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Almost complex structures in bundle spaces over almost contact manifolds

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Introduction. A tangent bundle of the differentiable manifold M endowed with a linear connection has the almost complex structure J associated with the linear connection which was defined and studied by T. Nagano [8], C. J. Hsu [3], S. Tachibana and M. Okumura [17] and P. Dombrowski [1], etc.. It was shown that J is integrable if and only if the torsion and curvature tensor of the connection vanish. If we take as a metric L for the tangent bundle the one defined by S. Sasaki [13] using a metric of the base manifold M, then (J, L) is an almost hermitian structure, and in the case where J is associated with the Levi-Civita connection, (J, L) is an almost Kählerian structure which is not Kählerian unless M is locally flat ([1], [17]).

However, if M itself has an almost complex structure or almost contact structure, the tangent bundle has a naturally related almost complex structure, associating with an arbitrary connection, whose integrability condition does not necessarily involve the local flatness of the connection. In § 2, we assume that M has an almost contact structure and treat the almost complex structure associated with that. Its integrability conditions will be given in Proposition 2.1. Especially if we adopt as the connection the Levi-Civita connection for the associated metric of the almost contact structure, the integrability conditions are the normality of the structure and a certain relation of the curvature tensor.

On the other hand, A. Morimoto [6] made a study of the almost complex structure in the product space of almost contact manifolds, and Y. Ogawa [11] studied the almost complex structure in the principal fiber bundle whose structural group is a 1-dimensional abelian group. In § 3, we consider an almost complex structure in the principal fiber bundle with an odd dimensional Lie group. Some of the results in [11] will be generalized slightly. A converse of his theorem is proved. In § 4, we show that, if the principal fiber bundle with a 1-dimensional Lie group G has a G-invariant almost complex structure, then its base space has an almost contact structure.

In the last section, returning to the tangent bundle of an almost contact manifold, a coordinate expression of J is given. From this it is shown that the tangent bundle of an almost contact manifold has an almost complex structure which depends only on the almost contact structure of the base space and its integrability condition is equivalent to the normality of the almost contact structure.

Here I express my hearty thanks to Professor S. Sasaki for his kind advices.

1. Preliminaries.

For a differentiable (C^∞) manifold M, FM, $\mathfrak{X}M$, and TM denote the ring of all differentiable functions on M, FM-module of differentiable vector fields on M, and the total space of the tangent bundle of M respectively. Z_p for $Z \in \mathfrak{X}M$, $p \in M$, is the value of Z at p. If $\mu : M \to M'$ is a differentiable map, we denote the differential of μ by the same letter and its dual by μ^*.

i. *An almost complex structure.* A tensor field $J : \mathfrak{X}M \to \mathfrak{X}M$ is an almost complex structure if $J^2X = -X$ for $X \in \mathfrak{X}M$. The Nijenhuis tensor N is by definition

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for $X, Y \in \mathfrak{X}M$. J is said to be integrable if $N = 0$.

ii. *An almost contact structure.* A (1,1)-tensor ϕ, a vector field ξ and a 1-form η define an almost contact structure in M ($\dim M \neq 1$) if

$$\phi^2 u = -u + \eta(u)\xi ,$$
$$\eta(\xi) = 1 ,$$

for $u \in \mathfrak{X}M$. Several tensors which we put below were defined and studied in [16], $\mathfrak{L}$ denoting the operation of Lie derivative,

$$S(u, v) = [u, v] + \phi[\phi u, v] + \phi[u, \phi v] - [\phi u, \phi v] + \{v \cdot \eta(u) - u \cdot \eta(v)\}\xi ,$$
$$S_1(u, v) = \mathfrak{L}(\phi u)\eta \cdot v - \mathfrak{L}(\phi v)\eta \cdot u ,$$
$$S_2(u) = \mathfrak{L}(\xi)\phi \cdot u ,$$
$$S_3(u) = \mathfrak{L}(\xi)\eta \cdot u ,$$

for $u, v \in \mathfrak{X}M$. It is known that $S = 0$ implies $S_1 = S_2 = S_3 = 0$ and such an almost contact structure is called to be normal. An associated metric g is one such that $\eta(u) = g(\xi, u)$ and

$$g(u, v) = g(\phi u, \phi v) + \eta(u) \cdot \eta(v)$$

for $u, v \in \mathfrak{X}M$.

