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Transformation groups satisfying some local metric conditions

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If a locally compact group G acts effectively on a manifold M, then is G necessarily a Lie group? Considerably many investigations have been directed to this well known problem. The purpose of this paper is to show that it is affirmatively answered under some local metric conditions: locally Lipschitzian (cf. Definition 1) or locally similar (cf. Definition 3).

The proof is reduced to show that if G is zero-dimensional compact then for an open normal subgroup G' of G there exists a G'-invariant local metric ρ^* in M such that $\rho^*(x, y) \leqq c \cdot \rho(x, y)$ holds locally, where c is a constant and ρ is a local euclidean metric.

In this paper a *manifold* means a separable, metric, connected, and locally euclidean space.

§ 1. Locally Lipschitzian transformation groups.

DEFINITION 1. A topological transformation group G acting on a manifold M is said to be *locally Lipschitzian* if for a coordinate neighborhood U_a of each point a in M there exist a neighborhood V of the identity of G and a neighborhood U of the point a as follows: 1) $V(U) \subset U_a$, 2) $\rho(g(x), g(y)) \leqq c \cdot \rho(x, y)$ for all $g \in V$ and all $x, y \in U$, where ρ is a euclidean distance function in U_a and c is a constant.

The following Lemma 1 shows that classical transformation groups acting on manifolds are locally Lipschitzian.

LEMMA 1. *In the above if G is locally compact and, in place of the condition 2), 2)′ the local coordinate functions of $g(x)$ have partial derivatives with respect to x that are continuous at $(g; x)$ simultaneously, then G is locally Lipschitzian.*

PROOF. Let n be the dimension of M. Choose a compact neighborhood V of the identity of G and a neighborhood U' of the point a that satisfy the conditions 1) and 2)′, and an open convex neighborhood U of a such that $\bar{U}$ is compact and $\bar{U} \subset U'$. The coordinate functions $g_i(x)$, $i=1, 2, \cdots, n$, are totally differentiable with respect to $x \in \bar{U}$ i.e., for any point $x+h$ (vector sum) sufficiently near x,

$$g_i(x+h)-g_i(x)=\sum_{j=1}^{n}\frac{\partial g_i(x)}{\partial x_j}h_j+\varepsilon_i(h)\,, \qquad i=1,2,\cdots,n\,,$$

where $\varepsilon_i(h)/\|h\|\to 0$ as $\|h\|\to 0$. Hence

$$\frac{\rho(g(x+h),g(x))}{\rho(x+h,x)}=\left\{\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\frac{\partial g_i(x)}{\partial x_j}\frac{h_j}{\|h\|}+\frac{\varepsilon_i(h)}{\|h\|}\right]^2\right\}^{\frac{1}{2}}.$$

Let $x+h$ approach to x along the straight line whose direction cosines are $l_1, l_2, \cdots, l_n$, then

$$\lim_{\|h\|\to 0}\frac{\rho(g(x+h),g(x))}{\rho(x+h,x)}=\left\{\sum_{i,j=1}^{n}\left[\frac{\partial g_i(x)}{\partial x_j}l_j\right]^2\right\}^{\frac{1}{2}}\leqq\left\{\sum_{i,j=1}^{n}\left[\frac{\partial g_i(x)}{\partial x_j}\right]^2\right\}^{\frac{1}{2}}.$$

The right-hand side of this inequality is independent of $l_1, l_2, \cdots, l_n$ and continuous on the compact set $V\times\bar{U}$. Let c be the maximal value it attains, then

$$\lim_{\|h\|\to 0}\frac{\rho(g(x+h),g(x))}{\rho(x+h,x)}\leqq c\,, \qquad \text{for all } g\in V \text{ and all } x\in U.$$

Now let x, y be any distinct points in U. For any positive number ε there exist a finite number of points $z_1, z_2, \cdots, z_m$ (arranged in this order) in the segment $\overline{xy}$ such that

$$\frac{\rho(g(z_i),g(z_{i+1}))}{\rho(z_i,z_{i+1})}\leqq c+\varepsilon\,, \qquad i=0,1,2,\cdots,m\,, \text{ where } z_0=x\,,\ z_{m+1}=y\,.$$

Hence

$$\rho(g(x),g(y))\leqq\rho(g(x),g(z_1))$$
$$+\rho(g(z_1),g(z_2))+\cdots+\rho(g(z_m),g(y))\leqq(c+\varepsilon)\rho(x,y)\,.$$

Since ε is arbitrary, we have

$$\rho(g(x),g(y))\leqq c\cdot\rho(x,y)\,, \qquad \text{for all } g\in V \text{ and all } x,y\in U\,.$$

(If $x=y$, the inequality holds trivially.)

The following Lemma 2, 3 are well known and can be proved easily.

