G, it can be reduced to the case in which $\xi_0<0$, $p=0$. To prove (II)~(IV) we must define

$T_t; t \geqq 0$ in F. First we define them on $\mathfrak{R}$ as

(7) $$T_t \sum_{i=1} G(\varphi_i)x_i = \sum_i G(\delta_t * \varphi_i)x_i \qquad (t>0,\ \varphi_i \in \mathcal{D}_0,\ x_i \in E),$$

$$T_0 \sum_i G(\varphi_i)x_i = \sum_i G(\varphi_i)x_i .$$

Let Γ_2 be a parametrix such that,

(8) $$\delta^{(N+2)} * \Gamma_2 = \delta_0 + \omega_2 ,$$

where N is the integer in lemma 1, $\omega_2 \in \mathcal{D}_0$, and Γ_2 belongs to $C_0^N(R^1)$. From (8) we have, for any integer $k>0$ and $y = \sum_{i=1}^{m} G(\varphi_i)x_i$

$$\| (-A)^k T_t \sum_i G(\varphi_i)x_i \| = \| \sum_i G(\delta_0^{(k)} * \delta_t * \varphi_i)x_i \|$$

$$\leqq \| \sum_i G(\delta^{(k+N+2)} * \Gamma_2 * \delta_t * \varphi_i)x_i \| + \| \sum_i G(\delta^{(k)} * \omega_2 * \varphi_i)x_i \| .$$

As $\Gamma_2 * \delta_t \in C_0^N(R^1)$, we can apply Corollary to Lemma 1 in which φ is replaced by $\Gamma_2 * \delta_t$. Thus we obtain

(9) $$\| T_t y \|_k \leqq C(\| A^{k+N+2} \sum_i G(\varphi_i)x_i \|)(\sum_{l=0}^{N} \| \Gamma_2^{(l)} \|_{L^1}) + C \| \sum_i G(\varphi_i)x_i \|$$

$$\leqq C'(\| y \|_{k+N+2} + \| y \|)$$

where C' is independent of t.

Thus we can extend T_t continuously to F. The inequality (9) shows that $\{T_t\}$ is equi-continuous. For any $\sum_i G(\varphi_i)x_i$ in $\mathfrak{R}$, we have

$$\lim_{t \downarrow 0} \| T_t \sum_i G(\varphi_i)x_i - \sum_i G(\varphi_i)x_i \|_{(k)} = \lim_{t \downarrow 0} \| G(\delta_0^{(k)} * (\delta_t - \delta_0) * \varphi)x \| .$$

As $\delta_0^{(k)} * (\delta_t - \delta_0) * \varphi \to 0$ in $\mathcal{D}_0$, the left-hand side tends to zero. $\mathfrak{R}$ being dense in F, the equi-continuity implies the strong continuity at the origin. The semi-group property of T_t is easily proved. In fact,

$$T_{t+s}(\sum_i G(\varphi_i)x_i) = \sum_i G(\delta_{t+s} * \varphi_i)x_i = \sum_i G(\delta_t * \delta_s * \varphi_i)x_i$$

$$= T_t \sum_i G(\delta_s * \varphi_i)x_i = T_t \cdot T_s \sum_i G(\varphi_i)x_i .$$

It is easily verified that A generates T_t. Thus theorem 2 is proved.

Conversely, we can prove

THEOREM 3. *A closed linear operator A in E generates an E.D.S.G. if there exist $2p_0$ and a linear set F in D_{A^∞} satisfying* (I) *through* (IV) *of Theorem* 2.

PROOF. Considering $A - 2p_0 I$ instead of A, we may assume $p_0 = 0$. From (I) and (IV) we can find an integer N and a constant C, such that, for any x in F,

(10) $$\| T_t x \| \leqq C \| x \|_N = C \| A^N x \| .$$

For any x in F and $p=\xi+i\eta$, ξ being sufficiently large,

$$(pI-A)^{-1}x=\int_0^\infty e^{-pt}T_t x\,dt .$$

(Cf. K. Yosida [6] p. 240.) Thus

$$(pI-A)^{-1}x=\int_0^\infty e^{-pt}A^N T_t A^{-N}x\,dt ,$$

$$=\int_0^\infty e^{-pt}\frac{d^N}{dt^N}(T_t A^{-N}x)\,dt .$$

Integrating by part, we obtain

$$(pI-A)^{-1}x=p^N\int_0^\infty e^{-pt}T_t A^{-N}x dt+\sum_{p=0}^{N-1}p^k A^{-k-1}x .$$