iii. *The tangent bundle.* For the linear connection ∇ and $u, v \in \mathfrak{X}M$, let $\nabla_v u$ be the covariant derivative of u with respect to v. If we consider u as a map $u: M \to TM$, we get the image $u(v)$ of v by its differential $u: TM \to TTM$. And it is verified that any point of TTM is expressed in the form $u(v)$ for some u and v. Then the connection map $K: TTM \to TM$ is determined by

$$\nabla_v u = K \cdot u(v) .$$

The expression of K by local coordinates is used in § 5, for the details see [1] and [4]. The almost complex structure J of TM defined in [1] and [17] is as follows

$$\pi_z(JX) = -K_z X, \quad K_z(JX) = \pi_z X$$

for $X \in \mathfrak{X}TM$, $z \in TM$, where π is a projection $TM \to M$. The integrability condition of J is the local flatness of the connection. If g is a Riemannian metric for M, the metric L defined by

$$L(X, Y) = g(\pi X, \pi Y) + g(KX, KY)$$

for $X, Y \in \mathfrak{X}TM$, together with J, defines an almost hermitian structure on TM. And if K comes from the Levi-Civita connection for g, the pair (J, L) is almost Kählerian, which is not Kählerian unless M is locally flat with respect to g. On the other hand, if M has an almost complex structure h, TM has a natural almost complex structure J' such that

$$\pi_z(J'X) = h_{\pi z}\pi_z X, \quad K_z(J'X) = h_{\pi z} K_z X$$

for $X \in \mathfrak{X}TM$, $z \in TM$, where K is a connection map with respect to the linear connection ∇. J' is integrable if and only if

$$h : \text{integrable}, \tag{1.1}$$

$$R(u, v) - R(hu, hv) + h \cdot \{R(u, hv) + R(hu, v)\} = 0, \tag{1.2}$$

$$\nabla_{hu} h - h \cdot \nabla_u h = 0 \tag{1.3}$$

for $u, v \in \mathfrak{X}M$, R denoting the curvature tensor of ∇.

If (h, g) is a hermitian structure and ∇ is the Levi-Civita connection for g, then (1.1) and (1.3) hold. Moreover if (h, g) is a Kählerian structure, then (1.2) is automatically satisfied too. That is to say, the tangent bundle of a Kählerian manifold has a complex structure as above.

In § 2, we treat the case where M has an almost contact structure, and we utilize the followings (see P. Dombrowski [1]): By u^* and v° we denote the horizontal and vertical lift of u and v of $\mathfrak{X}M$ respectively, which are characterised by $K(u^*) = 0$, $\pi(u^*) = u$, $K(v^\circ) = v$ and $\pi(v^\circ) = 0$. Then the relations

$$[u^\circ, v^\circ] = 0 ,$$

$$[u^*, v^\circ] = (\nabla_u v)^\circ ,$$

$$\pi_z([u^*, v^*]) = [u, v]_{\pi z} ,$$

$$K_z([u^*, v^*]) = R_{\pi z}(u, v) z ,$$

for $z \in TM$, hold good, where K is a connection map with respect to the linear connection ∇ and R is the curvature tensor of ∇.

