J. Math. Soc. Japan
Vol. 17, No. 4, 1965

On the ζ-functions of a total matric algebra over the field of rational numbers

By Moto-o KINOSHITA

(Received Jan. 28, 1965)

Introduction

Iwasawa and Tate [8, 15] reconstructed the theory of Hecke's L-function as a theory of the ζ-function, attached to a number field k, with a character of the idele class group of k. Since then, it has been expected to generalize this theory to the case of the ζ-function of a simple algebra over the rational number field $\boldsymbol{Q}$. Let A and G be the adele ring and the idele group of a simple algebra over $\boldsymbol{Q}$ respectively. Fujisaki [2, 3] solved the problem for the ζ-function with an abelian character of G. The theory of Fujisaki includes the results of Hey and Eichler [7, 1]. Godement [4] showed the possibility of applying the Iwasawa-Tate method to the ζ-function, attached to a division algebra, with a "non-abelian character" of G. Tamagawa [14] developed the theory of Euler product. He determined an explicit form of the local ζ-function, attached to a simple algebra, with a zonal spherical function. And he proved that the ζ-function of a division algebra, defined as an infinite product of local ζ-functions, satisfies a functional equation. From the theory of Maass [9] on the Dirichlet series corresponding to a non-holomorphic automorphic function on upper half-plane, we can extract a theory of the ζ-function, attached to the total matric algebra of degree 2 over $\boldsymbol{Q}$, with a zonal spherical function.

On the other hand, Hecke [6] gave the theory of constructing Dirichlet series with Euler product and functional equation out of a modular form. Shimura [12] generalized this theory to the case of the automorphic form of Hilbert type by means of the Iwasawa-Tate method. In other words, the ζ-function of a quaternion algebra, with a spherical function, not necessarily of class 1, was treated.

The purpose of the present paper is to prove that the ζ-function of a total matric algebra over $\boldsymbol{Q}$ is defined as an infinite product of local ζ-functions, is meromorphic on the whole z-plane and satisfies a functional equation, if the "character" is a zonal spherical function determined by a certain automorphic function on G (cf. § 6, Theorem).

We shall sketch the contents of the paper. Let p be a prime number or ∞. We denote by A_p the completion at p of the total matric algebra over $\boldsymbol{Q}$ of degree n. The group of all invertible elements in A_p is denoted by G_p.

For a zonal spherical function ω_p on G_p relative to a maximal compact subgroup of G_p, we introduce a local ζ-function at p, $\zeta_p(z, \omega_p)$, with a certain weight function ϕ_p (cf. (1.1)). The weight function ϕ_∞ is defined by

$$\phi_\infty(x) = \exp(-\pi tr(x^t x)),\ x \in A_\infty .$$

For the proof of the theorem, we need a trick, suggested by Shimura [**12**] (pp. 270-272). We consider another local ζ-function at ∞ with a modified weight function of the form

$$\psi_\infty = w\phi_\infty, \qquad w = \sum_{s=0}^{n} M_s c_0 c_s .$$

Here, the M_s are real numbers, and the c_s are the functions on A_∞ defined by

$$\det(x^t x - T 1_n) = \sum_{s=0}^{n} (-1)^s c_s(x) T^s,\ x \in A_\infty ,$$

where T is an indeterminate.

We choose M_s so that the Fourier transform of ψ_∞ is equal to ψ_∞. In § 2, § 3 and § 4, we shall show that it suffices to take

$$M_s = (-1)^s \frac{(s+2)!}{2} \frac{1}{(2\pi)^s}$$

by rather technical computations. The starting-point is Proposition 2, cited from Maass [**10**], p. 4. In § 2, we reduce the problem to that of finding a sequence N_s $(s = 0, 1, 2, \cdots)$ which satisfies the linear equations

$$\sum_{s=0}^{m} B_s(t, m) N_s = \begin{cases} 0 & (1 \leqq t \leqq m), \\ (-1)^m N_m & (t = 0) \end{cases}$$

for all non-negative integers m. The coefficients $B_s(t, m)$ are defined inductively, but it is difficult to express them in a simple formula by s, t and m. In § 3, we calculate two auxiliary integrals. Making use of the one, we compute $B_s(m, m)$ in Proposition 8, § 4. On the other hand, Proposition 9 enables us to express in some sense the coefficients $B_s(t, m)$ by $B_s(m-l, m-l)$. We see, in Proposition 10, that a sequence $N_s = (s+2)!$ satisfies the above equations. The other integral in § 3 is a means to calculate the local ζ-function at ∞ with the modified weight function thus obtained (Proposition 11).

In § 5, we define a function on A of type Z by $(Z1), \cdots, (Z5)$. This definition is analogous to that of Tate [**15**] except $(Z5)$. Then, we define a global ζ-function with a weight function of type Z. The condition $(Z5)$ allows us to apply the Iwasawa-Tate method to the global ζ-function, as is seen in

Proposition 12.

In § 6, we state the theorem. For the proof, we first construct the function ψ of type Z by the usual weight functions at $p \neq \infty$ and by the modified weight function at ∞. We deduce a functional equation of

$$\prod_p \zeta_p(z, \omega_p)$$

from that of the global ζ-function with the weight function ψ.

Notations

As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ are the sets of all integers, rational numbers, real numbers and complex numbers respectively. For a real number x, we denote by $[x]$ the largest integer n which satisfies $n \leqq x$. For a complex number z, $\bar{z}$ is the complex conjugate of z.

Let p be a prime number or ∞. For $p \neq \infty$, we denote by $\boldsymbol{Q}_p$ the field of all p-adic numbers, and by $\boldsymbol{Z}_p$ the ring of all p-adic integers. We put $\boldsymbol{Q}_\infty = \boldsymbol{R}$. Let x be an element of $\boldsymbol{Q}_p$. In the case $p \neq \infty$, we denote by $\{x\}_p$ the main part of the p-adic expansion of x. So, if we expand x in the form

$$x = \sum_{i \geqq n_0} a_i p^i, \quad 0 \leqq a_i \leqq p-1, \quad a_{n_0} \neq 0,$$

then we have

$$\{x\}_p = \sum_{i<0} a_i p^i .$$

We put $\mathrm{ord}_p(x) = n_0$ and $|x|_p = p^{-\mathrm{ord}_p(x)}$. In the case $p = \infty$, $|x|_\infty$ means the usual absolute value of x.

Let R be a commutative ring. We denote by $M_n(R)$ the ring of all matrices of degree n over R. For an element x of $M_n(R)$, the symbols $tr(x)$ and ${}^t x$ mean the trace of the matrix x and the transposed matrix of x respectively. If R has an identity, $GL(n, R)$ denotes the group of all matrices in $M_n(R)$ whose determinants are the invertible elements of R. The neutral element of $GL(n, R)$ is denoted by 1_n.

Let S be a set, and T be a subset of S. The characteristic function of T on S is denoted by $\chi_{T,S}$, and sometimes by χ_T, if there is no fear of confusion. When S is a finite set, we denote by $\sharp S$ the number of all elements of S.

Let S be a topological space. We shall frequently use the following notations.

$C(S)$: the set of all complex valued continuous functions on S.

$L(S)$: the set of all functions in $C(S)$ with compact carrier.

When the space S has a measure, $L_1(S)$ denotes the set of all complex valued integrable functions on S.

Let A be a ring, and f a function on A. For an invertible element a of

A, we define operators L_a and R_a by

$$(L_a f)(x)=f(a^{-1}x), \qquad (R_a f)(x)=f(xa), \qquad x\in A\,.$$

§ 1. Local ζ-functions

Let us put $A_p=M_n(\boldsymbol{Q}_p)$, and for $p\neq\infty$, $O_p=M_n(\boldsymbol{Z}_p)$. A_p is a locally compact topological ring, and O_p is an open compact subring of A_p. We define a unitary character χ_p of A_p by

$$\chi_p(x)=\begin{cases}\exp(2\pi\sqrt{-1}\{tr(x)\}_p) & (p\neq\infty)\,,\\ \exp(-2\pi\sqrt{-1}\,tr(x)) & (p=\infty)\end{cases}$$

for $x\in A_p$. We have $\chi_p(O_p)=1$. Obviously, $\chi_p(xy)=\chi_p(yx)$ for every $x, y\in A_p$. A_p is self-dual by the mapping

$$A_p\times A_p\ni(x, y)\to\chi_p(xy)\in\boldsymbol{C}\,.$$

We denote by dx a Haar measure of A_p. Then, we have

$$d(ax)=d(xa)=|\det a|_p^n dx$$

for every element a of A_p such that $\det a\neq 0$. The Fourier transform of a function φ_p in $L_1(A_p)$ is denoted by $\hat{\varphi}_p$:

$$\hat{\varphi}_p(y)=\int_{A_p}\varphi_p(x)\chi_p(xy)dx \qquad \text{for} \quad y\in A_p\,.$$

We normalize the measure dx in such a way that the total volume of O_p is equal to 1 for $p\neq\infty$, and that $dx=\prod_{i,j}dx_{ij}$ for every element $x=(x_{ij})$ of A_∞.

We set

$$\phi_p(x)=\begin{cases}\chi_{O_p}(x) & (p\neq\infty)\,,\\ \exp(-\pi tr(x^t x)) & (p=\infty)\end{cases} \tag{1.1}$$

for $x\in A_p$. Then, we have $\phi_p\in C(A_p)\cap L_1(A_p)$ and $\hat{\phi}_p=\phi_p$.

Put $G_p=GL(n, \boldsymbol{Q}_p)$, $U_p=GL(n, \boldsymbol{Z}_p)$ for $p\neq\infty$ and $U_\infty=O(n, \boldsymbol{R})$. Inducing to G_p the topology of A_p, G_p is a unimodular locally compact group, and U_p is a maximal compact subgroup of G_p. U_p is an open subset of G_p for $p\neq\infty$. Let Z_p be the centre of G_p.

We denote by du the Haar measure on U_p, such that the total volume of U_p is equal to 1. A non-zero function ω_p in $C(G_p)$ is called zonal spherical function relative to U_p, or simply spherical function, if the condition

$$\int_{U_p}\omega_p(gug')du=\omega_p(g)\omega_p(g') \quad \text{for all } g, g'\in G_p \tag{1.2}$$

is satisfied. For spherical functions, we refer to [**5**], [**11**], [**13**], [**14**]. We have $\omega_p(1_n)=1$. A spherical function ω_p is called positive-definite, if it satisfies the condition

$$\iint_{G_p\times G_p} \omega_p(gh^{-1})f_p(g)\overline{f_p(h)}dgdh \geqq 0$$

for all $f_p \in L(G_p)$, where dg is a Haar measure on G_p. We denote by Ω_p the set of all spherical functions, and by Ω_p^+ the totality of positive-definite spherical functions. If ω_p is a positive-definite spherical function, then we have

$$|\omega_p(g)| \leqq 1, \qquad \overline{\omega_p(g)} = \omega_p(g^{-1}) \text{ for all } g \in G_p .$$

Moreover, we denote by $\tilde{\Omega}_p$ the set of all ω_p in Ω_p which satisfy the condition

$$\omega_p(\zeta g) = \omega_p(g) \qquad \text{for all } \zeta \in Z_p,\ g \in G_p .$$

We note that the above definitions and properties of spherical functions are valid when G_p is a general unimodular locally compact group and U_p is a compact subgroup of G_p.

Spherical functions are parametrized by n complex numbers as follows. Let T_p be the set of all upper triangular matrices in G_p whose diagonal elements are integral powers of p or positive numbers according as $p \neq \infty$ or $p = \infty$. Every element g of G_p can be written uniquely in the form:

$$g = ut,\ u \in U_p,\ t \in T_p, \text{ or } g = t_1 u_1,\ t_1 \in T_p,\ u_1 \in U_p .$$

With n complex numbers $s_1, \cdots, s_n$, we associate a character $\alpha_{s_1,\cdots,s_n}$ of T_p:

$$\alpha_{s_1,\cdots,s_n}(t) = \prod_{i=1}^{n} |t_{ii}|_p^{-s_i+(i-1)},\ t = (t_{ij}) \in T_p .$$

The character $\alpha_{s_1,\cdots,s_n}$ is extensible to a function on G_p by putting

$$\alpha_{s_1,\cdots,s_n}(ut) = \alpha_{s_1,\cdots,s_n}(t)$$

for $u \in U_p$, $t \in T_p$. Then, the function $\omega_{s_1,\cdots,s_n}$ on G_p, defined by

$$\omega_{s_1,\cdots,s_n}(g) = \int_{U_p} \alpha_{s_1,\cdots,s_n}(g^{-1}u)du, \qquad g \in G_p ,$$

is spherical. Conversely, for every spherical function ω_p on G_p, there exist complex numbers $s_1, \cdots, s_n$ such that $\omega_p = \omega_{s_1,\cdots,s_n}$. By the above definition, we have $\overline{\omega_{s_1,\cdots,s_n}} = \omega_{\bar{s}_1,\cdots,\bar{s}_n}$ and

$$\omega_{s_1,\cdots,s_n}(g)\,|\det g|_p^z = \omega_{s_1+z,\cdots,s_n+z}(g) . \tag{1.3}$$

We denote by $\mathfrak{S}_n$ the symmetric group of degree n. Spherical functions $\omega_{s_1,\cdots,s_n}$ and $\omega_{s_1',\cdots,s_n'}$ coincide with each other, if and only if

$$s_{\sigma(i)} \equiv s_i' \quad \left(\text{mod.}\ \frac{2\pi\sqrt{-1}}{\log p}\right),\quad i = 1, \cdots, n$$

for some element σ of $\mathfrak{S}_n$, where $2\pi\sqrt{-1}/\log p$ means zero for $p = \infty$. In particular, $\omega_{s_1,\cdots,s_n}=1$, if and only if

$$s_{\sigma(i)}\equiv i-1 \left(\text{mod.}\frac{2\pi\sqrt{-1}}{\log p}\right),\quad i=1,\cdots,n$$

for some $\sigma\in\mathfrak{S}_n$. The condition

$$\sum_{i=1}^{n} s_i\equiv\frac{n(n-1)}{2}\quad\left(\text{mod.}\frac{2\pi\sqrt{-1}}{\log p}\right) \tag{1.4}$$

is necessary and sufficient for $\omega_{s_1,\cdots,s_n}$ to be in $\tilde{\Omega}_p$. If a spherical function $\omega_{s_1,\cdots,s_n}$ belongs to Ω_p^+, then we have

$$\bar{s}_i\equiv n-1-s_{\sigma(i)}\left(\text{mod.}\frac{2\pi\sqrt{-1}}{\log p}\right),\quad i=1,\cdots,n \tag{1.5}$$

for some $\sigma\in\mathfrak{S}_n$.

