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Hardy-Littlewood majorants in function spaces

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1. Throughout this paper, we term $\boldsymbol{X}$ a *Banach function space*[1], if $\boldsymbol{X}$ is a *normed linear space of integrable functions over the interval* (0, 1) *satisfying*

(i) $|g| \leqq |f|$[2], $f \in \boldsymbol{X}$ *implies* $g \in \boldsymbol{X}$ *and* $\|g\| \leqq \|f\|$;

(ii) $0 \leqq f_n \uparrow_{n=1}^{\infty} f$ *implies* $\sup_{n \geqq 1} \|f_n\| = \|f\|$;

(iii) $0 \leqq f_n \uparrow_{n=1}^{\infty}$ *with* $\sup_{n \geqq 1} \|f_n\| < +\infty$ *implies* $\bigcup_{n=1}^{\infty} f_n \in \boldsymbol{X}$[3].

We shall call the rorm fulfiling (i) and (ii) to be *semi-continuous*. $\boldsymbol{X}$ is said to have the *Rearrangement Invariant property*[4] (or shortly *RIP*), if each function g equimeasurable to a function $f \in \boldsymbol{X}$ also belongs to $\boldsymbol{X}$ and $\|g\| = \|f\|$.

Let f be an integrable function on (0, 1). The *Hardy-Littlewood majorant* $\theta(f)$ of f is the function defined by

$$\theta(f)(x) = \sup_{0 \leqq y \leqq 1} \int_x^y \frac{f(t)}{y-x} dt \quad (x \in (0, 1)), \tag{1}$$

provided it exists almost everywhere. G. H. Hardy and J. E. Littlewood have shown that if $f \in \boldsymbol{L}^p (1 < p)$, then $\theta(f)$ is defined and belongs to $\boldsymbol{L}^p$ also [**9**]. Here, in accordance with G. Lorentz [**3**], we shall say that $\boldsymbol{X}$ has the *Hardy-Littlewood property*, and shall denote by $\boldsymbol{X} \in HLP$, if $f \in \boldsymbol{X}$ implies $\theta(f) \in \boldsymbol{X}$. In his paper cited above, G. Lorentz discussed this property for Banach function spaces having *RIP*[5], and presented necessary and sufficient conditions in order that $\boldsymbol{X} \in HLP$, in case $\boldsymbol{X}$ is an Orlicz space $\boldsymbol{L}_{\Phi}$ or a space $\Lambda(\phi)$.

The aim of this note is to give a necessary and sufficient condition in order that a general Banach function space $\boldsymbol{X}$ with *RIP* have the Hardy-Little-

1) Here we deal with Banach spaces consisting of real functions. For an exposition of Banach function spaces see [**4**].

2) $|g| \leqq |f|$ means that $g(t) \leqq f(t)$ holds almost everywhere in (0, 1).

3) A norm satisfying (iii) is called monotone complete. If a norm is monotone complete, it is complete.

4) On account of Theorem 3 in [**8**], we may replace this condition by the weak rearrangement invariant property (this requires only $g \in X$, if g is equimeasurable to an $f \in X$) throughout this paper.

5) In his paper Banach function spaces are introduced in terms of Köthe spaces.

wood property (Theorem 1). As a consequence, it shall be shown that the results of [3] in case of $\boldsymbol{X}=\boldsymbol{L}_{\Phi}$ or $\boldsymbol{X}=\Lambda(\phi)$, which are simplified so as to bear directly on Φ or ϕ, can be derived easily from this condition.

Finally we shall establish a generalization of the Hardy-Littlewood property for Banach function spaces consisting of integrable functions on a finite measure space (E, Ω, μ).

2. In the sequel, let $(\boldsymbol{X}, \|\cdot\|)$ be always a Banach function space consisting of integrable functions over $(0, 1)$ which has *RIP*. We shall denote by f^* the decreasing rearrangement of $|f|$, and by $\bar{f}$ the function defined by

$$\bar{f}(x)=\int_0^x \frac{f(t)}{x}dt \qquad (x\in(0,1)). \tag{2}$$

It is clear that $\theta(f)=\bar{f}$, if f is positive decreasing.