2. Almost complex structures of the tangent bundle of an almost contact manifold.

Let ϕ, ξ and η be the structure tensors of M. The definition of an almost complex structure J of TM is

$$\pi_z(JX) = \phi_{\pi z}\pi_z X + \eta_{\pi z}(K_z X)\xi_{\pi z},$$

$$K_z(JX) = \phi_{\pi z}K_z X - \eta_{\pi z}(\pi_z X)\xi_{\pi z},$$

for $X \in \mathfrak{X}TM$. As for the horizontal lift u^* and vertical lift v° of u and v of $\mathfrak{X}M$, the operation of J is characterized by

$$Ju^* = (\phi u)^* - (\eta(u)\xi)^\circ,$$

$$Jv^\circ = (\phi v)^\circ + (\eta(v)\xi)^*.$$

A. *Integrability*. Now, turning to account the identities in § 1, we calculate the value of $N_z(X, Y)$ for $X, Y \in \mathfrak{X}TM$. It suffices to perform in the following cases for $u, v \in \mathfrak{X}M$ such that $\eta(u)$ and $\eta(v)$ are constant, because we are able to replace u by $\bar{u} = \eta_{\pi z}(u)\xi - \phi^2 u$ for fixed $\pi z \in M$. Abbreviating πz in the right hand side, we get

$$\begin{aligned}\pi_z N_z(u^*, v^*) = S(u, v) &+ \eta\{R(\phi u, v)z + R(u, \phi v)z\}\xi \\ &+ \eta(u)\cdot\eta(\nabla_v\xi)\xi - \eta(v)\cdot\eta(\nabla_u\xi)\xi,\end{aligned} \tag{2.1}$$

$$\begin{aligned}K_z N_z(u^*, v^*) &= R(u, v)z - R(\phi u, v)z \\ &+ \phi\{R(\phi u, v)z + R(u, \phi v)z\} \\ &+ S_1(u, v)\xi + \eta(u)\{\phi\cdot\nabla_v\xi - \nabla_{\phi v}\xi\} \\ &- \eta(v)\{\phi\cdot\nabla_u\xi - \nabla_{\phi u}\xi\},\end{aligned} \tag{2.2}$$

where we have utilized that $S_1(u, v) = -\eta\{[u, \phi v] + [\phi u, v]\}$.

$$\begin{aligned}\pi_z K_z(u^*, v^\circ) = -(\nabla_{\phi u}\eta)(v)\cdot\xi &- (\nabla_u\eta)(\phi v)\cdot\xi \\ &+ \eta(v)\{S_2(u) + (\eta\cdot R(u, \xi)z)\xi\},\end{aligned} \tag{2.3}$$

$$\begin{aligned}K_z N_z(u^*, v^\circ) = \eta(v)\{\phi\cdot R(u, \xi)z &- R(\phi u, \xi)z\} \\ -\eta(u)\cdot\eta(v)\nabla_\xi\xi &+ \phi\cdot\nabla_u\phi\cdot v - \nabla_{\phi u}\phi\cdot v \\ -\eta(v)S_3(u)\xi &- (\nabla_u\eta)(v)\cdot\xi,\end{aligned} \tag{2.4}$$

$$\pi_z N_z(u^\circ, v^\circ) = \eta(v)(\nabla_\xi\eta\cdot u)\xi - \eta(u)(\nabla_\xi\eta\cdot u)\xi, \tag{2.5}$$

$$K_z N_z(u^\circ, v^\circ) = \eta(v)\cdot\nabla_\xi\phi\cdot u - \eta(u)\nabla_\xi\phi\cdot v. \tag{2.6}$$

Therefore we have

PROPOSITION 2.1. *An almost complex structure of TM associated with the almost contact structure of M and an arbitrary linear connection ∇ is integrable if and only if the followings are satisfied*

$$S(u, v)+\eta(u)\cdot\eta(\nabla_v\xi)\xi-\eta(v)\cdot\eta(\nabla_u\xi)\xi=0\,, \tag{2.7}$$

$$S_1(u, v)+\eta(u)(\phi\cdot\nabla_v\xi-\nabla_{\phi v}\xi)-\eta(v)(\phi\cdot\nabla_u\xi-\nabla_{\phi u}\xi)=0\,, \tag{2.8}$$

$$(\nabla_{\phi u}\eta)(v)\cdot\xi+(\nabla_u\eta)(\phi v)\cdot\xi-\eta(u)S_2(u)=0\,, \tag{2.9}$$