We normalize the measure dg on G_p so that for $p\neq\infty$, the total volume of U_p is equal to 1, and that for $p=\infty$, we have

$$dg=2^n du\left(\prod_{i=1}^{n} t_{ii}^{-i}dt_{ii}\right)\left(\prod_{i<j} dt_{ij}\right),$$

where $g=ut$, $u\in U_\infty$, $t=(t_{ij})\in T_\infty$.

Let ω_p be a spherical function on G_p. Let ϕ_p be the function on A_p as (1.1). The following proposition is a special case of a result of Tamagawa [14].

PROPOSITION 1. *If $\omega_p=\omega_{s_1,\cdots,s_n}$, then the integral*

$$\zeta_p(z,\omega_p)=\int_{G_p}\phi_p(g)\omega_p(g^{-1})\,|\det g|_p^z dg$$

converges for $\operatorname{Re} z>\operatorname{Max}_i(\operatorname{Re} s_i)$. *The function $\zeta_p(z,\omega_p)$ of z is continued to a meromorphic function on the whole z-plane, which is called the local ζ-function at p with weight function ϕ_p. We have*

$$\zeta_p(z,\omega_p)=\begin{cases}\prod_{i=1}^{n}(1-p^{s_i}p^{-z})^{-1} & (p\neq\infty),\\ \pi^{-\frac{n}{2}z+\frac{1}{2}\sum_{i=1}^{n}s_i}\prod_{i=1}^{n}\Gamma\left(\frac{z-s_i}{2}\right) & (p=\infty).\end{cases}$$

Now, we ask if the infinite product $\prod_p\zeta_p(z,\omega_p)$ converges in some region of z-plane, if it is continued to a meromorphic function on the whole z-plane and if it satisfies a functional equation under some assumptions on $\{\omega_p\}_{p\leqq\infty}$. When $n\neq 1$, these questions shall not be solved by the immediate application of the Iwasawa-Tate method to the idele group of $M_n(\boldsymbol{Q})$. As was suggested by Shimura [12] pp. 270-272, we need another local ζ-function at ∞, with a slightly modified weight function, which will allow us to apply that method.

Let $X=(X_{ij})$ be an (n, n) matrix of n^2 indeterminates X_{ij}; let $w(X)$ be a polynomial of X_{ij} over $\boldsymbol{C}$. We define a function w on A_∞ by

$$A_\infty \ni x \rightarrow w(x) \in \boldsymbol{C},$$

which will be called the polynomial on A_∞. The Gauss transform w^* of a polynomial w on A_∞ is defined by

$$w^*(x)=\int_{A_\infty} w(x+y)\phi_\infty(y)dy, \qquad x \in A_\infty .$$

The function w^* is a polynomial on A_∞. Put $\tilde{w}(x)=w^*(-\sqrt{-1}\,{}^t x)$. Then, we get easily

$$\widehat{w\phi_\infty}=\tilde{w}\phi_\infty .$$

We adopt a modified weight function of the form $w\phi_\infty$.

Actually, we use a polynomial of more restricted type as follows. Let T be an indeterminate. We define a polynomial c_s $(s=0, \cdots, n)$ on A_∞ by

$$\det(x\,{}^t x - T1_n)=\sum_{s=0}^{n}(-1)^s c_s(x)T^s, \qquad x \in A_\infty .$$

In particular, $c_0(x)=(\det x)^2$, $c_n(x)=1$. We put

$$\psi_\infty = w\phi_\infty, \qquad w=\sum_{s=0}^{n} M_s c_0 c_s, \qquad M_s \in \boldsymbol{R}. \tag{1.6}$$

Then, we have

$$\begin{cases} \psi_\infty(uxv)=\psi_\infty(x) & \text{for} \quad u, v \in U_\infty,\ x \in A_\infty , \\ \psi_\infty(x)=0 & \text{for} \quad x \in A_\infty,\ \det x=0 . \end{cases}$$

In the following sections, we shall seek a function ψ_∞ of the form (1.6) satisfying the requirements

i) $\hat{\psi}_\infty=\psi_\infty$,

ii) the local ζ-function at ∞ with the weight function ψ_∞ is not identically zero.

§ 2. Some properties on determinants

In § 2, § 3 and § 4, we consider only functions on A_∞ or G_∞, so we omit the suffix ∞. Let w be a polynomial on A. For integers i, j $(1 \leqq i, j \leqq n)$, we define an operator $\partial/\partial(ij)$ by

$$\frac{\partial w}{\partial(ij)}(x)=\frac{\partial w(x)}{\partial x_{ij}}, \qquad x \in A .$$

Put $\Delta=\sum_{i,j}(\partial/\partial(ij))^2$.

We cite the following proposition from Maass [**10**], p. 4.

PROPOSITION 2. *For every polynomial w on A, we have*

$$w^*=\sum_{k=0}^{\infty}\frac{1}{(4\pi)^k k!}\Delta^k w .$$

Since $\Delta^k w=0$ for sufficiently large k, the right-hand side of the above relation is a finite sum.

Put $c_\mu=0$ for $\mu>n$.

PROPOSITION 3. *We have*

$$\Delta(c_r c_s)=2\{(r+1)^2 c_{r+1}c_s+(s+1)^2 c_r c_{s+1}+4\sum_{t=0}^{r}(-r+s+2t+1)c_{r-t}c_{s+t+1}\}$$

for integers r and s such that $0\leqq r,\ s\leqq n$.

We need some preliminaries for the proof of the proposition. We denote by I_n the set of integers $\{1,\cdots,n\}$. Let s be a non-negative integer; let $\{i_1,\cdots,i_s\}$ and $\{j_1,\cdots,j_s\}$ be two indexed sets of s integers in I_n.

Suppose x is an element of A. If $i_1,\cdots,i_s$ are mutually different and $j_1,\cdots,j_s$ are also mutually different, we denote by

$$d(i_1\cdots i_s;\ j_1\cdots j_s)(x)$$

the minor of $\det x$ formed by removing $i_1,\cdots,i_s$-rows and $j_1,\cdots,j_s$-columns; we put

$$e(i_1\cdots i_s;\ j_1\cdots j_s)(x)=\operatorname{sign}(i_1\cdots i_s)\operatorname{sign}(j_1\cdots j_s)d(i_1\cdots i_s;\ j_1\cdots j_s)(x) .$$

If $i_\alpha=i_\beta$ or $j_\alpha=j_\beta$ for different integers α, β such that $1\leqq\alpha,\ \beta\leqq s$ (in particular, when $s>n$), we set

$$d(i_1\cdots i_s;\ j_1\cdots j_s)(x)=e(i_1\cdots i_s;\ j_1\cdots j_s)(x)=0 .$$

From the definition of c_s, we obtain the equality

$$c_s=\sum_{(s)}d(i_1\cdots i_s;\ j_1\cdots j_s)^2 , \tag{2.1}$$

where $\sum_{(s)}$ means $\sum_{\substack{i_1<\cdots<i_s\\ j_1<\cdots<j_s}}$.

Suppose $i_1<\cdots<i_s$, $j_1<\cdots<j_s$. Let i and j be integers in I_n. Put

$$\mu=\#\{i_\alpha;\ 1\leqq\alpha\leqq s,\ i_\alpha<i\} ,\qquad \nu=\#\{j_\alpha;\ 1\leqq\alpha\leqq s,\ j_\alpha<j\} .$$

Then,

$$e(ii_1\cdots i_s;\ jj_1\cdots j_s)=(-1)^{\mu+\nu}d(ii_1\cdots i_s;\ jj_1\cdots j_s) . \tag{2.2}$$

Therefore, we obtain the relation

$$\frac{\partial}{\partial(ij)}d(i_1\cdots i_s;\ j_1\cdots j_s)=(-1)^{i+j}e(ii_1\cdots i_s;\ jj_1\cdots j_s) . \tag{2.3}$$

t follows from these results that

$$\Delta c_s = 2(s+1)^2 c_{s+1}\,. \tag{2.4}$$

Indeed, by (2.1) and (2.3), we have

$$\begin{aligned}\Delta c_s &= \sum_{i,j}\sum_{(s)} \frac{\partial}{\partial(ij)}\{2d(i_1\cdots i_s\,;\,j_1\cdots j_s)(-1)^{i+j}e(ii_1\cdots i_s\,;\,jj_1\cdots j_s)\}\\ &= 2\sum_{i,j}\sum_{(s)} d(ii_1\cdots i_s\,;\,jj_1\cdots j_s)^2\\ &= 2\sum_{\mu,\nu=1}^{s+1}\sum_{\substack{i_1<\cdots<i_{\mu-1}<i<i_\mu<\cdots<i_s\\ j_1<\cdots<j_{\nu-1}<j<j_\nu<\cdots<j_s}} d(ii_1\cdots i_s\,;\,jj_1\cdots j_s)^2\\ &= 2(s+1)^2 c_{s+1}\,.\end{aligned}$$

This completes the proof of (2.4).

We write $d(x)=\det x$, $d(ij)=d(i\,;j)$. The equality

$$e(i_1\cdots i_s\,;\,j_1\cdots j_s)=\frac{1}{d^{s-1}}\begin{vmatrix} d(i_1j_1) & \cdots & d(i_1j_s)\\ \vdots & & \vdots\\ d(i_sj_1) & \cdots & d(i_sj_s)\end{vmatrix}$$

holds for $s\neq 0$. So we have for $s\neq 0$

$$\begin{aligned}&\sum_{\mu=1}^{s}(-1)^{\mu+\nu}d(i_\mu j_\nu)e(i_1\cdots \hat{i}_\mu\cdots i_s\,;\,j_1\cdots\hat{j}_\nu\cdots j_s)\\ &=\sum_{\nu=1}^{s}(-1)^{\mu+\nu}d(i_\mu j_\nu)e(i_1\cdots \hat{i}_\mu\cdots i_s\,;\,j_1\cdots\hat{j}_\nu\cdots j_s)\\ &=de(i_1\cdots i_s\,;\,j_1\cdots j_s).\end{aligned} \tag{2.5}$$

LEMMA 1. *Let $a(i,j)$ be a polynomial on A for every i and j in I_n. Then, we have*

$$\begin{aligned}&\sum_{i,j}(-1)^{i+j}a(i,j)\frac{\partial c_s}{\partial(ij)}\\ &=2\sum_{(s+1)}\sum_{\mu,\nu=1}^{s+1}(-1)^{\mu+\nu}a(i_\mu,j_\nu)d(i_1\cdots\hat{i}_\mu\cdots i_{s+1}\,;\,j_1\cdots\hat{j}_\nu\cdots j_{s+1})\\ &\qquad\times d(i_1\cdots i_{s+1}\,;\,j_1\cdots j_{s+1})\,.\end{aligned}$$

In particular,

$$\sum_{i,j}(-1)^{i+j}d(ij)\frac{\partial c_s}{\partial(ij)}=2(s+1)dc_{s+1}\,.$$

PROOF. Making use of (2.1), (2.3) and (2.2), we get the equalities

$$\begin{aligned}&\sum_{i,j}(-1)^{i+j}a(i,j)\frac{\partial c_s}{\partial(ij)}\\ &=2\sum_{i,j}(-1)^{i+j}a(i,j)\sum_{(s)}(-1)^{i+j}d(i_1\cdots i_s\,;\,j_1\cdots j_s)e(ii_1\cdots i_s\,;\,jj_1\cdots j_s)\end{aligned}$$

$$=2\sum_{\mu,\nu=1}^{s+1}(-1)^{\mu+\nu}\sum_{\substack{i_1<\cdots<i_{\mu-1}<i<i_\mu<\cdots<i_s\\ j_1<\cdots<j_{\nu-1}<j<j_\nu<\cdots<j_s}} a(i,j)d(i_1\cdots i_s\,;j_1\cdots j_s)d(ii_1\cdots i_s\,;jj_1\cdots j_s)$$

$$=2\sum_{(s+1)}\sum_{\mu,\nu=1}^{s+1}(-1)^{\mu+\nu}a(i_\mu,j_\nu)d(i_1\cdots\hat{i}_\mu\cdots i_{s+1}\,;j_1\cdots\hat{j}_\nu\cdots j_{s+1})$$
$$\times d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})\,.$$

They give us the first part of Lemma 1. As to the second part, we put $a(i,j)=d(ij)$ in the first part, then the result follows readily from (2.5).