LEMMA 2. *Let G be a compact transformation group acting on a metric space M with a distance function $\rho(x,y)$. Then we can define a G-invariant metric ρ^*, which defines on M an equivalent topology to the original one, by*

$$\rho^*(x,y)=\int_G\rho(g(x),g(y))dg\,, \qquad x,y\in M\,,$$

where dg means a left invariant Haar measure in G.

LEMMA 3. *Let G be a compact transformation group acting on a metric space M, $\tilde{M}$ be the orbit space of M under G, and p be the natural projection of M onto $\tilde{M}$. If a G-invariant metric ρ^* is defined on M which gives an equivalent topology to the original one then we can define a new metric $\tilde{\rho}$ in $\tilde{M}$, which gives an equivalent topology to the natural one, by*

$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho^*(G(x), G(y)), \qquad \tilde{x}, \tilde{y} \in \tilde{M}, \qquad x \in p^{-1}(\tilde{x}), \qquad y \in p^{-1}(\tilde{y}).$$

Now we prove the fundamental lemma for the proof of our theorem.

LEMMA 4. *Let K be a zero-dimensional compact transformation group acting on an n-manifold M, and $D(M; K)$ the orbit space of M under K. If K is locally Lipschitzian, then $D(M; K)$ is n-dimensional everywhere.*

PROOF. Let a be any point of M. Choose a coordinate neighborhood U_a of a. Let $\rho(x, y)$ be a euclidean distance function defined on U_a. Then there exist an open normal subgroup G of K and an n-dimensional euclidean closed cube U containing the point a as an interior point such that $G(U) \subset U_a$ and $\rho(g(x), g(y)) \leqq c \cdot \rho(x, y)$ for all g in G and all x, y in U, where c is a constant. We can define a G-invariant metric ρ^*, which gives on $G(U)$ an equivalent topology to the original one, by

$$\rho^*(x, y) = \int_G \rho(g(x), g(y)) dg, \qquad x, y \in G(U), \text{ with } \int_G dg = 1,$$

where dg means the left invariant Haar measure in G. Then

$$\rho^*(x, y) \leqq c \cdot \rho(x, y) \text{ for all } x, y \in U, \text{ and } 0 < c < +\infty.$$

Consequently it is easy to see that U has $(n+1)$-dimensional Hausdorff measure zero with respect to the metric ρ^* (cf. [2], p. 103). Let $\tilde{U}$ denotes the orbit space of $G(U)$ under G, then $\tilde{U}$ has $(n+1)$-dimensional Hausdorff measure zero with respect to such a metric $\tilde{\rho}$ as the one defined in Lemma 3. Consequently $\dim \tilde{U} \leqq n$ with respect to the topology by $\tilde{\rho}$ (cf. [2], p. 104), and so with respect to the natural topology.

Now we can see that $\dim \tilde{U} = n$. In fact suppose

$$\dim G(U) - \dim \tilde{U} = m, \qquad m > 0.$$

Let π be the natural projection of $G(U)$ onto $\tilde{U}$. Since π is a closed mapping, there is a point y of $G(U)$ such that $\dim G(y) \geqq m$ (cf. [2], pp. 91, 92). On the other hand, any orbit $G(x)$ is homeomorphic to the factor space G/G_x respectively by Arens' theorem ([6], p. 65) and G/G_x is zero-dimensional (cf. [3] or [7]). Hence any orbit $G(x)$ is zero-dimensional. This contradicts with the hypothesis that m is positive.

Let K^* be the factor group K/G, and $\tilde{M}$ the orbit space of M under G. Then the group K^* is a finite transformation group acting on $\tilde{M}$ in a natural way. Let π be the natural projection of M onto $\tilde{M}$, and $\tilde{a} = \pi(a)$. There exists a neighborhood $\tilde{U}^*$ of $\tilde{a}$ such that 1) $\tilde{U}^* \subset \tilde{U}$, 2) $\tilde{U}^*$ has the same topological property as $\tilde{U}$, and 3) $\tilde{U}^*$ intersects with each K^*-orbit at most one point. Let M' be the orbit space of $\tilde{M}$ under K^*, and $\tilde{\pi}$ the natural projection of $\tilde{M}$ onto M'. Put $U' = \tilde{\pi}(\tilde{U}^*)$ and $a' = \tilde{\pi}(\tilde{a})$. The set U' is homeomorphic to $\tilde{U}^*$ under the mapping $\tilde{\pi}$, and so U' is n-dimensional. Since U' can be chosen arbitrarily small and the point a' can be taken arbitrarily at first, the space M' is n-dimensional everywhere. The space M' is homeomorphic to the orbit space $D(M; K)$. Hence our lemma has been proved.