$$\phi\cdot\nabla_u\phi\cdot v-\nabla_{\phi u}\phi\cdot v-(\nabla_u\eta)(v)\cdot\xi-\eta(v)S_3(u)\cdot\xi-\eta(u)\cdot\eta(v)\nabla_\xi\xi=0\,, \tag{2.10}$$

$$\eta(v)(\nabla_\xi\eta)u-\eta(u)(\nabla_\xi\eta)v=0\,, \tag{2.11}$$

$$\eta(v)\cdot\nabla_\xi\phi\cdot u-\eta(u)\cdot\nabla_\xi\phi\cdot v=0\,, \tag{2.12}$$

$$R(u, v)-R(\phi u, \phi v)+\phi\cdot\{R(\phi u, v)+R(u, \phi v)\}=0 \tag{2.13}$$

for $u, v\in\mathfrak{X}M$.

PROOF. *Necessity:* We separate each term into two parts according as it contains z or not, then (2.7)$\sim$(2.13) follow.

Sufficiency: It is enough to show the next relations

$$\eta\{R(\phi u, v)+R(u, \phi v)\}=0\,, \tag{2.14}$$

$$\phi\cdot R(u, \xi)-R(\phi u, \xi)=0\,. \tag{2.15}$$

Replace v by ξ in (2.13) and operate η, then we have $\eta\cdot R(u, \xi)=0$. Similarly replace u in (2.13) by ϕu and operate η, then $\eta\{R(\phi u, v)-R(\phi^2u, \phi v)\}=0$ follows and we have (2.14). Next we put $v=\xi$ in (2.13), getting

$$R(u, \xi)+\phi\cdot R(\phi u, \xi)=0\,.$$

If we operate ϕ to the last equation, it turns to (2.15). Q. E. D.

The conditions in Proposition 2.1 are too much complicated, and so let g be an associated Riemannian metric to the almost contact structure and consider the Levi-Civita connection D for g. Here we prepare lemmas.

LEMMA. *The neccessary and sufficient condition that an almost contact metric structure is normal is that*

$$\phi\cdot D_u\phi\cdot v-D_{\phi u}\phi\cdot v-(D_u\eta)(v)\cdot\xi=0 \tag{2.16}$$

is valid for any u and v of $\mathfrak{X}M$.

This is stated in [9] in somewhat different form. In order to see the necessity (only which we need later), it is enough to verify

$$\begin{aligned}g(S(u, v), r)-g(S(v, r), u)+g(S(r, u), v)\\ =2g(r, \phi\cdot D_u\phi\cdot v-D_{\phi u}\phi\cdot v-(D_u\eta)v\cdot\xi)\end{aligned}$$

for $u, v, r \in \mathfrak{X}M$.

Now, by this lemma if $S=0$, the relation (2.16) replaced u by ξ implies $\phi \cdot D_\xi \phi = 0$ and, as $\eta \cdot D_\xi \phi = -D_\xi \eta \cdot \phi = 0$, we have $D_\xi \phi = 0$.

LEMMA. *For the Levi-Civita connection D, the normality of the structure yields* (2.8) *and* (2.9).

PROOF. The relation $S_2(u)=0$ is written in the following from

$$D_\xi \phi \cdot u - D_{\phi u} \xi + \phi \cdot D_u \xi = 0 . \tag{2.17}$$

Using $D_\xi \phi = 0$ we obtain (2.8). To verify (2.9), we consider inner product of $D_{\phi u}\xi - \phi \cdot D_u \xi$ $(=0)$ and v of $\mathfrak{X}M$

$$g(D_{\phi u}\xi, v) + g(D_u \xi, \phi v) = 0 ,$$

where we have utilized $g(u, \phi v) = -g(\phi u, v)$. If we notice $g(\xi, v) = \eta(v)$, we see that $(D_{\phi u}\eta)v + (D_u \eta) \cdot \phi v = 0$.