LEMMA 2. *For integers α and β in I_n, we have*

$$\sum_{i,j}(-1)^{i+j}d(i\beta)d(\alpha j)\frac{\partial c_s}{\partial(ij)}=2d(\alpha\beta)dc_{s+1}-d^2(-1)^{\alpha+\beta}\frac{\partial c_{s+1}}{\partial(\alpha\beta)}\,.$$

PROOF. From Lemma 1 and (2.5),

$$\sum_{i,j}(-1)^{i+j}d(i\beta)d(\alpha j)\frac{\partial c_s}{\partial(ij)}$$

$$=2\sum_{(s+1)}\sum_{\mu,\nu=1}^{s+1}(-1)^{\mu+\nu}d(i_\mu\beta)d(\alpha j_\nu)d(i_1\cdots\hat{i}_\mu\cdots i_{s+1}\,;j_1\cdots\hat{j}_\nu\cdots j_{s+1})$$
$$\times d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})$$

$$=2d\sum_{(s+1)}\sum_{\nu=1}^{s+1}d(\alpha j_\nu)e(i_1\cdots i_{s+1}\,;j_1\cdots j_{\nu-1}\beta j_\nu\cdots j_{s+1})d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})$$

$$=2d\sum_{(s+1)}\Big(-\sum_{\nu=1}^{s+1}(-1)^{1+\nu+1}d(\alpha j_\nu)e(i_1\cdots i_{s+1}\,;\beta j_1\cdots\hat{j}_\nu\cdots j_{s+1})\Big)$$
$$\times d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})$$

$$=2d\sum_{(s+1)}(d(\alpha\beta)d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})-de(\alpha i_1\cdots i_{s+1}\,;\beta j_1\cdots j_{s+1}))$$
$$\times d(i_1\cdots i_{s+1}\,;j_1\cdots j_{s+1})\,.$$

The last expression is equal to

$$2d(\alpha\beta)dc_{s+1}-d^2(-1)^{\alpha+\beta}\frac{\partial c_{s+1}}{\partial(\alpha\beta)}$$

by (2.3). This completes the proof of Lemma 2.

LEMMA 3. *Let r be a positive integer and s be a non-negative integer. Then,*

$$\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}=4(-r+s+1)c_rc_{s+1}+\sum_{i,j}\frac{\partial c_{r-1}}{\partial(ij)}\frac{\partial c_{s+1}}{\partial(ij)}\,.$$

PROOF. We have first by the use of (2.1) and (2.3)

$$\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}$$
$$=2\sum_{i,j}\sum_{(r)}d(i_1\cdots i_r\,;j_1\cdots j_r)(-1)^{i+j}e(ii_1\cdots i_r\,;jj_1\cdots j_r)\frac{\partial c_s}{\partial(ij)}\,.$$

On the other hand, when $i_1<\cdots<i_r$, $j_1<\cdots<j_r$, the relations (2.5) give us the equalities

$$\begin{aligned} & e(ii_1\cdots i_r\,;\,jj_1\cdots j_r) \\ &= \frac{1}{d}d(ij)d(i_1\cdots i_r\,;\,j_1\cdots j_r)+\frac{1}{d}\sum_{\nu=1}^{r}(-1)^{\nu}d(ij_\nu)e(i_1\cdots i_r\,;\,jj_1\cdots \hat{j}_\nu\cdots j_r) \\ &= \frac{1}{d}d(ij)d(i_1\cdots i_r\,;\,j_1\cdots j_r)-\frac{1}{d^2}\sum_{\mu,\nu=1}^{r}(-1)^{\mu+\nu}d(ij_\nu)d(i_\mu j) \\ &\qquad \times d(i_1\cdots \hat{i}_\mu\cdots i_r\,;\,j_1\cdots \hat{j}_\nu\cdots j_r)\,. \end{aligned}$$

Accordingly, we can convert the sum

$$\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}$$

into the sum of two expressions:

$$\frac{2}{d}\sum_{(r)}\sum_{i,j}d(i_1\cdots i_r\,;\,j_1\cdots j_r)^2(-1)^{i+j}d(ij)\frac{\partial c_s}{\partial(ij)}\,, \tag{2.6}$$

$$-\frac{2}{d^2}\sum_{(r)}\sum_{\mu,\nu=1}^{r}\sum_{i,j}(-1)^{i+j}d(ij_\nu)d(i_\mu j)\frac{\partial c_s}{\partial(ij)} \times(-1)^{\mu+\nu}d(i_1\cdots \hat{i}_\mu\cdots i_r\,;\,j_1\cdots \hat{j}_\nu\cdots j_r)d(i_1\cdots i_r\,;\,j_1\cdots j_r)\,. \tag{2.7}$$

The expression (2.6) is reduced, by Lemma 1, to

$$4(s+1)c_r c_{s+1}\,.$$

Taking into account Lemma 2, we again devide (2.7) into the sum of two expressions:

$$-\frac{4}{d}\sum_{(r)}\sum_{\mu,\nu=1}^{r}d(i_\mu j_\nu)c_{s+1}(-1)^{\mu+\nu}d(i_1\cdots \hat{i}_\mu\cdots i_r\,;\,j_1\cdots \hat{j}_\nu\cdots j_r) \times d(i_1\cdots i_r\,;\,j_1\cdots j_r)\,, \tag{2.8}$$

$$2\sum_{(r)}\sum_{\mu,\nu=1}^{r}(-1)^{\mu+\nu}(-1)^{i_\mu+j_\nu}\frac{\partial c_{s+1}}{\partial(i_\mu j_\nu)}d(i_1\cdots \hat{i}_\mu\cdots i_r\,;\,j_1\cdots \hat{j}_\nu\cdots j_r) \times d(i_1\cdots i_r\,;\,j_1\cdots j_r)\,. \tag{2.9}$$

By the relations (2.5) and (2.1), we have

$$(2.8)=-4r\sum_{(r)}d(i_1\cdots i_r\,;\,j_1\cdots j_r)^2 c_{s+1}=-4rc_r c_{s+1}\,.$$

Further, Lemma 1 gives us the relations

$$(2.9)=\sum_{i,j}(-1)^{i+j}\left\{(-1)^{i+j}\frac{\partial c_{s+1}}{\partial(ij)}\right\}\frac{\partial c_{r-1}}{\partial(ij)}=\sum_{i,j}\frac{\partial c_{r-1}}{\partial(ij)}\frac{\partial c_{s+1}}{\partial(ij)}\,.$$

Summing up these results, we get

$$\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}=4(-r+s+1)c_rc_{s+1}+\sum_{i,j}\frac{\partial c_{r-1}}{\partial(ij)}\frac{\partial c_{s+1}}{\partial(ij)}.$$

This completes the proof.

PROOF OF PROPOSITION 3. We have

$$\begin{aligned}\Delta(c_rc_s)&=\Big\{\Delta(c_r)c_s+c_r\Delta(c_s)+2\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}\Big\}\\&=2\Big\{(r+1)^2c_{r+1}c_s+(s+1)^2c_rc_{s+1}+\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}\Big\}.\end{aligned}$$

So it is enough to prove the formula

$$\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}=4\sum_{t=0}^{r}(-r+s+2t+1)c_{r-t}c_{s+t+1}, \tag{2.10}$$

which we shall show by induction on r.

We have by Lemma 1.

$$\sum_{i,j}\frac{\partial c_0}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}=2d\sum_{i,j}(-1)^{i+j}d(ij)\frac{\partial c_s}{\partial(ij)}=4(s+1)c_0c_{s+1}.$$

This is the formula (2.10) for $r=0$. Let $r\geqq 1$. Suppose

$$\sum_{i,j}\frac{\partial c_{r-1}}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}=4\sum_{t=0}^{r-1}(-r+s+2t+2)c_{r-1-t}c_{s+t+1}.$$

We get from Lemma 3

$$\begin{aligned}\sum_{i,j}\frac{\partial c_r}{\partial(ij)}\frac{\partial c_s}{\partial(ij)}&=4(-r+s+1)c_rc_{s+1}+4\sum_{t=0}^{r-1}(-r+s+2t+3)c_{r-1-t}c_{s+t+2}\\&=4\sum_{t=0}^{r}(-r+s+2t+1)c_{r-t}c_{s+t+1}.\end{aligned}$$

This concludes the proof of Proposition 3.

On the ground of Proposition 3 and Proposition 2, we next define two sequences.

Sequence A. We shall define $A_l^k(s)$ for all integers k, l and s. For k, l, s not satisfying $k\geqq l\geqq 0$ and $s\geqq 0$, we put

$$A_l^k(s)=0.$$

For k, l, s satisfying $k\geqq l\geqq 0$ and $s\geqq 0$, we define $A_l^k(s)$ inductively by

$$\begin{cases}A_0^0(s)=1\\ A_l^k(s)=A_l^{k-1}(s)(k-l)^2+A_{l-1}^{k-1}(s)(s+l)^2+4(-k+2l+s)\sum_{\mu=0}^{l-1}A_\mu^{k-1}(s).\end{cases}$$

Sequence B. We shall define $B_s(t, m)$ for all integers s, t and m. For s, t, m satisfying $0\leqq s,\ t\leqq m$, we put

$$B_s(t,m)=\frac{1}{1+\delta_{t,m}}W_{t+m-s}\{A_{t-s}^{t+m-s}(s)+A_{m-s}^{t+m-s}(s)\}\,,$$

where $\delta_{t,m}$ is the Kronecker's symbol and $W_k=(-1)^k/k!$. For s, t, m not satisfying $0\leqq s$, $t\leqq m$, we put

$$B_s(t,m)=0\,.$$

We explain the rôle of these sequences in the following Proposition 4 and Proposition 5.

PROPOSITION 4. *Let s be an integer such that $0\leqq s\leqq n$. For every non-negative integer k, we have*

$$\Delta^k(c_0c_s)=2^k\sum_{l=0}^{k}A_l^k(s)c_{k-l}c_{s+l}\,. \tag{2.11}$$

In particular, $\Delta^k(c_0c_s)=0$ for $k>2n-s$.

PROOF. Since $A_0^0(s)=1$, (2.11) is valid for $k=0$. Suppose

$$\Delta^{k-1}(c_0c_s)=2^{k-1}\sum_{l=0}^{k-1}A_l^{k-1}(s)c_{k-1-l}c_{s+l}\,.$$

By means of Proposition 3, we devide $\Delta^k(c_0c_s)$ into the sum of the following three expressions:

$$2^k\sum_{l=0}^{k-1}A_l^{k-1}(s)(k-l)^2c_{k-l}c_{s+l}=2^k\sum_{l=0}^{k}A_l^{k-1}(s)(k-l)^2c_{k-l}c_{s+l}\,,$$

$$2^k\sum_{l=0}^{k-1}A_l^{k-1}(s)(s+l+1)^2c_{k-1-l}c_{s+l+1}=2^k\sum_{l=0}^{k}A_{l-1}^{k-1}(s)(s+l)^2c_{k-l}c_{s+l}\,,$$

$$2^k\sum_{\mu=0}^{k-1}A_\mu^{k-1}(s)\cdot 4\sum_{t=0}^{k-1-\mu}(-k+2\mu+s+2t+2)c_{k-1-\mu-t}c_{s+\mu+t+1}\,.$$

Put $l=\mu+t+1$, then the third expression is equal to

$$2^k\sum_{\mu=0}^{k-1}\sum_{l=\mu+1}^{k}A_\mu^{k-1}(s)\cdot 4(-k+2l+s)c_{k-l}c_{s+l}$$

$$=2^k\sum_{l=0}^{k}4(-k+2l+s)\sum_{\mu=0}^{l-1}A_\mu^{k-1}(s)c_{k-l}c_{s+l}\,.$$

Thus,

$$\Delta^k(c_0c_s)=2^k\sum_{l=0}^{k}\{A_l^{k-1}(s)(k-l)^2+A_{l-1}^{k-1}(s)(s+l)^2+4(-k+2l+s)\sum_{\mu=0}^{l-1}A_\mu^{k-1}(s)\}c_{k-l}c_{s+l}$$

$$=2^k\sum_{l=0}^{k}A_l^k(s)c_{k-l}c_{s+l}\,.$$

Since $c_\mu=0$ for $\mu>n$, we get the second part of the proposition.