LEMMA 1 (Lorentz [3]). *$\boldsymbol{X}\in HLP$ if and only if for every positive decreasing $f\in\boldsymbol{X}$, $\bar{f}$ belongs also to $\boldsymbol{X}$. Furthermore, there exists a constant $K>0$ such that*

$$\|\theta(f)\|\leqq K\|f\| \qquad \textit{for all } f\in\boldsymbol{X} \tag{3}$$

holds in this case.

Here we note that the latter part of the lemma can be proved directly as follows. If $\boldsymbol{X}\in HLP$, the functional $\rho: f\to\rho(f)=\|\theta(f)\|$ $(f\in\boldsymbol{X})$ satisfies i) $\rho(\alpha f)=\alpha\rho(f)$ for all positive number α; ii) $0\leqq f\leqq g$ implies $\rho(f)\leqq\rho(g)$. On account of the condition (ii) in **1** and the relation $|\theta(f)|\leqq\theta(|f|)$, we need only to show that (3) holds for all positive $f\in\boldsymbol{X}$. Now suppose that there exists no positive number satisfying (3) for all positive $f\in\boldsymbol{X}$. We can then find a sequence of positive functions $\{f_n\}_{n=1}^{\infty}$ of $\boldsymbol{X}$ such that $\rho(f_n)\geqq n$ with $\|f_n\|=1/2^n$ $(n=1, 2, \cdots)$. Since $\boldsymbol{X}$ is complete, $\sum_{n=1}^{\infty} f_n=f_0\in\boldsymbol{X}$ and $0\leqq f_n\leqq f_0$ holds for each n. Thus we obtain $\|\theta(f_0)\|=\rho(f_0)\geqq\rho(f_n)\geqq n$ $(n=1, 2, \cdots)$, which is a contradiction.

For any $f\in\boldsymbol{X}$ and $0<\alpha\leqq 1$, let $f_{(\alpha)}$ denote the function

$$f_{(\alpha)}(x)=f(\alpha x) \qquad (x\in(0,1)). \tag{4}$$

If f is positive decreasing, we have for any $\alpha, \beta (0<\alpha, \beta\leqq 1)$ and $\xi\geqq 0$

$$f_{(\alpha)}\geqq f_{(\beta)}, \quad \text{if} \quad \alpha\leqq\beta; \tag{5}$$

$$\{f_{(\alpha)}\}_{(\beta)}=f_{(\alpha\cdot\beta)}, \tag{6}$$

$$\{\xi f\}_{(\alpha)}=\xi\{f_{(\alpha)}\}. \tag{7}$$

The following sufficient condition for $\boldsymbol{X}\in HLP$ was given in [3].

LEMMA 2. *For $\boldsymbol{X}\in HLP$ it is sufficient that for some constant K and for all $f\in\boldsymbol{X}$,*

$$\int_0^1 \|f_{(\alpha)}\|\, d\alpha \leqq K\|f\| . \tag{8}$$

We shall show below that (8) is, in fact, a necessary condition for *HLP* at the same time (Corollary 1).

We write $f \prec g$, if $\int_0^x f^*(t)dt \leqq \int_0^x g^*(t)dt$ holds for every $x \in (0, 1)$. Since $\boldsymbol{X}$ has *RIP*, $f \prec g$ implies $\|f\| \leqq \|g\|$. Also we write $f \sim g$, whenever f is equimeasurable to g. Then it follows easily that $f^* = g^*$ if $f \sim g$, and that $f \prec g$, $f \sim f'$, and $g \sim g'$ imply $f' \prec g'$. Let $\boldsymbol{X}'$ denote the *conjugate space of* $\boldsymbol{X}$, i.e. the totality of measurable functions g for which $\|g\|' = \sup_{\|f\| \leqq 1, f \in X} |\langle f, g\rangle| < \infty$, where $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ ($f \in \boldsymbol{X}, g \in \boldsymbol{X}'$). $\boldsymbol{X}'$ is also a Banach function space having *RIP*, and can be considered to be included in $\boldsymbol{X}^*$, the Banach dual of $\boldsymbol{X}$. It is well known that $\|\cdot\|$ on $\boldsymbol{X}$ is reflexive, i.e. [**4, 6**]