Remembering (I), (10) and the trivial inequality $\| A^{-N}x \|_N \leqq C \| x \|$, we have

$$\| (pI-A)^{-1}x \| \leqq |p|^N\int_0^\infty e^{-\xi t}\| T_t A^{-N}x \| dt+\sum_{k=0}^{N-1}|p|^k \| A^{-k-1}x \|$$

$$\leqq C|p|^N\frac{1}{\xi}\| x \| + \sum_{k=0}^{N-1}|p|^k \| x \| .$$

Hence we have proved that for any $x\in F$,

(11) $$\| (pI-A)^{-1}x \| \leqq C(1+|p|^N)\| x \|,\ p=\xi+i\eta, \qquad \xi \text{ is sufficiently large.}$$

Since F is dense in E, $(pI-A)^{-1}\in L(E, E)$ and (11) holds for any x in E. This completes the proof.

§ 4. Holomorphic E. D. S. G.

Theorem 2 and 3 suggest us to generalize the notion of holomorphic semi-group (cf. Yosida [6]) to that of holomorphic E. D. S. G.

DEFINITION 5. An E. D. S. G. G is said to be holomorphic in the sector $\Sigma=\left\{t: |\arg t|< \alpha,\ 0<\alpha<\frac{\pi}{2}\right\}$, if G induces an equi-continuous holomorphic semi-group T_t in F in this sector, where F is the Fréchet space introduced in Theorem 2 and 3.

THEOREM 4. *A closed linear operator A in E generates an E. D. S. G. G which is holomorphic in the sector $\Sigma=\left\{t: |\arg t|\leqq \alpha,\ 0<\alpha<\frac{\pi}{2}\right\}$, if and only if A satisfies the following two conditions:*

(1°) *there exists a real β such that, for any $\varepsilon>0$ and any p in the sector $\Sigma'=\left\{p: |\arg (p-\beta)|<\theta=\frac{\pi}{2}+\alpha-\varepsilon\right\}$ we have $(pI-A)^{-1}\in L(E, E)$ with the* *estimate*

$$\|(pI-A)^{-1}\| \leqq C(1+|p|)^N ,$$

where C and N are positive constants independent of $p \in \Sigma'$.

(2°) D_A, *the domain of A, is dense in E.*

PROOF. Necessity of (1°) and (2°).

Multiplying T_t by $\exp(-\beta-\gamma)t$, $\gamma>0$, we can reduce the problem to the case in which $\beta<0$. Let $T_t: t \geqq 0$ be the equi-continuous semi-group of operators in F generated by A. Thus for x in F, and p in Σ', we have

$$(pI-A)^{-1}x=\int_0^\infty e^{-pt}T_t x dt .$$

As $T_t x$ is holomorphic in Σ and $e^{-pt}T_t x$ rapidly tends to zero at infinity, we can change the path of integration and obtain the following

$$(pI-A)^{-1}x=\int_0^\infty s_0 e^{-pts_0}T_{ts_0}x dt$$

where

$$s_0=\begin{cases} e^{-i\alpha} & \text{if } \operatorname{Im} p \geqq 0 , \\ e^{i\alpha} & \text{if } \operatorname{Im} p<0 . \end{cases}$$

As in the proof of Theorem 3, by partial integration, we have

$$(pI-A)^{-1}x=p^N\int_0^\infty s_0 e^{-pts_0}T_{ts_0}A^{-N}x dt+\sum_{k=0}^{N-1}p^k s_0 A^{-k-1}x .$$

Since $\operatorname{Re} pts_0>0$, we have

$$\|(pI-A)^{-1}x\| \leqq |p^N|\Big(\int_0^\infty e^{-\operatorname{Re} pts_0}dt\Big)\|x\|+\sum_{k=0}^{N-1}|p^k|\|x\| \leqq C(1+|p|^N)\|x\| .$$

F being dense in E, this inequality holds for any x in E. The necessity of (2°) is trivial.

Sufficiency of (1°) and (2°).

By Theorem 1, A generates an E. D. S. G. Let F be the linear subspace introduced in Theorem 2 and $\{T_t; t \geqq 0\}$ be the equi-continuous semi-group on F generated by A. We have only to show the analyticity of $T_t x$ for $x \in F$.