PROPOSITION 5. *For real numbers N_s $(0\leqq s\leqq n)$, we have*

$$\widetilde{\sum_{s=0}^{n}(-1)^s N_s \frac{c_0 c_s}{(2\pi)^s}} = \sum_{0\leqq\rho\leqq\sigma\leqq n}\left\{\sum_{s=0}^{\sigma} B_s(\rho,\sigma)N_s\right\}\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}.$$

PROOF. Since $\Delta^k(c_0c_s)=0$ for $k>2n-s$,

$$\begin{aligned}(c_0c_s)^* &= \sum_{k=0}^{2n-s}\frac{1}{k!(4\pi)^k}\Delta^k(c_0c_s)\\ &= \sum_{k=0}^{2n-s}\sum_{l=0}^{k}\frac{1}{k!(2\pi)^k}A_l^k(s)c_{k-l}c_{s+l}\end{aligned}$$

by Proposition 2 and Proposition 4. On the other hand,

$$d(i_1\cdots i_s;\, j_1\cdots j_s)(-\sqrt{-1}\,{}^t x) = (\sqrt{-1})^{n-s}d(j_1\cdots j_s;\, i_1\cdots i_s)(x),$$

so

$$c_s(-\sqrt{-1}\,{}^t x) = (-1)^{n-s}c_s(x).$$

Hence, we get

$$\widetilde{(-1)^s\frac{c_0c_s}{(2\pi)^s}} = \sum_{k=0}^{2n-s}\sum_{l=0}^{k}\frac{(-1)^k}{k!}A_l^k(s)\frac{c_{k-l}c_{s+l}}{(2\pi)^{k+s}}.$$

Put $\rho=k-l$, $\sigma=s+l$. Since $c_\mu=0$ for $\mu>n$, it follows that

$$\begin{aligned}\widetilde{(-1)^s\frac{c_0c_s}{(2\pi)^s}} &= \sum_{\rho=0}^{n}\sum_{\sigma=s}^{n}W_{\rho+\sigma-s}A_{\sigma-s}^{\rho+\sigma-s}(s)\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}\\ &= \sum_{\sigma=s}^{n}\sum_{\rho=0}^{\sigma}+\sum_{\rho=s+1}^{n}\sum_{\sigma=s}^{\rho-1}.\end{aligned}$$

Recalling that $A_\alpha^\beta(s)=0$ for $\alpha<0$, we rewrite the second sum in the last expression as follows:

$$\begin{aligned}\text{the second sum} &= \sum_{\sigma=s+1}^{n}\sum_{\rho=s}^{\sigma-1}W_{\rho+\sigma-s}A_{\rho-s}^{\rho+\sigma-s}(s)\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}\\ &= \sum_{\sigma=s}^{n}\sum_{\rho=0}^{\sigma-1}W_{\rho+\sigma-s}A_{\rho-s}^{\rho+\sigma-s}(s)\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\widetilde{(-1)^s\frac{c_0c_s}{(2\pi)^s}} &= \sum_{\sigma=s}^{n}\sum_{\rho=0}^{\sigma}\frac{1}{1+\delta_{\rho,\sigma}}W_{\rho+\sigma-s}\{A_{\rho-s}^{\rho+\sigma-s}(s)+A_{\sigma-s}^{\rho+\sigma-s}(s)\}\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}\\ &= \sum_{\sigma=s}^{n}\sum_{\rho=0}^{\sigma}B_s(\rho,\sigma)\frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}}.\end{aligned}$$

Now, Proposition 5 readily follows from this.

Thus, from Proposition 5, we shall obtain a polynomial w on A of the form (1.6) such that $\tilde{w}=w$, if we get a sequence N_s $(s=0,1,2,\cdots)$ which satisfies the equations

$$\sum_{s=0}^{m} B_s(t, m)N_s = \begin{cases} 0 & (1 \leqq t \leqq m), \\ (-1)^m N_m & (t=0) \end{cases} \tag{2.12}$$

for every non-negative integer m. But it is difficult to express the coefficients $B_s(t, m)$ in a simple formula by s, t, m, immediately from the definitions of sequences A and B. We shall keep this difficulty out of the way in the following sections.

§ 3. Certain integrals

In this section, we calculate the integrals

$$\int_A c_0(x)c_s(x)\phi(x)\,dx, \quad \int_G c_0(g)c_s(g)\phi(g)\omega_{s_1,\cdots,s_n}(g^{-1})dg$$

for later use.

We first consider differential operators on G. Let φ be a differentiable function on G. For integers i and j $(1 \leqq i, j \leqq n)$, we define an operator D_{ij} by

$$(D_{ij}\varphi)(g) = \sum_{\mu=1}^{n} g_{i\mu}\frac{\partial\varphi(g)}{\partial g_{j\mu}}, \qquad g=(g_{ij}) \in G.$$

Let s be an integer such that $0 \leqq s \leqq n$. Put $k=n-s$. Let $L_{\alpha\beta}$ be a differential operator on G for every α and β $(1 \leqq \alpha, \beta \leqq k)$. We put

$$\begin{vmatrix} L_{kk} & \cdots & L_{k1} \\ \vdots & & \vdots \\ L_{1k} & \cdots & L_{11} \end{vmatrix} = \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sign}(\sigma) L_{\sigma(k)\,k} \cdots L_{\sigma(1)\,1}.$$

We define a differential operator D_s by

$$D_s = \sum_{i_1<\cdots<i_k} \begin{vmatrix} D_{i_k i_k}+(k-1) & \cdots\cdots & D_{i_k i_1} \\ \vdots & \ddots & \vdots \\ \vdots & D_{i_2 i_2}+1 & \vdots \\ D_{i_1 i_k} & \cdots\cdots\cdots & D_{i_1 i_1} \end{vmatrix}.$$

PROPOSITION 6. *The operator D_s commutes with the translations R_a and L_u for all a in G and u in U. Moreover, we have*

$$D_s\phi = (-2\pi)^{n-s}c_s\phi.$$

PROOF. Let $g=(g_{ij})$ and $a=(a_{ij})$ be elements of G. Put $h=ga$. Let $(a^{-1})_{i,j}$ be the (i, j) element of the matrix a^{-1}. Then, we have

$$(D_{ij}R_a\varphi)(g) = \sum_{\mu=1}^{n} g_{i\mu}\frac{\partial\varphi(ga)}{\partial g_{j\mu}} = \sum_{\mu=1}^{n} g_{i\mu}\sum_{\nu=1}^{n}\left(\frac{\partial\varphi(g)}{\partial g_{j\nu}}\right)_{g=h} a_{\mu\nu}.$$

Hence,

$$(R_a^{-1}D_{ij}R_a\varphi)(g) = \sum_{\mu,\nu,\sigma=1}^{n} g_{i\sigma}(a^{-1})_{\sigma\mu}\frac{\partial\varphi(g)}{\partial g_{j\nu}}a_{\mu\nu} = \sum_{\nu=1}^{n} g_{i\nu}\frac{\partial\varphi(g)}{\partial g_{j\nu}} = (D_{ij}\varphi)(g).$$

Thus, we get $D_{ij}R_a = R_a D_{ij}$, hence $D_s R_a = R_a D_s$.

Let $u = (u_{ij})$ be an element of U. Put $h = u^{-1}g$. Similarly as above, we have

$$L_u^{-1} D_{ij} L_u \varphi = \sum_{\alpha,\beta=1}^{n} u_{i\alpha}(D_{\alpha\beta}\varphi) u_{j\beta} .$$

We consider the (n, n) matrices (D_{ij}) and $(L_u^{-1} D_{ij} L_u)$. Then, the following relation holds:

$$(L_u^{-1} D_{ij} L_u) = u(D_{ij}) u^{-1} . \tag{3.1}$$

On the other hand, we have

$$(D_{i_l j_l} \cdots D_{i_1 j_1}\varphi)(g) = \sum_{\mu_1,\cdots,\mu_l=1}^{n} g_{i_l \mu_l} \cdots g_{i_1 \mu_1} \frac{\partial^l \varphi(g)}{\partial g_{j_l \mu_l} \cdots \partial g_{j_1 \mu_1}}$$

for $i_1 \neq j_2, \cdots, i_1 \neq j_l; \cdots ; i_{l-1} \neq j_l$. The auxiliary symbols Δ_{ij} are used instead of D_{ij}, to indicate by their composition the same result for every $i_1, \cdots, i_l$, $j_1, \cdots, j_l$:

$$(\Delta_{i_l j_l} \cdots \Delta_{i_1 j_1}\varphi)(g) = \sum_{\mu_1,\cdots,\mu_l=1}^{n} g_{i_l \mu_l} \cdots g_{i_1 \mu_1} \frac{\partial^l \varphi(g)}{\partial g_{j_l \mu_l} \cdots \partial g_{j_1 \mu_1}}$$

(cf. Weyl [**16**], p. 39). Then, we have by (3.1)

$$\sum_{i_1<\cdots<i_k} \begin{vmatrix} L_u^{-1}\Delta_{i_1 i_1} L_u & \cdots & L_u^{-1}\Delta_{i_1 i_k} L_u \\ \vdots & & \vdots \\ L_u^{-1}\Delta_{i_k i_1} L_u & \cdots & L_u^{-1}\Delta_{i_k i_k} L_u \end{vmatrix} = \sum_{i_1<\cdots<i_k} \begin{vmatrix} \Delta_{i_1 i_1} & \cdots & \Delta_{i_1 i_k} \\ \vdots & & \vdots \\ \Delta_{i_k i_1} & \cdots & \Delta_{i_k i_k} \end{vmatrix}$$

From this relation, we can derive the result $L_u^{-1} D_s L_u = D_s$ by the same argument as in Weyl [**16**], p. 40.

Next, a formal computation gives us the formula

$$\sum_{j_1<\cdots<j_k} \begin{vmatrix} g_{i_1 i_1} & \cdots & g_{i_1 j_k} \\ \vdots & & \vdots \\ g_{i_k j_1} & \cdots & g_{i_k j_k} \end{vmatrix} \begin{vmatrix} \dfrac{\partial}{\partial g_{i_1 j_1}} & \cdots & \dfrac{\partial}{\partial g_{i_k j_1}} \\ \vdots & & \vdots \\ \dfrac{\partial}{\partial g_{j_1 j_k}} & \cdots & \dfrac{\partial}{\partial g_{i_k j_k}} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{\mu=1}^{n} g_{i_1 \mu} \dfrac{\partial}{\partial g_{i_1 \mu}} & \cdots & \sum_{\mu=1}^{n} g_{i_1 \mu} \dfrac{\partial}{\partial g_{i_k \mu}} \\ \vdots & & \vdots \\ \sum_{\mu=1}^{n} g_{i_k \mu} \dfrac{\partial}{\partial g_{i_1 \mu}} & \cdots & \sum_{\mu=1}^{n} g_{i_k \mu} \dfrac{\partial}{\partial g_{i_k \mu}} \end{vmatrix} .$$

So we have

$$(D_s\phi)(g)=\sum_{i_1<\cdots<i_k}\left\{\begin{vmatrix} \Delta_{i_1i_1} & \cdots & \Delta_{i_1i_k} \\ \vdots & & \vdots \\ \Delta_{i_ki_1} & \cdots & \Delta_{i_ki_k}\end{vmatrix}\phi\right\}(g)$$

$$=\sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_k}}\begin{vmatrix} g_{i_1j_1} & \cdots & g_{i_1j_k} \\ \vdots & & \vdots \\ g_{i_kj_1} & \cdots & g_{i_kj_k}\end{vmatrix}\cdot\left\{\begin{vmatrix} \frac{\partial}{\partial g_{i_1j_1}} & \cdots & \frac{\partial}{\partial g_{i_kj_1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial g_{i_1j_k}} & \cdots & \frac{\partial}{\partial g_{i_kj_k}}\end{vmatrix}\phi(g)\right\}$$

$$=(-2\pi)^k\sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_k}}\begin{vmatrix} g_{i_1j_1} & \cdots & g_{i_1j_k} \\ \vdots & & \vdots \\ g_{i_kj_1} & \cdots & g_{i_kj_k}\end{vmatrix}^2\phi(g)$$

$$=(-2\pi)^{n-s}c_s(g)\phi(g)\,. \qquad \text{q. e. d.}$$

LEMMA 4. *Let $\alpha_1,\cdots,\alpha_n$ be complex numbers. Let φ be a function on G which is expressed in the form*

$$\varphi(g)=t_{11}^{\alpha_1}\cdots t_{nn}^{\alpha_n} \text{ for } g=tu,\ t=(t_{ij})\in T,\ u\in U\,.$$

Then, for $\alpha_i\neq -1$ $(l\leqq i\leqq n)$,

$$D_s\varphi=\sum_{i_1<\cdots<i_k}\left\{\prod_{\mu=1}^{k}(\alpha_{i_\mu}+\mu-1)\right\}\varphi\,.$$

PROOF. We have for $i\geqq j$

$$(D_{ij}\varphi)(g)=\sum_{\mu}g_{i\mu}\frac{\partial\varphi(g)}{\partial g_{j\mu}}=\sum_{\sigma,\tau,\mu}g_{i\sigma}u_{\tau\sigma}u_{\tau\mu}\frac{\partial\varphi(g)}{\partial g_{j\mu}}$$

$$=\sum_{\tau}t_{i\tau}\frac{\partial t_{11}^{\alpha_1}\cdots t_{nn}^{\alpha_n}}{\partial t_{j\tau}}=t_{ij}\frac{\partial t_{11}^{\alpha_1}\cdots t_{nn}^{\alpha_n}}{\partial t_{jj}}\,.$$

In particular,

$$D_{ij}\varphi=0 \qquad \text{for } i>j\,,$$

$$D_{ii}\varphi=\alpha_i\varphi\,.$$

From them, we obtain the result:

$$D_s\varphi=\sum_{i_1<\cdots<i_k}\begin{vmatrix} D_{i_ki_k}+(k-1) & \cdots & D_{i_ki_1} \\ \vdots & \ddots & \vdots \\ D_{i_1i_k} & \cdots\cdots & D_{i_1i_1}\end{vmatrix}\varphi$$

$$=\sum_{i_1<\cdots<i_k}\left\{\prod_{\mu=1}^{k}(D_{i_\mu i_\mu}+\mu-1)\right\}\varphi$$

$$=\sum_{i_1<\cdots<i_k}\left\{\prod_{\mu=1}^{k}(\alpha_{i_\mu}+\mu-1)\right\}\varphi\,. \qquad \text{q. e. d.}$$

PROPOSITION 7. *We have*

i) $\displaystyle\int_A c_0(x)c_s(x)\phi(x)dx=\frac{1}{(2\pi)^{2n-s}}\binom{n}{s}n!\frac{(n+2)!}{(s+2)!}\,,$

ii) $\int_G c_0(g)c_s(g)\phi(g)\omega_{s_1,\cdots,s_n}(g^{-1})dg$

$$=\frac{(-1)^s}{(2\pi)^{2n-s}}\int_G \phi(g)\omega_{s_1,\cdots,s_n}(g^{-1})dg\Big\{\prod_{i=1}^n s_i\Big\}\sum_{i_1<\cdots<i_k}\prod_{\mu=1}^k(s_{i_\mu}-i_\mu-2+\mu)$$

for $\operatorname{Re} s_i<0$.