$$\|f\| = \sup_{\|g\|' \leqq 1, g \in X'} |\langle f, g\rangle| \qquad (f \in \boldsymbol{X}),$$

and $|\langle f, g\rangle| \leqq \langle f^*, g^*\rangle$ holds for each $f \in \boldsymbol{X}$ and $g \in \boldsymbol{X}'$. For any measurable set $\mathbf{e} \subset (0, 1)$, we define a linear operator $A_{\mathbf{e}}$ by the formula:

$$A_{\mathbf{e}} f = \left(\frac{1}{d(\mathbf{e})} \int_{\mathbf{e}} f(x)dx\right) \chi_{\mathbf{e}} , \tag{9}$$

where $\chi_{\mathbf{e}}$ is the characteristic function of the set $\mathbf{e}$ and $d(\mathbf{e})$ denotes the Lebesgue measure of $\mathbf{e}$. Since $A_{\mathbf{e}} f \prec f$ for any positive $f \in \boldsymbol{X}$, $\|A_{\mathbf{e}} f\| \leqq \|f\|$ holds. Furthermore, we have

LEMMA 3. *If f_i ($i = 0, 1, 2$) are all positive decreasing functions belonging to $\boldsymbol{X}$ and $f_0 = f_1 + f_2$. Then*

$$\|f_0\| \geqq \|A_{(0,\xi)} f_1 + f_2\| \tag{10}$$

holds for any $0 < \xi \leqq 1$.

PROOF. For any positive decreasing $c \in \boldsymbol{X}'$ with $\|c\|' = 1$, one obtains

$$\langle f_0, c\rangle = \langle f_1, c\rangle + \langle f_2, c\rangle \geqq \langle A_{(0,\xi)} f_1, c\rangle + \langle f_2, c\rangle = \langle A_{(0,\xi)} f_1 + f_2, c\rangle ,$$

because of $f_1 \succ A_{(0,\xi)} f_1$. Hence it follows that $\|f_0\| \geqq \|A_{(0,\xi)} f_1 + f_2\|$, since the norm $\|\cdot\|$ is reflexive, and both f_0 and $A_{(0,\xi)} f_1 + f_2$ are positive decreasing.

We can now prove our main result:

THEOREM 1. *$X \in HLP$ if and only if there exist positive numbers K and P ($0 < p < 1$) such that*

$$\|f_{(\alpha)}\| \leqq K\alpha^{-p} \|f\| \tag{*}$$

holds for every $\alpha \in (0, 1]$.

Since the function $g(\alpha) = \alpha^{-p}$ is integrable over $(0, 1)$ for any p with $0 < p < 1$, the sufficiency of the theorem is obvious by virtue of Lemma 2.

Therefore, we merely need to prove the necessity. First we shall prove the following lemma.

LEMMA 4. *If there exists an* $\varepsilon_0 (0<\varepsilon_0<1)$ *such that*

$$(\sharp) \qquad \|f\| \leqq 1 \quad \text{and} \quad n \leqq 2\varepsilon_0^{-1} \quad \text{imply} \quad \|f_{(n^{-1})}\| \leqq \varepsilon_0^{-1},$$

then $(*)$ *holds for some* p *with* $0<p<1$.