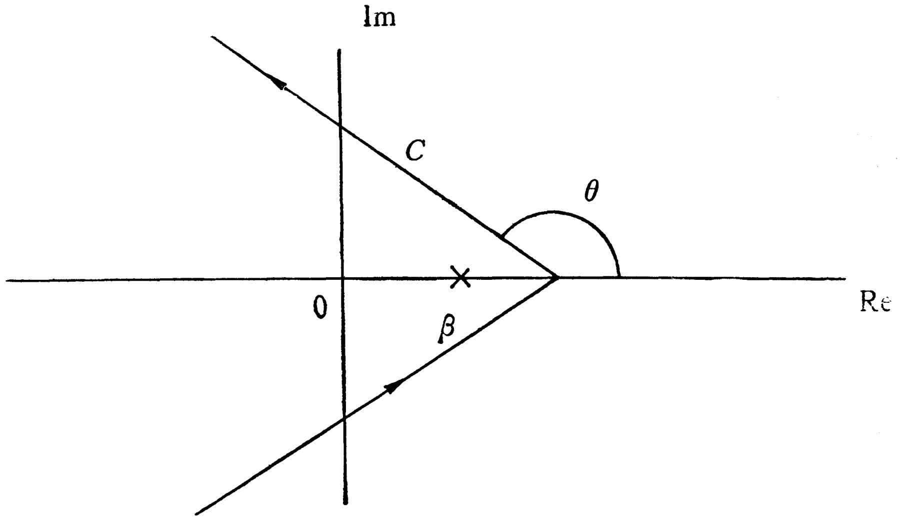
For $x \in F \subset D_{A^\infty}$,

$$(pI-A)^{-1}x=\frac{I}{p}x+\frac{A}{p^2}x+\cdots+\frac{A^{N+2}}{p^{N+3}}x+\frac{I}{p^{N+3}}(pI-A)^{-1}A^{N+3}x .$$

Hence, inverting $(pI-A)^{-1}x=\int_0^\infty e^{-pt}T_t x dt$, we obtain by (1°)

$$T_t x=\frac{1}{2\pi i}\int_C e^{pt}\Big(\sum_{k=1}^{N+3}\frac{A^{k-1}}{p^k}x+\frac{1}{p^{N+3}}(pI-A)^{-1}A^{N+3}x\Big)dp$$

where the curve C is as in the figure:



Thus we obtain

$$(12)\qquad T_t x = \sum_{k=1}^{N+3} \frac{t^{k-1}A^{k-1}}{\Gamma(k)} x + \frac{1}{2\pi i}\int_C e^{pt}\frac{1}{p^{N+3}}(pI-A)^{-1}A^{N+3}x\,dp.$$

By (i) we have

$$\left\| \frac{1}{p^{N+3}}(pI-A)^{-1}A^{N+3}x \right\|_l \leqq C|p|^{-N-3}(1+|p|^N)\|x\| \qquad \text{for any integer } l>0.$$

Because of (12), this implies that $T_t x$ is holomorphic in Σ.

COROLLARY. *Let G be an E.D.S.G. holomorphic in the sector $\Sigma=\{t:|\arg t|<\alpha\}$, in the sense of definition 4. Then G is equal to a function holomorphic in Σ.*

PROOF. If x is in F and if t is in the sector, then

$$T_t x = \frac{1}{2\pi i}\int_C e^{pt}(pI-A)^{-1}x\,dp$$

where the curve C is as in the proof of Theorem 4. From (1°) of theorem 4,

$$\|T_t x\| \leqq C\|x\|\int_0^\infty e^{-\mathrm{Re}(tp)}(1+|p|^N)d|p| \leqq C(t)\|x\|.$$

where $C(t)$ depends on t but not on x.

Since F is dense in E, $\{T_t: t\geqq 0\}$ can be extended to the whole space E continuously. Similar argument proves that the extended T_t is a holomorphic function of t in that sector Σ.

REMARK. This corollary does not give any information about the behaviour of the semi-group at the origin.

REMARK added during the proof reading:

Professor J. L. Lions kindly noticed to the author's attention that G. Da Prato and U. Mosco [7], [8], had already introduced the notion of holomorphic distribution semi-groups. Their definition of analytic E. D. S. G. is not the same as ours. But from Theorem 4 of this paper, it is easy to see that these two different definitions are equivalent. See G. Da Prato and U. Mosco [7], [8].

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