PROOF. i) Let g be an element of G. We put $\varphi(g)=|\det g|^{-(n+2)}$. Then,

$$\int_A c_0(x)\phi(gx)dx=(\det g)^{-2}\,|\det g|^{-n}\int_A c_0(x)\phi(x)dx=\frac{n\,!}{(2\pi)^n}\varphi(g).$$

On the other hand, we have by Proposition 6

$$D_sR_x\phi=R_xD_s\phi=(-2\pi)^{n-s}R_x(c_s\phi)\qquad\text{for}\quad x\in G.$$

From these results, we get

$$(-2\pi)^{n-s}\int_A c_0(x)c_s(gx)\phi(gx)dx=\frac{n\,!}{(2\pi)^n}(D_s\varphi)(g),$$

hence

$$\int_A c_0(x)c_s(x)\phi(x)dx=(-1)^{n-s}\frac{n\,!}{(2\pi)^{2n-s}}(D_s\varphi)(1_n).$$

So it is enough to prove the formula

$$(D_s\varphi)(1_n)=(-1)^{n-s}\binom{n}{s}\frac{(n+2)\,!}{(s+2)\,!}.$$

We have

$$\varphi(g)=(t_{11}\cdots t_{nn})^{-(n+2)}\quad\text{for}\quad g=tu,\ t=(t_{ij})\in T,\ u\in U.$$

Applying Lemma 4 to the function φ, we obtain

$$\begin{aligned}D_s\varphi&=\sum_{i_1<\cdots<i_k}\Big\{\prod_{\mu=1}^k(-n-2+\mu-1)\Big\}\varphi\\&=(-1)^k\binom{n}{k}\Big\{\prod_{\mu=1}^k(n+3-\mu)\Big\}\varphi\\&=(-1)^{n-s}\binom{n}{s}\frac{(n+2)\,!}{(s+2)\,!}\varphi.\end{aligned}$$

This proves i). The proof of ii) is similar to that of i). We write ω, ω' and α' instead of $\omega_{s_1,\cdots,s_n}$, $\omega_{s_1-2,\cdots,s_n-2}$ and $\alpha_{s_1-2,\cdots,s_n-2}$. For $\operatorname{Re} s_i<0$, we have

$$\begin{aligned}&\int_G c_0(h)\phi(gh)\omega(h^{-1})dh\\&=\int_G c_0(g^{-1}h)\phi(h)\omega(h^{-1}ug)dh\quad(h\to g^{-1}u^{-1}h)\end{aligned}$$

$$=\int_U\Big(\int_G c_0(g^{-1}h)\phi(h)\omega(h^{-1}ug)\,dh\Big)du$$

$$=(\det g)^{-2}\omega(g)\int_G(\det h)^2\phi(h)\omega(h^{-1})dh$$

$$=\omega'(g)\pi^{-\frac{1}{2}\sum_{i=1}^{n}(s_i-2)}\prod_{i=1}^{n}\Gamma\Big(\frac{-s_i+2}{2}\Big)$$

$$=\frac{(-1)^n}{(2\pi)^n}\int_G\phi(h)\omega(h^{-1})dh\Big\{\prod_{i=1}^{n}s_i\Big\}\omega'(g)$$

by (1.2), Proposition 1 and the relation $\Gamma(z+1)=z\Gamma(z)$. Applying the operator D_s, we get the relation

$$(-2\pi)^{n-s}\int_G c_0(h)c_s(h)\phi(h)\omega(h^{-1})dh$$

$$=\frac{(-1)^n}{(2\pi)^n}\int_G\phi(h)\omega(h^{-1})dh\Big\{\prod_{i=1}^{n}s_i\Big\}(D_s\omega')(1_n)\,.$$

Thus, it remains to prove the formula

$$(D_s\omega')(1_n)=\sum_{i_1<\cdots<i_k}\prod_{\mu=1}^{k}(s_{i_\mu}-i_\mu-2+\mu)\,.$$

We call λ the right-hand side for brevity. Setting $\varphi(g)=\alpha'(g^{-1})$, we have

$$\varphi(g)=t_{11}^{s_1-2}\cdots t_{nn}^{s_n-(n+1)}\ \text{for}\ g=tu,\ t=(t_{ij})\in T,\ u\in U\,.$$

Consequently, it follows from Lemma 4 that

$$D_s\varphi=\sum_{i_1<\cdots<i_k}\Big\{\prod_{\mu=1}^{k}(s_{i_\mu}-(i_\mu+1)+\mu-1)\Big\}\varphi=\lambda\varphi\,.$$

Considering $L_uD_s=D_sL_u$, we get

(3.2) $$(D_sL_u\varphi)(g)=\lambda(L_u\varphi)(g)\,.$$

On the other hand,

$$\omega'(g)=\int_U\alpha'(g^{-1}u)du=\int_U\varphi(u^{-1}g)du=\int_U(L_u\varphi)(g)du\,.$$

Hence, integrating the both sides of (3.2) on U, we obtain

$$(D_s\omega')(g)=\lambda\omega'(g),\quad\text{and}\quad(D_s\omega')(1_n)=\lambda\,.$$

§ 4. A self-reciprocal function on A.

We return to the equations (2.12). The result of the preceding section enables us to calculate the coefficient of the form $B_s(m,m)$.

PROPOSITION 8. *For non-negative integers s and m such that $s\leqq m$, we have*

$$B_s(m, m)=\binom{m}{s}(-1)^s m!\,\frac{(m+2)!}{(s+2)!}.$$

PROOF. The proposition is trivial for $m=0$, so we suppose $m\geqq 1$. Proposition 5 gives us the relation

$$\frac{(-1)^s}{(2\pi)^s}\int_A c_0(-\sqrt{-1}\,{}^t x+y)c_s(-\sqrt{-1}\,{}^t x+y)\phi(y)dy=\sum_{\sigma=s}^{n}\sum_{\rho=0}^{\sigma}B_s(\rho, \sigma)\frac{c_\rho(x)c_\sigma(x)}{(2\pi)^{\rho+\sigma}},$$

where x is an element of A. Taking $x=0$, we get the equality

$$\frac{(-1)^s}{(2\pi)^s}\int_A c_0(y)c_s(y)\phi(y)dy=\frac{1}{(2\pi)^{2n}}B_s(n, n).$$

The integral of the left-hand side is equal to

$$\frac{1}{(2\pi)^{2n-s}}\binom{n}{s}n!\,\frac{(n+2)!}{(s+2)!}$$

by Proposition 7, i). From these results, we obtain readily

$$B_s(n, n)=\binom{n}{s}(-1)^s n!\,\frac{(n+2)!}{(s+2)!}. \qquad \text{q. e. d.}$$

The following proposition allows us to express in some sense the coefficients $B_s(t, m)$ by $B_s(m-l, m-l)$.

PROPOSITION 9. *There exist, for integers m and k ($0\leqq k\leqq[m/2]$), a sequence of real numbers $E(m, k\,;l)$ ($l=0, \cdots, k$), and for integers m and k($0\leqq k\leqq[(m-1)/2]$), a sequence of real numbers $F(m, k\,;l)$ ($l=0, \cdots, k$), which satisfy the following conditions*

$$E(m, k\,;0)=1,$$

$$\left\{\frac{m!}{(m-2k)!}\right\}^2 B_s(m-2k, m) \tag{4.1}$$

$$=\sum_{l=0}^{k}E(m, k\,;l)\left\{\prod_{i=0}^{2k-1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l),$$

$$F(m, k\,;0)=-1,$$

$$\left\{\frac{m!}{(m-2k-1)!}\right\}^2 B_s(m-2k-1, m) \tag{4.2}$$

$$=\sum_{l=0}^{k}F(m, k\,;l)\left\{\prod_{i=0}^{2k-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l),$$

for every integer s such that $0\leqq s\leqq m$.

LEMMA 5. *For integers t and m ($1\leqq t\leqq m$), there exists a sequence of integers $L(t, m\,;\mu)$ ($\mu=0, 1, 2, \cdots$), which satisfies the conditions*

$$L(m, m; \mu)=0,$$

$$t^2 B_s(t-1, m)=-(t+m-s)B_s(t, m)+\sum_{\mu\geqq 0} L(t, m; \mu)B_s(t+\mu, m-\mu-1),$$

for every integer s such that $0\leqq s\leqq m$.

PROOF. In the case $t=m$, we have

$$\begin{aligned}-(2m-s)B_s(m, m) &= -(2m-s)W_{2m-s}A_{m-s}^{2m-s}(s)\\ &= W_{2m-1-s}\{m^2 A_{m-s}^{2m-1-s}(s)+m^2 A_{m-1-s}^{2m-1-s}(s)\}\\ &= m^2 B_s(m-1, m)\end{aligned}$$

by the definitions of sequences A and B. Putting $L(m, m; \mu)=0$, we get the proof in the case $t=m$. In the case $1\leqq t\leqq m-1$, we have similarly

$$\begin{aligned}&-(t+m-s)B_s(t, m)\\ &=-(t+m-s)W_{t+m-s}\{A_{t-s}^{t+m-s}(s)+A_{m-s}^{t+m-s}(s)\}\\ &=W_{t+m-s-1}\{m^2 A_{t-s}^{t+m-s-1}(s)+t^2 A_{t-s-1}^{t+m-s-1}(s)-4(m-t)\sum_{\mu=0}^{t-s-1} A_{\mu}^{t+m-s-1}(s)\\ &\qquad +t^2 A_{m-s}^{t+m-s-1}(s)+m^2 A_{m-s-1}^{t+m-s-1}(s)+4(m-t)\sum_{\mu=0}^{m-s-1} A_{\mu}^{t+m-s-1}(s)\}\\ &=W_{t+m-s-1}[t^2\{A_{t-1-s}^{t-1+m-s}(s)A_{m-s}^{t-1+m-s}(s)\}\\ &\qquad +m^2\{A_{t-s}^{t+m-1-s}(s)+A_{m-1-s}^{t+m-1-s}(s)\}+4(m-t)\sum_{\mu=t-s}^{m-s-1} A_{\mu}^{t+m-s-1}(s)]\\ &=t^2 B_s(t-1, m)+\{(1+\delta_{t,m-1})m^2+4(m-t)\}B_s(t, m-1)\\ &\qquad +4(m-t)\sum_{\mu\geqq 1} B_s(t+\mu, m-\mu-1).\end{aligned}$$

Thus, it is enough to set

$$\begin{cases} L(t, m; 0)=-\{(1+\delta_{t,m-1})m^2+4(m-t)\}, \\ L(t, m; \mu)=-4(m-t) \qquad \text{for} \quad \mu\geqq 1. \end{cases}$$

PROOF OF PROPOSITION 9. In the case $m=0$, we need only to prove the existence of the sequence E, i.e. to put $E(0, 0; 0)=1$. In the case $k=0$, set $E(m, 0; 0)=1$, $F(m, 0; 0)=-1$. The equality

$$m^2 B_s(m-1, m)=-(2m-s)B_s(m, m)$$

means the validity of the proposition.

Let $m\geqq 1$. Assume that we have already proved the proposition in the case $0, \cdots, m-1$ with arbitrary k and in the case m with a fixed k. We shall prove the proposition in the case m with $k+1$.

By lemma 5, we have

$$(m-2k-1)^2 B_s(m-2k-2, m) = -(2m-2k-1-s)B_s(m-2k-1, m)$$
$$+\sum_{\mu\geqq 0} L(m-2k-1, m; \mu)B_s(m-2k-1+\mu, m-1-\mu).$$

Multiply the both sides by $\{m!/(m-2k-1)!\}^2$. Put

$$M(m, k; \mu) = \left\{\frac{m!(m-2k-1+\mu)!}{(m-2k-1)!(m-1-\mu)!}\right\}^2 L(m-2k-1, m; \mu).$$

Then, we have

$$\left\{\frac{m!}{(m-2k-2)!}\right\}^2 B_s(m-2k-2, m)$$
$$= -(2m-2k-s-1)\left\{\frac{m!}{(m-2k-1)!}\right\}^2 B_s(m-2k-1, m)$$
$$+\sum_{\mu\geqq 0} M(m, k; \mu)\left\{\frac{(m-1-\mu)!}{(m-2k-1+\mu)!}\right\}^2 B_s(m-2k-1+\mu, m-1-\mu)$$
$$= -(2m-2k-s-1)\sum_{l=0}^{k} F(m, k; l)\left\{\prod_{i=0}^{2k-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l)$$
$$+\sum_{\mu\geqq 0} M(m, k; \mu)\sum_{l=0}^{k-\mu} E(m-1-\mu, k-\mu; l)\left\{\prod_{i=0}^{2k-2\mu-1-2l}(2m-2-2\mu-2l-s-i)\right\}$$
$$\times B_s(m-1-\mu-l, m-1-\mu-l).$$

The last equality comes of the assumptions (4.2) in the case m, k, and (4.1) in the case $m-1-\mu$, $k-\mu$. And the right-hand side is equal to the expression

$$\sum_{l=0}^{k} -F(m, k; l)\left\{\prod_{i=0}^{2k+1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l)$$
$$+\sum_{\mu=0}^{k} M(m, k; \mu)\sum_{l=\mu+1}^{k+1} E(m-1-\mu, k-\mu; l-1-\mu)$$
$$\times\left\{\prod_{i=0}^{2k+1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l),$$

hence to the expression

$$\sum_{l=0}^{k} -F(m, k; l)\left\{\prod_{i=0}^{2k+1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l)$$
$$+\sum_{l=1}^{k+1}\left\{\prod_{\mu=0}^{l-1} M(m, k; \mu)E(m-1-\mu, k-\mu; l-1-\mu)\right\}$$
$$\times\left\{\prod_{i=0}^{2k+1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l).$$

We put

$$E(m, k+1\,;\,l)=\begin{cases} -F(m, k\,;\,l)+\sum_{\mu=0}^{l-1} M(m, k\,;\,\mu)E(m-1-\mu, k-\mu\,;\,l-1-\mu) & (0\leqq l\leqq k)\,, \\ \sum_{\mu=0}^{l-1} M(m, k\,;\,\mu)E(m-1-\mu, k-\mu\,;\,l-1-\mu) & (l=k+1)\,. \end{cases}$$

Then, we have $E(m, k+1\,;\,0)=1$ and the relation (4.1) in the case $m, k+1$.