PROOF. We may assume that ε_0 satisfying $(\sharp)$ is taken so that $m = 2\varepsilon_0^{-1}$ is an integer. For any natural number ν and $f \in \boldsymbol{X}$ with $\|f\| \leqq 1$, one obtains that

$$(11) \qquad n \leqq m^{\nu} \quad \text{implies} \quad \|f_{(n^{-1})}\| \leqq \varepsilon_0^{-\nu}.$$

In fact, for $\nu=1$, this is valid on account of $(\sharp)$. Now suppose that this holds true for $\nu=k-1$. Then we have $\|f_{(m^{-\nu+1})}\| \leqq \varepsilon_0^{-\nu+1}$ for every f with $\|f\|=1$. But $\|\varepsilon_0^{\nu-1} f_{(m^{-\nu+1})}\| \leqq 1$ implies $\varepsilon_0^{-1} \geqq \|(\varepsilon_0^{\nu-1} f_{(m^{-\nu+1})})_{(m^{-1})}\| = \varepsilon_0^{\nu-1}\|f_{(m^{-\nu})}\|$ on account of (6) and (7). Consequently, we find $\|f_{(m^{-\nu})}\| \leqq \varepsilon_0^{-\nu}$, and (11) is proved by virtue of (5). Now let p be a positive number satisfying $(\varepsilon_0/2)^p = \varepsilon_0$. One sees that $0<p<1$ and $\|f\| \leqq 1$ implies $\|f_{(m^{-\nu})}\| \leqq \varepsilon_0^{-\nu} = m^{p\nu}$ for every $\nu \geqq 1$. Therefore, using (5) again, we see that $(*)$ holds for $K=m^p$.

PROOF OF THEOREM 1. Suppose that $\boldsymbol{X} \in HLP$. Since $(g_{(\alpha)})^* \prec (g^*)_{(\alpha)}$ holds for each α with $0<\alpha\leqq 1$ and each $g \in \boldsymbol{X}$, it is sufficient to show that $(*)$ holds for all positive decreasing $f \in \boldsymbol{X}$. Furthermore, in view of the semi-continuity of $\|\cdot\|$, it suffices only to prove that $(*)$ *holds for all positive decreasing step functions of* $\boldsymbol{X}$.

Now assume that this does not hold. By the preceding lemma, there exists no $\varepsilon_0>0$ for which $(\sharp)$ holds for all positive decreasing step functions of $\boldsymbol{X}$. Hence we can find a sequence of positive numbers $\{\varepsilon_\nu\}_{\nu=1}^{\infty}$ with $\varepsilon_\nu \downarrow_{\nu=1}^{\infty} 0$ and a sequence of positive decreasing step functions $\{f_\nu\}_{\nu=1}^{\infty} \subset \boldsymbol{X}$ with $\|f_\nu\| \leqq 1$ $(\nu=1,2,\cdots)$ such that $\|f_{\nu(m_\nu^{-1})}\| > \varepsilon_\nu^{-1}$ holds for each $\nu \geqq 1$, where $m_\nu \leqq 2\varepsilon_\nu^{-1}$. Let ν fix and $f_\nu = \sum_{i=1}^{k} \alpha_i \chi_{(0,\xi_i)}$, where $0<\xi_1<\xi_2<\cdots<\xi_k = \frac{1}{m_\nu}$[6] and $0<\alpha_i$ for all $1 \leqq i \leqq k$. From (2) we find

$$\bar{f}_\nu(x) = \frac{1}{x}\int_0^x f_\nu(t)dt = \sum_{i=1}^{k} \alpha_i \frac{1}{x}\int_0^x \chi_{(0,\xi_i)}(t)dt = \sum_{i=1}^{k} \alpha_i \bar{\chi}_{(0,\xi_i)} = \sum_{i=1}^{k} g_i,$$

where $g_i = \alpha_i \bar{\chi}_{(0,\xi_i)} (i=1,2,\cdots,k)$. Applying Lemma 3 we obtain[7]

$$\|\bar{f}_\nu\| \geqq \|A_{(0,m_\nu\xi_1)}g_1 + g_2 + \cdots + g_k\| \geqq \|A_{(0,m_\nu\xi_1)}g_1 + A_{(0,m_\nu\xi_2)}g_2 + g_3 + \cdots g_k\|$$
$$\geqq \cdots \geqq \|\sum_{i=1}^{k} A_{(0,m_\nu\xi_i)} g_i\|.$$

6) Since $(f_\nu \chi_{(0,m_\nu^{-1})})_{(m_\nu^{-1})} = f_{\nu(m_\nu^{-1})}$ holds, we may assume $f_\nu = f_\nu \chi_{(0,m_\nu^{-1})}$.