PROPOSITION 2.2. *An almost complex structure of TM associated with the almost contact metric structure and the Levi-Civita connection D is integrable if and only if the structure is normal and the curvature tensor satisfies the following*

$$R(u, v) - R(\phi u, \phi v) + \phi \cdot \{R(u, \phi v) + R(\phi u, v)\} = 0 \tag{2.18}$$

for any $u, v \in \mathfrak{X}M$.

PROOF. *Necessity:* For the Levi-Civita connection $\eta(D_v \xi) = 0$ for any v of $\mathfrak{X}M$ is valid and so $S=0$ follows from (2.7). (2.18) is the same as (2.13).

Sufficiency: By virtue of the preceding lemmas, all (2.7)$\sim$(2.12) follow from $S=0$.

COROLLARY. *If the almost contact metric structure of M is derived from a contact structure, then the almost complex structure J defined as above is not integrable.*

PROOF. Suppose that J is integrable, by Proposition 2.2 this contact metric structure is normal and we have ([**12**])

$$4\eta \cdot R(u, v) r = \eta(v) g(u, r) - \eta(u) g(v, r) . \tag{2.19}$$

However, (2.18) implies $\eta \cdot R(\xi, v) = 0$. And (2.19) leads the relation $g(v, r) = \eta(v) \cdot \eta(r)$ which is impossible in the case of $\dim M \geqq 3$.

B. *Almost hermitian metrics.* As a metric L in TM, following [**13**] we define

$$L_z(X, Y) = g_{\pi z}(\pi_z X, \pi_z Y) + g_{\pi z}(K_z X, K_z Y) \tag{2.20}$$

for $X, Y \in \mathfrak{X}TM$, $z \in TM$, where g is the associated metric. The pair (J, L) defines the almost hermitian structure of TM and the fundamental 2-form Ξ is

$$\Xi(X, Y) = g(\pi X, \phi \pi Y + \eta(KY)\xi) + g(KX, \phi KY - \eta(\pi Y)\xi) .$$

PROPOSITION 2.3. *This almost hermitian structure is an almost Kählerian structure if and only if the almost contact metric structure and the linear connection ∇ satisfy the followings,*

$$\phi \cdot R(u, v) = 0\,, \tag{2.21}$$

$$\eta(r)\eta \cdot R(u, v)z + \eta(u)\eta \cdot R(v, r)z + \eta(v)\eta \cdot R(r, u)z = 0\,, \tag{2.22}$$

$$\eta(r)d\eta(u, v) - \eta(u)\nabla_v\eta \cdot r + \eta(v)\nabla_u\eta \cdot r = 0\,, \tag{2.23}$$

$$d\zeta = 0, \qquad \nabla\zeta = 0\,, \tag{2.24}$$

for any $u, v, r, z \in \mathfrak{X}M$, where $\zeta(u, v) = g(u, \phi v)$.

For the proof, the standard formula is

$$(d\Xi)(X, Y, Z) = X \cdot \Xi(Y, Z) + Y \cdot \Xi(Z, X) + Z \cdot \Xi(X, Y)$$
$$- \Xi([X, Y], Z) - \Xi([Y, Z], X) - \Xi([Z, X], Y)\,,$$

for $X, Y, Z \in \mathfrak{X}TM$. We carry out the following computation assuming that $\eta(u)$, $\eta(v)$ and $\eta(r)$ are constant and abbreviating πz in the right hand side of the equations

$$(d\Xi)_z(u^*, v^*, r^*) = (d\zeta)(u, v, r) + \eta(r)\eta \cdot R(u, v)z + \eta(u)\eta \cdot R(v, r)z + \eta(v)\eta \cdot R(r, u)z\,, \tag{2.25}$$

where we have used the lemma below.

$$(d\Xi)_z(u^*, v^*, r^\circ) = \eta(r)(d\eta)(u, v) - g(R(u, v)z, \phi r) - \eta(u)\nabla_v\eta \cdot r + \eta(v)\nabla_u\eta \cdot r\,, \tag{2.26}$$

$$(d\Xi)_z(u^*, v^\circ, r^\circ) = g(v, \nabla_u\phi \cdot r) + (\nabla_u g)(v, \phi r)\,, \tag{2.27}$$

$$(d\Xi)_z(u^\circ, v^\circ, r^\circ) = 0\,. \tag{2.28}$$

Necessity: As the right hand side of (2.27) is equal to $(\nabla_u\zeta)(v, r)$, we have (2.21)$\sim$(2.24). The converse is clear.