Similarly, by Lemma 5 and the relation (4.1) in the case $m, k+1$ and the assumption (4.2) in the case $m-1-\mu$, $k-\mu$, we can prove the existence of the required sequence $F(m, k+1\,;\,l)$. This concludes the proof.

PROPOSITION 10. *For integers t and m $(0\leqq t\leqq m)$, we have*

$$\sum_{s=0}^{m} B_s(t, m)(s+2)\,!=\begin{cases} 0 & (1\leqq t\leqq m)\,, \\ (-1)^m(m+2)\,! & (t=0)\,. \end{cases}$$

PROOF. Put $N_s=(s+2)\,!$. By the relation (4.1), we have

$$\left\{\frac{m\,!}{(m-2k)\,!}\right\}^2 \sum_{s=0}^{m} B_s(m-2k, m)N_s$$

$$=\sum_{l=0}^{k} E(m, k\,;\,l)\sum_{s=0}^{m}\left\{\prod_{i=0}^{2k-1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l)N_s$$

for integers m and k $(0\leqq k\leqq [m/2])$. On the other hand, we get by Proposition 8 and the definition of the sequence B

$$B_s(m-l, m-l)N_s=\begin{cases} \binom{m-l}{s}(-1)^s(m-l)\,!\,N_{m-l} & (s\leqq m-l)\,, \\ 0 & (s>m-l)\,. \end{cases}$$

Therefore,

$$\sum_{s=0}^{m}\left\{\prod_{i=0}^{2k-1-2l}(2m-2l-s-i)\right\}B_s(m-l, m-l)N_s$$

$$=\sum_{s=0}^{m-l}\left[\left\{\prod_{i=0}^{2k-1-2l}(2m-2l-s-i)\right\}\binom{m-l}{s}(-1)^s\right](m-l)\,!\,N_{m-l}$$

$$=\left[\frac{d^{2k-2l}}{dx^{2k-2l}}\left\{\sum_{s=0}^{m-l}\binom{m-l}{s}x^{2m-2l-s}(-1)^s\right\}\right]_{x=1}(m-l)\,!\,N_{m-l}$$

$$=\left[\frac{d^{2k-2l}}{dx^{2k-2l}}\left\{x^{m-l}(x-1)^{m-l}\right\}\right]_{x=1}(m-l)\,!\,N_{m-l}$$

$$=\begin{cases} 0 & (m>2k \text{ or } m=2k, l>0)\,, \\ (m\,!)^2N_m & (m=2k, l=0)\,. \end{cases}$$

Hence, recalling that $E(m, k\,;\,0)=1$, we obtain

$$\left\{\frac{m\,!}{(m-2k)\,!}\right\}^2 \sum_{s=0}^{m} B_s(m-2k, m)N_s=\begin{cases} 0 & (m>2k)\,, \\ (m\,!)^2N_m & (m=2k)\,. \end{cases}$$

Similarly, we have by (4.2)

$$\left\{\frac{m\,!}{(m-2k-1)\,!}\right\}^2 \sum_{s=0}^{m} B_s(m-2k-1,\, m)N_s = \begin{cases} 0 & (m>2k+1)\,, \\ -(m\,!)^2 N_m & (m=2k+1)\,. \end{cases}$$

These results show that

$$\sum_{s=0}^{m} B_s(t,\, m)N_s = \begin{cases} 0 & (1 \leqq t \leqq m)\,, \\ (-1)^m N_m & (t=0)\,. \end{cases}$$

COROLLARY. *Put*

$$\psi = w\phi,\; w = \sum_{s=0}^{m} (-1)^s \frac{(s+2)\,!}{2} \frac{c_0 c_s}{(2\pi)^s}\,.$$

Then, the function ψ *on* A *is self-reciprocal; i.e. we have*

$$\hat{\psi} = \psi\,.$$

PROOF. By Proposition 5 and Proposition 10,

$$\begin{aligned} \tilde{w} &= \sum_{0 \leqq \rho \leqq \sigma \leqq n} \left\{ \sum_{s=0}^{\sigma} B_s(\rho,\, \sigma) \frac{(s+2)\,!}{2} \right\} \frac{c_\rho c_\sigma}{(2\pi)^{\rho+\sigma}} \\ &= \sum_{0 \leqq \sigma \leqq n} (-1)^\sigma \frac{(\sigma+2)\,!}{2} \frac{c_0 c_\sigma}{(2\pi)^\sigma} = w\,. \end{aligned}$$

So, $\hat{\psi} = \widehat{w\phi} = \tilde{w}\phi = w\phi = \psi$. q. e. d.

The next task is to find an explicit expression of the local ζ-function at ∞ with the weight function ψ.

PROPOSITION 11. *Put* $\omega = \omega_{s_1,\cdots,s_n}$. *The integral*

$$\int_G \psi(g)\omega(g^{-1})\,|\det g\,|^z dg$$

converges for $\mathrm{Re}\, z > \max_i (\mathrm{Re}\, s_i)$, *and is equal to*

$$\frac{1}{(2\pi)^{2n}} \zeta(z,\, \omega) \prod_{i=1}^{n} (z-s_i)(z-s_i-1)\,.$$

PROOF. We have by (1.3)

$$\omega_{s_1,\cdots,s_n}(g^{-1})\,|\det g\,|^z = \omega_{s_1-z,\cdots,s_n-z}(g^{-1})\,.$$

So it is enough to prove

$$\int_G \psi(g)\omega(g^{-1})dg = \frac{1}{(2\pi)^{2n}} \int_G \phi(g)\omega(g^{-1})dg \prod_{i=0}^{n} s_i(s_i+1)$$

for $\mathrm{Re}\, s_i < 0$. Now, by proposition 7, ii), the equality

$$\begin{aligned} &\int_G (-1)^s \frac{c_0(g)c_s(g)}{(2\pi)^s} \phi(g)\omega(g^{-1})dg \\ &= \frac{1}{(2\pi)^{2n}} \int_G \phi(g)\omega(g^{-1})dg \left\{ \prod_{i=1}^{n} s_i \right\} \sum_{i_1<\cdots<i_k} \prod_{\mu=1}^{n} (s_{i_\mu} - i_\mu - 2 + \mu) \end{aligned}$$

holds for $\operatorname{Re} s_i < 0$, where $k = n - s$. Therefore, we need only to prove the formula

$$\sum_{k=0}^{n} \frac{(n-k+2)!}{2} \sum_{i_1<\cdots<i_k} \prod_{\mu=1}^{k} (s_{i_\mu} - i_\mu - 2 + \mu) = \prod_{i=1}^{n} (s_i + 1). \tag{4.3}$$

Let λ_n be the left-hand side of (4.3). It is obvious that

$$\lambda_1 = 3 + (s_1 - 2) = s_1 + 1.$$

And we show that $\lambda_{n+1} = \lambda_n (s_{n+1} + 1)$. Indeed,

$$\begin{aligned}
\lambda_{n+1} &= \sum_{k=0}^{n+1} \frac{(n-k+3)!}{2} \sum_{1 \leqq i_1 < \cdots < i_k \leqq n+1} \prod_{\mu=1}^{k} (s_{i_\mu} - i_\mu - 2 + \mu) \\
&= \frac{(n+3)!}{2} \\
&\quad + \sum_{k=1}^{n+1} \frac{(n-k+3)!}{2} \Big\{ \sum_{1 \leqq i_1 < \cdots < i_{k-1} \leqq n} \prod_{\mu=1}^{k-1} (s_{i_\mu} - i_\mu - 2 + \mu) \Big\} (s_{n+1} - n - 3 + k) \\
&\quad + \sum_{k=1}^{n} \frac{(n-k+3)!}{2} \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \prod_{\mu=1}^{k} (s_{i_\mu} - i_\mu - 2 + \mu) \\
&= \frac{(n+3)!}{2} + \frac{(n+2)!}{2} (s_{n+1} - n - 2) \\
&\quad + \sum_{k=2}^{n+1} \frac{(n-k+3)!}{2} \Big\{ \sum_{1 \leqq i_1 < \cdots < i_{k-1} \leqq n} \prod_{\mu=1}^{k-1} (s_{i_\mu} - i_\mu - 2 + \mu) \Big\} (s_{n+1} - n - 3 + k) \\
&\quad + \sum_{k=1}^{n} \frac{(n-k+2)!}{2} \Big\{ \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \prod_{\mu=1}^{k} (s_{i_\mu} - i_\mu - 2 + \mu) \Big\} (n - k + 3) \\
&= \frac{(n+2)!}{2} (s_{n+1} + 1) \\
&\quad + \sum_{k=1}^{n} \frac{(n-k+2)!}{2} \Big\{ \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \prod_{\mu=1}^{k} (s_{i_\mu} - i_\mu - 2 + \mu) \Big\} (s_{n+1} + 1) \\
&= \lambda_n (s_{n+1} + 1).
\end{aligned}$$

This proves (4.3).

§5. Certain global ζ-functions

Let $\Lambda = \{\lambda\}$ be a set of indices. Suppose a unimodular locally compact group G_λ is associated with all λ, and a compact open subgroup H_λ of G_λ is associated with almost all λ. We denote by G the restricted direct product of G_λ with respect to H_λ. It is the set of all elements $g = (g_\lambda)$ of $\prod_\lambda G_\lambda$ such that $g_\lambda \in H_\lambda$ for almost all λ. Let S_0 denote the set of all indices λ in Λ with which the group H_λ is not associated. For every finite subset S of Λ containing S_0, put $G_S=\prod_{\lambda\in S}G_\lambda\times\prod_{\lambda\notin S}H_\lambda$. G_S may be considered as a subgroup of G. We have $G=\bigcup_S G_S$. Each G_S has its natural product topology and G is topologized as the inductive limit with respect to S. Then, G is a unimodular locally compact group.

Let f_λ be a function on G_λ for every λ. We assume that $f_\lambda(H_\lambda)=1$ for almost all λ. Putting $f(g)=\prod_\lambda f_\lambda(g_\lambda)$ for $g=(g_\lambda)\in G$, we define a function f on G, which we denote by $f=\prod_\lambda f_\lambda$.

Let dg_λ be a Haar measure on G_λ. We assume that the total volume of H_λ is equal to 1 for almost all λ. The restricted direct product dg of dg_λ has the following property:

If the above mentioned function f_λ further satisfies the conditions

$$f_\lambda\in L_1(G_\lambda) \text{ for all } \lambda,\ \prod_\lambda\int_{G_\lambda}|f_\lambda(g_\lambda)|\,dg_\lambda<\infty\,,$$

then we have

$$f=\prod_\lambda f_\lambda\in L_1(G),\quad \int_G f(g)dg=\prod_\lambda\int_{G_\lambda}f_\lambda(g_\lambda)dg_\lambda\,. \tag{5.1}$$

Of course, the infinite product of integrals converges absolutely. Further, if f_λ is in $C(G_\lambda)$ for all λ, then $f=\prod_\lambda f_\lambda$ is in $C(G)$.

We denote by A the adele ring of $\varDelta=M_n(\boldsymbol{Q})$; i.e. A is the restricted direct product of A_p with respect to O_p. By the canonical injection, $\varDelta$ may be considered as a discrete subgroup of A. It is known that the factor group $A/\varDelta$ is compact. We denote by A^∞ the set of all elements $x=(x_p)$ of A such that $x_p\in O_p$ for all $p\neq\infty$. It is an open subgroup of A. We have $A=A^\infty+\varDelta$.

We have $\chi_p(O_p)=1$ for $p\neq\infty$. We define a function $\chi=\prod_p\chi_p$ on A. It is a unitary character of A. Obviously, $\chi(xy)=\chi(yx)$ for all $x,y\in A$. By the mapping

$$A\times A\ni(x,y)\to\chi(xy)\in\boldsymbol{C}\,,$$

A is self-dual, and the annihilator of $\varDelta$ is again $\varDelta$.

Let dx_p be the Haar measure on A_p, normalized as in § 1. We denote by dx the restricted direct product of dx_p. There exists the canonical Haar measure $d\bar{x}$ on $A/\varDelta$ satisfying the relation

$$\int_A f(x)dx=\int_{A/\varDelta}\Big\{\sum_{\xi\in\varDelta}f(x+\xi)\Big\}d\bar{x}$$

for all $f\in L(G)$. We have

$$\int_{A/\varDelta}d\bar{x}=1\,. \tag{5.2}$$

The Fourier transform of a function φ in $L_1(A)$ is denoted by $\hat{\varphi}$:

$$\hat{\varphi}(y)=\int_A \varphi(x)\chi(xy)dx \qquad \text{for} \quad y\in A\,.$$

By the above definition, we have $\hat{\hat{\varphi}}(x)=\varphi(-x)$, if φ and $\hat{\varphi}$ are in $L_1(A)$. Let φ_p be a function on A_p satisfying the conditions

i) $\varphi_p,\ \hat{\varphi}_p \in C(A_p)\cap L_1(A_p)$ for all p,

ii) $\varphi_p=\chi_{O_p}$ for almost all p.