THEOREM 1. *Let G be a locally compact transformation group acting effectively on a manifold M. If G is locally Lipschitzian, then it is necessarily a Lie group.*

PROOF. A locally compact transformation group effectively acting on a manifold is necessarily finite-dimensional (Montgomery [**5**]). Therefore we can suppose without loss of generality that G is zero-dimensional compact by the structure theorem of finite-dimensional locally compact groups (cf. [**6**]). If G could contain a p-adic subgroup P, then the dimension of the orbit space of M under P would be higher than the dimension of M (Yang [8])[1]. This contradicts with Lemma 4. Consequently G has no arbitrarily small subgroups and so G is a Lie group (cf. [**6**]).

COROLLARY. *Let G be a locally compact transformation group acting effectively on a manifold M, and let each transformation of a neighborhood of the identity e of G be of class C^1 at each point a of M—it is not required that the above neighborhood of e is independent of a. Then G is necessarily a Lie group.*

PROOF. We can suppose without loss of generality that G is zero-dimensional compact. For any fixed point a in M there exist an open normal subgroup G' of G and an open spherical neighborhood U of the point a such that 1) $G'(U)$ is contained in a coordinate neighborhood of a, 2) each transformation g of G' is of class C^1 at each point of $G'(U)$, and 3) $\rho(g(x), x) <$ (radius of U) for all $g \in G'$ and all $x \in U$. Then $G'(U)$ is a connected manifold and the pair $(G', G'(U))$ satisfies the assumption of Lemma 1 (cf. [**6**], p. 197). Therefore G' is locally Lipschitzian on $G'(U)$ and so on M. Consequently G' is a finite group by Theorem 1 and G is so.

M. Kuranishi [**4**] has proved that if G is a locally compact effective transformation group of a manifold M of class C^1 and each transformation of G is of class C^1, then G must be a Lie group. The above Corollary is the localization of his result.

§ 2. Locally similar transformation groups.

DEFINITION 2. A transformation group G acting on a metric space M with a distance function $\rho(x, y)$ is said to be *similar* if

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1) I wish to express my thanks to Mr. H. Omori of Tokyo Metropolitan University for his valuable advice about the dimension of the orbit space under p-adic transformation group.

$$\rho(x, y) > \rho(x', y') \quad \text{implies} \quad \rho(g(x), g(y)) > \rho(g(x'), g(y')),$$

for any g in G and any x, x', y, y' in M.

We note that if G is similar then

$$\rho(x, y) = \rho(x', y') \quad \text{implies} \quad \rho(g(x), g(y)) = \rho(g(x'), g(y')),$$

and the converse is true if G is connected.

LEMMA 5. *Let G be a compact transformation group acting on a compact metric space M with a distance function $\rho(x, y)$. If G is similar, then we can define a G-invariant metric ρ^*, which gives on M an equivalent topology to the original one and $\rho^*(x, y) \leqq \rho(x, y)$ for any x, y in M.*

PROOF. The function f defined by

$$f(x, y) = \inf_{g \in G} \rho(g(x), g(y)), \qquad x, y \in M,$$

has the following properties: 1) $f(x, y) = 0$ if and only if $x = y$, 2) $f(x, y) = f(y, x) \geqq 0$, 3) the set $\{x : f(a, x) < \varepsilon\}$ is open for any positive number ε and any point a in M, 4) $f(x, z) \leqq 2 \operatorname{Max} [f(x, y), f(y, z)]$, 5) $f(g(x), g(y)) = f(x, y)$. Put

$$h(x, y) = \inf \{f(x, x_1) + f(x_1, x_2) + \cdots + f(x_n, y)\},$$

where inf is taken over all finite number of points $x_1, x_2, \cdots, x_n$ in M. Then

$$f(x, y)/4 \leqq h(x, y) \leqq f(x, y),$$

and a new metric is introduced in M by $h(x, y)$ (cf. [1]). Put

$$\rho^*(x, y) = \int_G h(g(x), g(y)) dg, \quad \text{with} \quad \int_G dg = 1,$$

where dg means the left invariant Haar measure in G. It is seen easily that ρ^* has the required properties.

DEFINITION 3. A topological transformation group G acting on a manifold M is said to be *locally similar* if for a coordinate neighborhood U_a of each point a in M there exist a neighborhood V of the identity of G and a neighborhood U of the point a with the following properties: 1) $V(U) \subset U_a$, 2) $\rho(x, y) > \rho(x', y')$ implies $\rho(g(x), g(y)) > \rho(g(x'), g(y'))$ for all g in V and all x, x', y, y' in U, where ρ is a euclidean distance function in U_a.

The following Lemma 6 and Theorem 2 can be proved slightly modifying the proof of Lemma 4 and Theorem 1 respectively.

LEMMA 6. *Let K be a zero-dimensional compact transformation group acting on an n-manifold M, and $\tilde{M}$ the orbit space of M under K. If K is locally similar, then $\tilde{M}$ is n-dimensional everywhere.*

THEOREM 2. *Let G be a locally compact transformation group acting effectively on a manifold M. If G is locally similar, then it is necessarily a Lie group.*

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