7) Note that all g_i $(1 \leqq i \leqq k)$ are positive decreasing.

On the other hand, it follows from the definition (9) that for each $i (1 \leqq i \leqq k)$

$$A_{(0,m_\nu\xi_i)}g_i = \frac{\alpha_i}{m_\nu\xi_i}\left(\int_0^{\xi_i} dx + \int_{\xi_i}^{m_\nu\xi_i} \frac{\xi_i}{x} dx\right)\chi_{(0,m_\nu\xi_i)}$$
$$= \frac{\alpha_i(1+\log m_\nu)}{m_\nu}\chi_{(0,m_\nu\xi_i)}.$$

This implies $\sum_{i=1}^k A_{(0,m_\nu\xi_i)}g_i = \frac{(1+\log m_\nu)}{m_\nu}\sum_{i=1}^k \alpha_i\chi_{(0,m_\nu\xi_i)} = \frac{(1+\log m_\nu)}{m_\nu}f_{\nu(m_\nu^{-1})}$, hence $\|\bar{f}_\nu\| \geqq \frac{(1+\log m_\nu)}{m_\nu}\|f_{\nu(m_\nu^{-1})}\| \geqq \frac{1+\log m_\nu}{m_\nu}\cdot \varepsilon_\nu^{-1} \geqq \frac{1+\log m_\nu}{2}$. Therefore, we have shown that both $\|f_\nu\| \leqq 1$ and $\|\bar{f}_\nu\| \geqq 1/2(1+\log m_\nu)$ hold for every $\nu \geqq 1$, which is, however, inconsistant with Lemma 1. Thus the proof is completed.

Q. E. D.

From Theorem 1 it follows immediately

COROLLARY 1. *The converse of Lemma* 3 *is also valid, i.e.* $X \in HLP$ *if and only if* (3) *holds for a constant* $K > 0$.

As a simple sufficient condition for $X \in HLP$ we have

COROLLARY 2. *If*

$$\sup_{\|f\| \leqq 1} \|f_{(\frac{1}{2})}\| < 2 \tag{12}$$

holds, then $X \in HLP$.

This can be derived in the quite same manner as the proof of Lemma 4 by showing that (12) implies (*).

For any $\alpha > 1$ and $f \in X$ we define $f^{(\alpha)}$ by

$$f^{(\alpha)}(x) = \begin{cases} f(\alpha x), & \text{for } x \in (0, \alpha^{-1}), \\ 0, & \text{otherwise}. \end{cases} \tag{13}$$

Now we consider the following condition on the norm $\|\cdot\|$ of X:

$$\|f^{(\alpha)}\| \leqq K\alpha^{-p}\|f\| \quad \text{for all } f \in X \text{ and } \alpha > 1, \tag{Δ}$$

where $K > 0$ and $p (0 < p < 1)$ are both constants.

In Banach function spaces with *RIP*, the conditions (*) and (Δ) are mutually dual, that is, we have

THEOREM 2. X *satisfies* (*) *if and only if* X' *satisfies* (Δ).

PROOF. Assume that (*) holds in X. For any $g \in X'$ and $\alpha > 1$,

$$\|g^{(\alpha)}\|' = \sup_{\|f\|=1, f \in X} |\langle f, g^{(\alpha)}\rangle| = \frac{1}{\alpha}\sup_{\|f\|=1} |\langle f_{(\alpha^{-1})}, g\rangle|$$
$$\leqq \frac{1}{\alpha}\sup_{\|f_{(\alpha^{-1})}\| = K\alpha^p} |\langle f_{(\alpha^{-1})}, g\rangle| = \|g\| K\alpha^{-(1-p)},$$

since $\|f_{(\alpha^{-1})}\| = K\alpha^p$ implies $\|f\| \geqq 1$, and $f = \{f^{(\alpha)}\}_{(\alpha^{-1})}$ holds. This shows that (Δ) holds in X' for $p' = 1-p$. The converse can be derived similarly. Q.E.D.