LEMMA. *In the notations above*

$$u_z^* \cdot \Xi(v^*, r^*) = u_{\pi z} \cdot g(v, \phi r)\,, \tag{2.29}$$

$$u_z^* \cdot \Xi(v^*, r^\circ) = 0\,, \tag{2.30}$$

$$u_z^* \cdot \Xi(v^\circ, r^\circ) = u_{\pi z} \cdot g(v, \phi r)\,, \tag{2.31}$$

$$u_z^\circ \cdot \Xi(v^*, r^*) = u_z^\circ \cdot \Xi(v^*, r^\circ) = u_z^\circ \cdot \Xi(v^\circ, r^\circ) = 0 \tag{2.32}$$

hold for any $u, v, r \in \mathfrak{X}M$ such that $\eta(v)$ and $\eta(r)$ are constant.

PROOF. Denote by $\bar{\varphi}_t$ and φ_t the local 1-parameter groups generated by u^* and u about $z \in TM$ and $\pi z \in M$ respectively, then $\varphi_t\pi z = \pi\bar{\varphi}_t z$. As $\Xi_z(v^*, r^*) = g_{\pi z}(v, \phi r)$, we obtain (2.29). Demonstrations for others shall be omited.

From Proposition 2.3 we get

PROPOSITION 2.4. *If K is defined by the Levi-Civita connection for g, then the almost hermitian structure is almost kählerian if and only if the almost contact metric structure satisfies $D\phi=0$ and has zero curvature tensor.*

In fact, from (2.27) $D\phi=0$ follows. And so $D\xi=0$ and $D\eta=0$. By Ricci identity for η we have $\eta\cdot R(u,v)z=0$. Combining this with (2.21) we get $R=0$.

COROLLARY. *If the almost hermitian structure* (*in Proposition* 2.4) *is almost kählerian, then it is necessarily kählerian.*

C. *Transformations.* We shall show first the following

PROPOSITION 2.5. *Let a diffeomorphism μ of M be an automorphism of the almost contact metric structure and assume that the connection map is of the Levi-Civita connection for g. Then the extended diffeomorphism $\bar\mu$ of μ to TM is an automorphism of the almost hermitian structure (J, L).*

PROOF. First we know that, if μ is an isometry, $\mu K=K\bar\mu$ ([4]). Then by definition (2.20) $\bar\mu$ is an isometry for L ([13]). And by $\mu\phi=\phi\mu$, $\eta^*\mu=\eta$ and $\mu\xi=\xi$ we have

$$\begin{aligned}\pi_{\bar\mu z}\bar\mu_z J_z X &= \mu_p\phi_p\pi_z X+\eta_p(K_z X)\cdot\mu_p\xi\\ &= \phi_{\mu p}\mu_p\pi_z X+(\mu^*\eta)_p(K_z X)\xi_{\mu p}\\ &= \phi_{\mu p}\pi_{\bar\mu z}\bar\mu_z X+\eta_{\mu p}(K_{\bar\mu z}\bar\mu_z X)\xi_{\mu p}\\ &= \pi_{\bar\mu z}J_{\bar\mu z}\bar\mu_z X\end{aligned}$$

for $X\in\mathfrak{X}TM$, $z\in TM$, $p=\pi z$. Similar calculation yeilds

$$K_{\bar\mu z}\bar\mu_z J_z X=K_{\bar\mu z}J_{\bar\mu z}\bar\mu_z X\,.$$

Therefore $\bar\mu J=J\bar\mu$.

REMARK: The same thing holds for an isomorphism $\mu: M\to N$ of two almost contact Riemannian manifolds.