We put $\varphi=\prod_p \varphi_p$. Then, the function φ belongs to $C(A)\cap L_1(A)$, and we have

$$\hat{\varphi}_p=\chi_{O_p} \quad \text{for almost all } p, \quad \hat{\varphi}=\prod_p \hat{\varphi}_p\,. \tag{5.3}$$

Let $x=(x_p)$ be an element of A. Then, $\det x_p \in \boldsymbol{Z}_p$ for almost all p. So the element $(\det x_p)$ of $\prod_p \boldsymbol{Q}_p$ is in the adele ring of $\boldsymbol{Q}$. We write

$$\det x=(\det x_p)\,.$$

The totality of invertible elements in A is denoted by G, on which we introduce the weakest topology, such that the mappings $G\ni x\to x\in A$ and $G\ni x\to x^{-1}\in A$ are both continuous. Then, G is equal to the restricted direct product of G_p with respect to U_p. G is called the idele group of $\varDelta=M_n(\boldsymbol{Q})$. By the canonical injection, $\Gamma=GL(n,\boldsymbol{Q})$ may be considered as a discrete subgroup of G. Put $U=\prod_p U_p$, then it is a maximal compact subgroup of G. Let Z be the centre of G. It is equal to the restricted direct product of Z_p with respect to $Z_p\cap U_p$. We denote by G^∞ the set of all elements $g=(g_p)$ of G satisfying $g_p\in U_p$ for all $p\neq\infty$. It is an open subgroup of G.

An element x of A belongs to G, if and only if $\det x$ is in the idele group of $\boldsymbol{Q}$. For an element $g=(g_p)$ of G, we put

$$\|\det g\|=\prod_p |\det g_p|_p\,.$$

We have $\|\det u\|=1$ for $u\in U$, and $\|\det\gamma\|=1$ for $\gamma\in\Gamma$. Furthermore, we have

$$d(gx)=d(xg)=\|\det g\|^n dx$$

for $g\in G$.

We denote by du_p the Haar measure on U_p, normalized as in § 1. We call du the direct product of du_p. Of course, the total measure of U is equal to 1. Let dg_p be the Haar measure on G_p, normalized as in § 1. We write dg the restricted direct product of dg_p. There exists on the homogeneous space G/Γ the canonical invariant measure $d\bar{g}$, such that the relation

$$\int_G f(g)dg=\int_{G/\Gamma}\Big\{\sum_{\gamma\in\Gamma} f(g\gamma)\Big\}d\bar{g}\,.$$

holds for all $f\in L(G)$. Let $L(G,U)$ be the set of all functions φ in $L(G)$ such that $\varphi(ugu')=\varphi(g)$ for all $u, u' \in U$ and $g \in G$. For φ in $L(G, U)$ and f in $C(G)$, we define the convolution $\varphi * f$ by

$$(\varphi * f)(g)=\int_G \varphi(gh^{-1})f(h)dh, \qquad g \in G.$$

We define a multiplication in $L(G, U)$ by the convolution, then it becomes a ring. It is known that the ring $L(G, U)$ is commutative.

We denote by Ω the set of all spherical functions on G relative to U, and by Ω^+ the totality of positive-definite spherical functions. We have $|\omega(g)| \leq 1$, $\overline{\omega(g)}=\omega(g^{-1})$ for $\omega \in \Omega^+$. We denote further by $\tilde{\Omega}$ the set of all ω in Ω such that $\omega(\zeta g)=\omega(g)$ for all $\zeta \in Z$ and $g \in G$. Every spherical function ω on G can be written uniquely in the form $\omega=\prod_p \omega_p$, $\omega_p \in \Omega_p$. Conversely, if ω_p is a spherical function on G_p for every p, then the function $\omega=\prod_p \omega_p$ on G is spherical. Moreover, a spherical function $\omega=\prod_p \omega_p$ belongs to Ω^+ (resp. $\tilde{\Omega}$), if and only if ω_p belongs to Ω_p^+ (resp. $\tilde{\Omega}_p$) for all p.

A function f in $C(G)$ will be called an automorphic function with respect to Γ, if the following conditions are satisfied:

i) $f(ug\gamma)=f(g)$ for all $u \in U$, $g \in G$, $\gamma \in \Gamma$,

ii) to every $\varphi \in L(G, U)$, corresponds a complex number λ_φ satisfying the relation $\varphi * f=\lambda_\varphi f$.

For a non-zero automorphic function f, there exists a unique spherical function ω, satisfying the condition

$$\int_U f(gug')du=\omega(g)f(g') \quad \text{for all } g, g' \in G.$$

Then, we say that f belongs to ω. We consider a spherical function in $\tilde{\Omega}^+=\Omega^+ \cap \tilde{\Omega}$ to which a non-zero automorphic function belongs. The set of all such spherical functions is called the spectrum of Γ in $\tilde{\Omega}^+$, and is denoted by $s(\Gamma)$.

If f is a non-zero automorphic function belonging to ω in the spectrum, then there exists an element h of G, such that

$$f(h) \neq 0, \qquad \| \det h \| = 1. \tag{5.4}$$

In fact, we have for all $\zeta \in Z$ and $g \in G$,

$$f(\zeta g)=f(u\zeta g)=f(\zeta ug)=\int_U f(\zeta ug)du=\omega(\zeta)f(g)=f(g).$$

On the other hand, there exists an element $g \in G$ such that $f(g) \neq 0$. We put $\zeta_p=1_n$ $(p \neq \infty)$, $\zeta_\infty=\| \det g \|^{-\frac{1}{n}} 1_n$, $\zeta=(\zeta_p)$ and $h=\zeta g$. Then, we have $f(h)=f(g) \neq 0$, $\| \det h \|=\| \det \zeta \| \times \| \det g \|=1$.

A function ϕ on A is called of type Z, if the following conditions are satisfied:

(Z1) $\phi, \hat{\phi} \in C(A) \cap L_1(A)$,

(Z2) $\phi(uxv) = \phi(x)$, $\hat{\phi}(uxv) = \hat{\phi}(x)$ for all $u, v \in U$, $x \in A$,

(Z3) there exists a real number σ_0, such that

$$\int_G |\phi(g)| \| \det g \|^\sigma | dg < \infty, \quad \int_G |\hat{\phi}(g)| \| \det g \|^\sigma | dg < \infty \quad \text{for } \sigma > \sigma_0$$

(Z4) $\sum_{\xi \in \Delta} \phi(g(x+\xi)h)$, $\sum_{\xi \in \Delta} \hat{\phi}(g(x+\xi)h)$ converge absolutely and uniformly on any compact subset of elements (g, x, h) in $G \times A \times G$,

(Z5) $\phi(x) = \hat{\phi}(x) = 0$ for every element x of A such that $\det x = 0$.

We see that ϕ is of type Z if and only if $\hat{\phi}$ is of type Z.

For a function ϕ of type Z and a spherical function ω in $s(\Gamma)$, we define a global ζ-function by the integral

$$\zeta_\phi(z, \omega) = \int_G \phi(g) \omega(g^{-1}) \| \det g \|^z dg .$$

We have $|\omega(g)| \leqq 1$ for all $g \in G$, so by (Z3), the above integral converges for $\operatorname{Re} z > \sigma_0$.

PROPOSITION 12. *For every ϕ of type Z and every ω in $s(\Gamma)$, the function $\zeta_\phi(z, \omega)$ is continued to an entire function. It satisfies the functional equation*

$$\zeta_\phi(z, \omega) = \zeta_{\hat{\phi}}(n-z, \bar{\omega}) .$$

PROOF. The "theta-formula"

$$\sum_{\gamma \in \Gamma} \phi(h^{-1}\gamma g) = \| \det hg^{-1} \|^n \sum_{\gamma \in \Gamma} \hat{\phi}(g^{-1}\gamma h), \quad g, h \in G \tag{5.5}$$

holds for every ϕ of type Z. Indeed, by the formulas $d(gx) = d(xg) = \|\det g\|^n dx$, we see easily that the function $L_h R_g \phi$ is in $C(A) \cap L_1(A)$. The Fourier transform of $L_h R_g \phi$ is equal to $\| \det hg^{-1} \|^n L_g R_h \hat{\phi}$, by the following calculation:

$$\begin{aligned} \int_A \phi(h^{-1}xg)\chi(xy)dx &= \| \det hg^{-1} \|^n \int_A \phi(x)\chi(hxg^{-1}y)dx \\ &= \| \det hg^{-1} \|^n \int_A \phi(x)\chi(xg^{-1}yh)dx \\ &= \| \det hg^{-1} \|^n \hat{\phi}(g^{-1}yh) . \end{aligned}$$

Therefore, from (Z4) and (5.2), we get by the Poisson formula

$$\sum_{\xi \in \Delta} \phi(h^{-1}\xi g) = \| \det hg^{-1} \|^n \sum_{\xi \in \Delta} \hat{\phi}(g^{-1}\xi h) .$$

If $\xi \notin \Gamma$, then $\det(h^{-1}\xi g) = \det(g^{-1}\xi h) = 0$, hence by (Z5) $\phi(h^{-1}\xi g) = \hat{\phi}(g^{-1}\xi h) = 0$. So we obtain the "theta-formula" (5.5).

Now, for a non-zero automorphic function f belonging to $\omega \in s(\Gamma)$, there exists an element $h \in G$ such that $f(h) \neq 0$, $\| \det h \| = 1$ (cf. (5.4)). By the use of this element h, we have for $\operatorname{Re} z > \sigma_0$

$$f(h)\zeta_\phi(z, \omega) = \int_G \phi(g)\omega(g^{-1})f(h) \| \det g \|^z dg$$

$$= \int_U \Big(\int_G \phi(g) f(g^{-1}uh) \| \det g \|^z dg \Big) du$$

$$= \int_U \Big(\int_G \phi(hg^{-1}) f(g) \| \det g \|^{-z} dg \Big) du \quad (g \to uhg^{-1})$$

$$= \int_G \phi(hg^{-1}) f(g) \| \det g \|^{-z} dg$$

$$= \int_{\| \det g \| \leqq 1} + \int_{\| \det g \| \geqq 1} .$$

We transform the two integrals of the right-hand side as follows:

$$\text{the first integral} = \int_{\| \det g \| \leqq 1} \phi(hg^{-1}) f(g) \| \det g \|^{-z} dg$$

$$= \int_{\| \det g \| \geqq 1} \phi(g) f(g^{-1}h) \| \det g \|^{z} dg \,,$$

$$\text{the second integral} = \int_{\| \det g \| \geqq 1} \phi(hg^{-1}) f(g) \| \det g \|^{-z} dg$$

$$= \int_{G/\Gamma, \| \det g \| \geqq 1} \sum_{\gamma \in \Gamma} \phi(h\gamma g^{-1}) f(g) \| \det g \|^{-z} d\bar{g}$$

$$= \int_{G/\Gamma, \| \det g \| \geqq 1} \sum_{\gamma \in \Gamma} \hat{\phi}(g\gamma h^{-1}) f(g) \| \det g \|^{n-z} d\bar{g}$$

$$= \int_{\| \det g \| \geqq 1} \hat{\phi}(gh^{-1}) f(g) \| \det g \|^{n-z} dg$$

$$= \int_{\| \det g \| \geqq 1} \hat{\phi}(g) f(gh) \| \det g \|^{n-z} dg \,.$$

We applied the "theta-formula" (5.5) in the transformations of the second integral. Consequently, we have for $\operatorname{Re} z > \sigma_0$

$$f(h)\zeta_\phi(z, \omega) = \int_{\| \det g \| \geqq 1} \phi(g) f(g^{-1}h) \| \det g \|^{z} dg \tag{5.6}$$

$$+ \int_{\| \det g \| \geqq 1} \hat{\phi}(g) f(g\,h) \| \det g \|^{n-z} dg \,.$$

Similarly, considering that $\hat{\hat{\phi}}(x) = \phi(-x) = \phi(x)$ and that $\overline{\omega(g)} = \omega(g^{-1})$, we have $f(h)\zeta\hat{\varphi}(z, \bar{\omega})=\int_G \hat{\phi}(gh^{-1})f(g)\,\|\det g\|^z dg$ and

$$f(h)\zeta\hat{\varphi}(z, \bar{\omega})=\int_{\|\det g\|\geqq 1}\phi(g)f(g^{-1}h)\,\|\det g\|^{n-z}dg + \int_{\|\det g\|\geqq 1}\hat{\phi}(g)f(gh)\,\|\det g\|^z dg \tag{5.7}$$

for $\mathrm{Re}\, z>\sigma_0$.

The first integral of (5.6) and the second integral of (5.7) converge for $\mathrm{Re}\, z>\sigma_0$. Since both integrals extend over $\|\det g\|\geqq 1$, they converge for all z. From this follows the convergence for all z of the second integral of (5.6) and of the first integral of (5.7). This means that the functions $\zeta\varphi(z, \omega)$ and $\zeta\hat{\varphi}(z, \bar{\omega})$ are continued to entire functions. The functional equation readily follows from (5.6) and (5.7).