3. Special classes of Banach function spaces with *RIP* are spaces $\Lambda(\phi)$

and Orlicz spaces $\boldsymbol{L}_{\Phi}$ (for an exposition of the theory of Orlicz spaces see [1]). We shall show below that the equivalent conditions for $X \in HLP$ presented in [3], when X is one of these spaces, can be derived from Theorem 1.

THEOREM 3. *An Orlicz space $\boldsymbol{L}_{\Phi}$ has the Hardy-Littlewood property if and only if the complementary function Ψ of Φ satisfies*

$$\Psi(2u) \leqq M\Psi(u) \qquad \text{for } u \geqq u_0 ,^{8)} \tag{14}$$

for some constants M and $u_0 \geqq 0$.

PROOF. As $(\boldsymbol{L}_{\Phi})' = \boldsymbol{L}_{\Psi}$, we shall prove that the norm of $\boldsymbol{L}_{\Psi}$ satisfies (Δ) if (14) holds for Ψ. If follows from (14) that the norm $\|\cdot\|_{\Psi}$ is finitely monotone (i.e. for any ε $(0<\varepsilon<1)$ there exists an $N>0$ such that $\|f_i\| \geqq \varepsilon$, $f_i \perp f_j$[9] $(i \neq j)$ and $n \geqq N$ imply $\|\sum_{i=1}^{n} f_i\| > 1$) [6, 8], hence by virtue of Theorem 6 in [7], there exists a lower semi-p-norm[10] $\|\cdot\|_0$ on $\boldsymbol{L}_{\Psi}$ equivalent to $\|\cdot\|_{\Psi}$. For any $f \in \boldsymbol{L}_{\Psi}$ and for any natural number n, we see obviously that f can be written as

$$f \sim \sum_{i=1}^{n} f_i \quad \text{with} \quad f_i \perp f_j \quad \text{for} \quad i \neq j \quad \text{and} \quad f_1 \sim f_2 \sim f_3 \sim \cdots \sim f_n = f^{(n)} .$$

Thus, we have for some fixed $\alpha > 0$[11]

$$\|f\|_0^p = \|\sum_{i=1}^{n} f_i\|_0^p \geqq \sum_{i=1}^{n} \|f_i\|_0^p \geqq \alpha \cdot n \|f^{(n)}\|_0^p ,$$

which implies $\|f^{(n)}\|_0 \leqq (\alpha n)^{1/p} \|f\|_0$. Because $\|\cdot\|_0$ and $\|\cdot\|_{\Psi}$ are mutually equivalent, we find that $\|\cdot\|_{\Psi}$ fulfils (Δ), hence $\boldsymbol{L}_{\Phi} \in HLP$.

Conversely, let (Δ) hold for $\boldsymbol{L}_{\Psi}$. If (14) fails to be true, we can find a sequence of positive elements $\{f_n\}_{n=1}^{\infty}$ such that both $\|f_n\|_{\Psi} \leqq 1$ and $\int_0^1 \Psi(2f_n)dx \geqq n$ hold for all $n \geqq 1$[12]. Then,

$$\int_0^1 \Psi((2f_n)^{(n)})dx = \frac{1}{n} \int_0^1 \Psi(2f_n)dx \geqq 1 ,$$

which implies $\frac{1}{2} \leqq \|(f_n)^{(n)}\|_{\Psi} \leqq K\left(\frac{1}{n}\right)^p \|f_n\|_{\Psi} \leqq K\left(\frac{1}{n}\right)^p$ for all $n \geqq 1$. But this is a contradiction. Q.E.D.

Let ϕ be a decreasing positive integrable function of (0, 1), which we shall

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8) (14) is equivalent to that $2l\Phi(u) \leqq \Phi(lu)$ holds for every $u \geqq u_0$ for some constants $l > 1$ and $u_0 \geqq 0$ [1].