§ 6. ζ-function of $\mathbf{M}_n(\boldsymbol{Q})$

With every p, we associate a spherical function on G_p:

$$\omega_p=\omega_{s_1(p),\cdots,s_n(p)} \quad (s_1(p), \cdots, s_n(p)\in \boldsymbol{C}).$$

Let $\zeta_p(z, \omega_p)$ be the local ζ-function with the weight function ϕ_p (cf. (1.1)). We consider the spherical function $\omega=\prod_p \omega_p$ on G.

THEOREM. *We assume that the spherical function ω is in the spectrum $s(\Gamma)$. Then, the infinite product*

$$\zeta(z, \omega)=\prod_p \zeta_p(z, \omega_p)$$

converges absolutely for $\mathrm{Re}\, z>n$. *And the function*

$$\zeta(z, \omega)\prod_{i=1}^{n}(z-s_i(\infty))(z-s_i(\infty)-1)$$

is continued to an entire function. The meromorphic function $\zeta(z, \omega)$ on the whole z-plane satisfies the functional equation

$$\zeta(z, \omega)=\zeta(n-z, \bar{\omega}).$$

REMARK. Since ω is in $s(\Gamma)$, we have $\omega_p\in\tilde{\Omega}_p$. So the relation

$$\sum_{i=1}^{n} s_i(p)\equiv\frac{n(n-1)}{2} \quad \left(\mathrm{mod.}\ \frac{2\pi\sqrt{-1}}{\log p}\right)$$

holds by (1.4). From this and Proposition 1, we have

$$\zeta_p(z, \omega_p) = \begin{cases} \prod_{i=1}^{n} (1-p^{s_i(p)}p^{-z})^{-1} & (p \neq \infty), \\ \pi^{-\frac{n}{2}z+\frac{n(n-1)}{4}} \prod_{i=1}^{n} \Gamma\left(\frac{z-s_i(\infty)}{2}\right) & (p=\infty). \end{cases}$$

Before the proof of the theorem, we need some preliminaries. We define a function ψ_∞ on A_∞ as in Corollary of Proposition 10. We put

$$\psi_p = \phi_p \ (p \neq \infty), \ \psi = \prod_p \psi_p.$$

PROPOSITION 13. *The function ψ on A is of type Z with $\sigma_0 = n$, and is self-reciprocal.*

PROOF. We have

$$\psi_p \in C(A_p) \cap L_1(A_p), \qquad \hat{\psi}_p = \psi_p$$

for all p. It is obvious for $p \neq \infty$; and for $p = \infty$, it follows from Proposition 7, i) and Corollary of Proposition 10. Hence, we have by (5.3)

$$\psi \in C(A) \cap L_1(A), \qquad \hat{\psi} = \psi,$$

which mean (Z1). We can easily verify (Z2) and (Z5). Furthermore, we have

$$\omega_{0,1,\cdots,n-1} = 1,$$

hence we have by Proposition 1 and Proposition 7, ii)

$$\int_{G_p} |\psi_p(g_p)| \, |\det g_p|_p^\sigma| \, dg_p = \int_{G_p} |\psi_p(g_p)| \, \omega_{0,1,\cdots,n-1}(g_p^{-1}) \, |\det g_p|_p^\sigma dg_p$$

$$\begin{cases} = \prod_{i=1}^{n} (1-p^{i-1}p^{-\sigma})^{-1} & (p \neq \infty), \\ \leqq \sum_{s=0}^{n} \frac{(s+2)!}{2} \frac{1}{(2\pi)^s} \int_{G_\infty} c_0(g_\infty) c_s(g_\infty) \phi_\infty(g_\infty) & \\ \qquad \times \omega_{0,1,\cdots,n-1}(g_\infty^{-1}) \, |\det g_\infty|_\infty^\sigma \, dg_\infty < \infty & (p=\infty) \end{cases}$$

for $\sigma > n-1$. Hence,

$$(6.1) \quad \begin{cases} \int_{G_p} |\psi_p(g_p)| \, |\det g_p|_p^\sigma| \, dg_p < \infty & \text{for } \sigma > n-1, \\ \prod_{p\neq\infty} \left(\int_{G_p} |\psi_p(g_p)| \, |\det g_p|_p^\sigma| \, dg_p\right) = \prod_{i=1}^{n} \zeta(\sigma-(i-1)) & \text{for } \sigma > n. \end{cases}$$

Applying (5.1) to these formulas and to the fact

$$\psi_p(u_p) \, |\det u_p|_p^\sigma = 1 \quad \text{for all } u_p \in U_p, \ p \neq \infty,$$

we obtain

$$\int_G |\psi(g)| \, \|\det g\|^\sigma| \, dg < \infty \qquad \text{for } \sigma > n.$$

This is the condition (Z3) in the case $\sigma_0=n$.

Let $a=(a_p)$ and $b=(b_p)$ be elements of G. We denote by $a_p(i,j)$ (resp. $b_p(i,j)$) the (i,j) element of a_p (resp. b_p). Since we have $a_p, b_p\in U_p$ for almost all p, we can associate with every $p\neq\infty$ a non-negative integer $n_p(a,b)$ satisfying the conditions

i) $n_p(a,b)=0$ for almost all $p\neq\infty$,

ii) $|a_p(i,j)|_p,\ |b_p(i,j)|_p\leqq p^{n_p(a,b)}$ for all $p\neq\infty$.

Hence, for $y_p=(y_p(i,j))\in a_pO_pb_p$, we have

$$|y_p(i,j)|_p\leqq p^{2n_p(a,b)}.$$

We denote by $\mathfrak{a}(a,b)$ the ideal of $\boldsymbol{Q}$ generated by the rational number $\prod_{p\neq\infty}p^{-2n_p(a,b)}$. Then, considering that $\phi_p=\chi_{O_p}$ for all $p\neq\infty$, the following inferences hold: For $x=(x_p)\in A^\infty$, $g=(g_p)\in G^\infty a^{-1}$, $h=(h_p)\in b^{-1}G^\infty$ and $\xi=(\xi_{ij})\in\Delta$,

$$\begin{aligned}&\phi_p(g_p(x_p+\xi)h_p)=1 && \text{for all } p\neq\infty\\ &\Leftrightarrow x_p+\xi\in g_p^{-1}O_ph_p^{-1}=a_pO_pb_p && \text{for all } p\neq\infty\\ &\Rightarrow|\xi_{ij}|_p\leqq p^{2n_p(a,b)} && \text{for all } p\neq\infty\\ &\Rightarrow\xi\in M_n(\mathfrak{a}(a,b)).\end{aligned}$$

From them, we get

$$\sum_{\xi\in\Delta}|\phi(g(x+\xi)h)|\leqq\sum_{\xi\in M_n(\mathfrak{a}(a,b))}|\phi_\infty(g_\infty(x_\infty+\xi)h_\infty)| \tag{6.2}$$

for $g\in G^\infty a^{-1}$, $h\in b^{-1}G^\infty$ and $x\in A^\infty$. We define a function λ on A_∞ by

$$\lambda(x_\infty)=\exp(-\sum_{i,j}|x_{ij}|),\quad x_\infty=(x_{ij})\in A_\infty.$$

Then, there exists a constant $K>0$, such that

$$|\phi_\infty(x_\infty)|\leqq K\lambda(x_\infty) \tag{6.3}$$

for sufficiently large $|x_{ij}|$. Let $\mathfrak{a}$ be an ideal of $\boldsymbol{Q}$. We see easily that the series

$$\sum_{\xi\in M_n(\mathfrak{a})}\lambda(g_\infty(x_\infty+\xi)h_\infty)$$

converges uniformly on any compact subset of elements $(g_\infty, x_\infty, h_\infty)$ in $G_\infty\times A_\infty\times G_\infty$. Therefore, (Z4) follows from (6.2), (6.3) and the relation $A=\Delta+A^\infty$.

PROOF OF THEOREM. By proposition 13 and Proposition 12, the global ζ-function $\zeta_\phi(z,\omega)$ has the following properties:

$$\zeta_\phi(z,\omega)=\int_G\phi(g)\omega(g^{-1})\|\det g\|^z dg\qquad\text{for }\operatorname{Re}z>n, \tag{6.4}$$

$$\zeta_\phi(z,\omega)=\zeta_{\hat\phi}(n-z,\bar\omega). \tag{6.5}$$

On the other hand, we have

$$(6.6)\quad \begin{cases} \psi_p(u_p)\omega_p(u_p^{-1})\,|\det u_p|_p^z=1 & \text{for } u_p\in U_p,\ p\neq\infty\,,\\ \int_{G_p}|\psi_p(g_p)w_p(g_p^{-1})\,|\det g_p|_p^z\,|\,dg_p<\infty & \text{for } \operatorname{Re} z>n-1\,,\\ \prod_p\int_{G_p}|\psi_p(g_p)\omega_p(g_p^{-1})\,|\det g_p|_p^z\,|\,dg_p<\infty & \text{for } \operatorname{Re} z>n\,, \end{cases}$$

and

$$(6.7)\quad \int_{G_p}\psi_p(g_p)\omega_p(g_p^{-1})\,|\det g_p|_p^z dg_p = \begin{cases} \zeta_p(z,\omega_p) & (p\neq\infty)\,,\\ \frac{1}{(2\pi)^{2n}}\zeta_\infty(z,\omega_\infty)\prod_{i=1}^n(z-s_i(\infty))(z-s_i(\infty)-1) & (p=\infty) \end{cases}$$

for $\operatorname{Re} z>n-1$. Indeed, since $|\omega_p(g_p)|\leqq 1$ for all element g_p of G_p, the formulas (6.6) follow from (6.1). The formula (6.7) holds for sufficiently large $\operatorname{Re} z$ by Proposition 1 and Proposition 11. The left-hand side integral of (6.7) converges for $\operatorname{Re} z>n-1$ by (6.6), so the formula (6.7) holds for $\operatorname{Re} z>n-1$.

Therefore, we have by (6.4), (6.6), (5.1) and (6.7)

$$\zeta_\psi(z,\omega)=\prod_p\int_{G_p}\psi_p(g_p)\omega_p(g_p^{-1})\,|\det g_p|_p^z\,dg_p = \frac{1}{(2\pi)^{2n}}\prod_p\zeta_p(z,\omega_p)\prod_{i=1}^n(z-s_i(\infty))(z-s_i(\infty)-1)$$

for $\operatorname{Re} z>n$. And the infinite product

$$\zeta(z,\omega)=\prod_p\zeta_p(z,\omega_p)$$

converges absolutely for $\operatorname{Re} z>n$. Hence, we see that the function

$$\zeta(z,\omega)\prod_{i=1}^n(z-s_i(\infty))(z-s_i(\infty)-1)$$

is continued to an entire function. We write (6.5) in the form

$$\zeta(z,\omega)\prod_{i=1}^n(z-s_i(\infty))(z-s_i(\infty)-1) = \zeta(n-z,\bar\omega)\prod_{i=1}^n(n-z-\overline{s_i(\infty)})(n-z-\overline{s_i(\infty)}-1)\,.$$

Considering (1.5), we have

$$\prod_{i=1}^n(z-s_i(\infty))(z-s_i(\infty)-1)=\prod_{i=1}^n(n-z-\overline{s_i(\infty)})(n-z-\overline{s_i(\infty)}-1)\,.$$

Hence, we obtain the result

$$\zeta(z, \omega)=\zeta(n-z, \bar{\omega}) .$$

Waseda University

References

[1] M. Eichler, Allgemeine Kongruenzklasseneinteilungen der Idealen einfacher Algebren über algebraischen Zahlkörpern und ihrer L-Reihen, J. Reine Angew. Math., **179** (1938), 227-251.

[2] G. Fujisaki, On the zeta-functions of the simple algebra over the field of rational numbers, J. Fac. Sci. Univ. Tokyo. Sect. I, **7** (1958), 567-604.

[3] G. Fujisaki, On the L-functions of the simple algebra over the field of rational numbers, J. Fac. Sci. Univ. Tokyo. Sect. I, **9** (1962), 293-311.

[4] R. Godement, Les fonctions ζ des algèbres simples, I, II, Séminaire Bourbaki, 1958/1959, Exposée, p. 171, 176.

[5] Harish-Chandra, Spherical functions on a semi-simple Lie group, I, II, Amer. J. Math., **80** (1958), 241-310, 553-613.

[6] E. Hecke, Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, I, II, Math. Ann., **114** (1937), 1-28, 316-351.

[7] K. Hey, Analytische Zahlentheorie in System hyperkomplexer Zahlen, Diss., Hamburg, 1929.

[8] K. Iwasawa, A note on L-functions, Proc. Int. Congress. Math. Cambr. Mass. U. S. A., 1950, p. 322.

[9] H. Maass, Über eine neue Art von nicht analytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichung, Math. Ann., **121** (1949), 141-183.

[10] H. Maass, Zetafunktionen mit Grössencharakteren und Kugelfunktionen, Math. Ann., **134** (1957), 1-32.

[11] I. Satake, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields, Publications Mathematiques, No. 18, 1963.

[12] G. Shimura, On the Dirichlet series and abelian varieties attached to automorphic forms, Ann. of Math., **72** (1962), 237-294.

[13] T. Tamagawa, On Selberg's trace formula, J. Fac. Sci. Univ. Tokyo Sect. I, **8** (1960), 363-386.

[14] T. Tamagawa, On the ζ-functions of a division algebra, Ann. of Math., **77** (1963), 387-405.

[15] J. Tate, Fourier analysis in number fields and Hecke's zeta-functions, Thesis, Princeton Univ., 1950.

[16] H. Weyl, Classical groups, Princeton Univ. Press, 1939.