9) $f \perp g$ means that $|f| \cap |g| = 0$, i.e. $f(t)g(t) = 0$ a.e..

10) A norm is called a lower semi-p-norm, if $f \perp g$ implies $\|f+g\|^p \geqq \|f\|^p + \|g\|^p$.

11) Since $\|\cdot\|_{\Psi}$ and $\|\cdot\|_0$ are mutually equivalent, there exists $\alpha > 0$ such that $f \sim g$ implies $\|f\|_0 \geqq \alpha \|g\|_0$.

12) $\|f\|_{\Psi} = \inf\left\{\frac{1}{|\xi|} : \int_0^1 \Psi(\xi f)\,dx \leqq 1\right\}$.

assume zero for $x>1$, and let $\Phi(x)=\int_0^x \phi(t)dt$. The space $\Lambda(\phi)$ consists of all functions $f(x)$ such that

$$\|f\|_\Lambda=\int_0^1 \phi(x)\cdot f^*(x)dx<\infty\,. \tag{15}$$

The dual space of a space $\Lambda(\phi)$ is $M(\phi)$ consisting of functions f with

$$\|f\|_M=\sup_a \left(\frac{1}{\Phi(a)}\right)\int_0^a f^*(t)dt<\infty\,. \tag{16}$$

Obviously $\Lambda(\phi)$ and $M(\phi)$[13] are Banach function spaces having *RIP* [**2**].

THEOREM 4. *A space $\Lambda(\phi)$ has the Hardy-Littlewood property if and only if*

$$\limsup_{u\to 0} \Phi(2u)/\Phi(u)<2\,. \tag{17}$$

PROOF. Assume that (17) holds. Then, we see easily that there exists an $\varepsilon>0$ such that $\Phi(2u)<(2-\varepsilon)\Phi(u)$ holds for all $0<u$. Let $f=\sum_{i=1}^k \alpha_i\chi_{(0,\xi_i)}$ be a positive decreasing step function with $\|f\|_\Lambda=1$. Since $\|\chi_{(0,\xi)}\|_\Lambda=\Phi(\xi)$ for all ξ with $0<\xi<1$ and $\|f\|_\Lambda=\langle f,\phi\rangle$, we find immediately $\|f_{(1/2)}\|_\Lambda=\langle f_{(1/2)},\phi\rangle=\sum_{i=1}^k \alpha_i\Phi(2\xi_i)$. This implies $\|f_{(1/2)}\|_\Lambda\leqq(2-\varepsilon)\sum_{i=1}^k\alpha_i\Phi(\xi_i)\leqq(2-\varepsilon)\|f\|_\Lambda$, and in view of the semi-continuity of $\|\cdot\|_\Lambda$, one can derive easily that (12) holds. Consequently, we obtain $\Lambda(\phi)\in HLP$ by Corollary 2.

Suppose conversely that $\Lambda(\phi)\in HLP$, or equivalently that (*) holds for some constants K and p $(0<p<1)$. We can then find an $\alpha_0>0$ and a p' $(0<p\leqq p'<1)$ such that $\|f_{(\alpha)}\|_\Lambda\leqq\alpha^{-p'}\|f\|_\Lambda$ holds for all $0\leqq\alpha\leqq\alpha_0$ and $f\in\Lambda(\phi)$. Now choose a natural number n as $2^{-n}<\alpha_0$, and an $\varepsilon>0$ as $(2-2^n\varepsilon)>2^{p'}$. If (17) is false, there exists a ξ with $\xi<2^{-n}$ for which $\Phi(2\xi)>(2-\varepsilon)\Phi(\xi)$ holds. But this implies $\Phi(\xi)\geqq(2-2\varepsilon)\Phi(\xi/2)$, for otherwise we would have $\Phi(2\xi)-\Phi(\xi)\leqq 2(\Phi(\xi)-\Phi(\xi/2))<\Phi(\xi)-\Phi(\xi/2)+(1-2\varepsilon)\Phi(\xi/2)\leqq(1-\varepsilon)\Phi(\xi)$, which contradicts the choice of ξ[14]. Therefore, repeating this argument n times, we get

$$\Phi(2\xi)\geqq(2-\varepsilon)\Phi(\xi)\geqq(2-\varepsilon)(2-2\varepsilon)\Phi(\xi/2)\geqq\cdots\geqq(2-2^n\varepsilon)^n\Phi(\xi/2^{n-1})\,.$$

On the other hand, by virtue of $\Phi(2^n\cdot\xi/2^{n-1})=\|\chi_{(0,\xi/2^{n-1})(2^{-n})}\|_\Lambda$ we have

$$\Phi(2\xi)=\Phi(2^n\xi/2^{n-1})\leqq 2^{np'}\Phi(\xi/2^{n-1})<(2-2^n\varepsilon)^n\Phi(\xi/2^{n-1})\,,$$

hence a contradiction. Thus we have shown $\limsup_{u\to 0}\Phi(2u)/\Phi(u)\leqq(2-\varepsilon)$, and this proves our assertion. Q.E.D.

REMARK. *For an arbitrary Banach function space X with RIP $\theta(f)$ belongs*

13) A necessary and sufficient condition for $M(\phi)\in HLP$ is also given in [**3**].

14) This fact is due to [**3**]. But the proof becomes somewhat simpler than that of [**3**].

to X, *if* $f(x)\log(1/x)\in X$.

This fact has been shown in [**2**; Theorem 7] in case of $X=\Lambda(\phi)$, and the proof is similarly obtained, since for any positive decreasing $c\in X'$ and any $f\in X$ one obtains

$$\langle\theta(f), c\rangle \leqq 2\langle \bar{f}^*, c\rangle^{15)} = 2\int_0^1 f^*(t)dt\int_t^1 \frac{c(x)}{x}dx$$
$$\leqq 2\int_0^1 (f^*(t)\log(1/t))c(t)dt\,.$$

4. Let (E, Ω, μ) be a non-atomic finite measure space with a countably additive non-negative measure μ on a σ-field Ω of E, and let $X=X(E)$ be a Banach function space of integrable functions over E, which has *RIP*.

Now, we consider the following condition on X:[16)]

$$(\Theta)\qquad \bigcup_{0\leqq\alpha\leqq 1} A_{\mathbf{e}_\alpha}|f| \in X$$

for any $f\in X$ and for any system of measurable sets $\{\mathbf{e}_\alpha\}_{0\leqq\alpha\leqq 1}$ satisfying $\mathbf{e}_\alpha\subset\mathbf{e}_\beta$ for $\alpha\leqq\beta$. As is easily shown, this property can be considered to be what corresponds to *HLP* in the case when E is an interval $(0, a)$ of real numbers. In fact, if f is positive decreasing and $\mathbf{e}_\alpha=(0, \alpha a)$ for all $\alpha\in(0, 1)$, then $\bigcup_{0\leqq\alpha\leqq 1} A_{\mathbf{e}_\alpha}f$ coincides with $\bar{f}$.

Lastly we shall describe a necessary and sufficient condition in order that X satisfies (Θ). For any $0\leqq\alpha\leqq 1$ and $0\leqq f\in X$, we denote by $S(f;\alpha)$ the set of all $0\leqq g\in X$ satisfying $\mu\{x: g(x)>r\}=\alpha\cdot\mu\{x: f(x)>r\}$ for all $r\geqq 0$. Then, we can prove

THEOREM 5. *$X(E)$ satisfies (Θ) if and only if there exist positive numbers K and p $(0<p<1)$ such that $\|f\|\leqq K\alpha^{-p}\|g\|$ holds for all $0\leqq f\in X$, $g\in S(f;\alpha)$ and $0<\alpha\leqq 1$.*

The proof being quite analogous to that of Theorem 1, we omit it.

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[2] G. Lorentz, On the theory of spaces Λ, Pacific J. Math., 1 (1951), 411–429.

15) $\theta(f)<2f^*$ holds for each f [**9**].

16) For this formulation, the author expresses his appreciation to Professor I. Amemiya for his advice.

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