Next, about the action of u^* or v° to J or L as an infinitesimal transformation, we have

PROPOSITION 2.6. *Let J be associated with an almost contact structure and a linear connection ∇. Then u^* is almost analytic if and only if*

$$\nabla_u\phi=0,\qquad \mathfrak{L}(u)\phi=0\,, \tag{2.33}$$

$$\eta(r)[u,\xi]+(\nabla_u\eta)(r)\cdot\xi=0\,, \tag{2.34}$$

$$(\mathfrak{L}(u)\eta\cdot r)\xi+\eta(r)\cdot\nabla_u\xi=0\,, \tag{2.35}$$

$$R(u,r)+\phi\cdot R(u,\phi r)=0\,, \tag{2.36}$$

for any $r\in\mathfrak{X}M$. v° is almost analytic if and only if

$$\nabla_r v + \phi \cdot \nabla_{\phi r} v = 0 \tag{2.37}$$

for any $r \in \mathfrak{X}M$.

PROOF. Assuming that $\eta(r)$ is constant, one computes straightforwardly, getting

$$\pi_z(\mathfrak{L}(u^*)J_z \cdot r^\circ) = \eta(r)[u, \xi] + (\nabla_u \eta)(r) \cdot \xi ,$$

$$K_z(\mathfrak{L}(u^*)J_z \cdot r^\circ) = \nabla_u \phi \cdot r + \eta(r) \cdot R(u, \xi)z ,$$

$$\pi_z(\mathfrak{L}(u^*)J_z \cdot r^*) = \mathfrak{L}(u)\phi \cdot r - \eta(R(u, r)z)\xi ,$$

$$K_z(\mathfrak{L}(u^*)J_z \cdot r^*) = R(u, \phi r)z - \eta(r) \cdot \nabla_u \xi$$
$$- \phi \cdot R(u, r)z - (\mathfrak{L}(u)\eta \cdot r)\xi .$$

Likewise one has

$$\mathfrak{L}(v^\circ)J_z \cdot r^\circ = -\eta(r)(\nabla_\xi v)^\circ ,$$

$$\mathfrak{L}(v^\circ)J_z \cdot r^* = (\phi \cdot \nabla_r v)^\circ - (\nabla_{\phi r} v)^\circ + (\eta(\nabla_r v) \cdot \xi)^* .$$

COROLLARY. *If J is associated with an almost contact metric structure satisfying $D_\xi \phi = 0$ with respect to the Levi-Civita connection D, then ξ° is almost analytic if and only if $S_2 = 0$.*

In fact, suppose that $S_2 = 0$, or (2.17) holds. Then by $D_\xi \phi = 0$, we have $\phi \cdot D_r \xi - D_{\phi r} \xi = 0$ and therefore

$$D_r \xi - \eta(D_r \xi)\xi + \phi \cdot D_{\phi r} \xi = 0 .$$

As $\eta(D_r \xi) = 0$, (2.37) follows. The converse is similar.

REMARK 1. If the almost contact metric structure is normal, we have $D_\xi \phi = 0$, and $S_2 = 0$.

REMARK 2. If the almost contact metric structure is a contact metric structure such that ξ is a Killing vector field, we have also $D_\xi \phi = 0$ and $S_2 = 0$.

COROLLARY. *If an almost complex structure J associated with an almost contact metric structure and the Levi-Civita connection D is integrable, then ξ^* is analytic.*

Because, from Proposition 2.2, we see that the almost contact metric structure is normal, then by the preceding Corollary, ξ° is (almost) analytic. As J is integrable, $J\xi^\circ = \xi^*$ is analytic too.

PROPOSITION 2.7. *With respect to the metric L (2.20), u^* is a Killing vector field if and only if u is a Killing vector field with respect to the metric g and satisfies $\nabla_u g = 0$ and $R(u, r) = 0$ (R is of ∇) for any $r \in \mathfrak{X}M$. The necessary and sufficient condition that v° is a Killing vector field for L is that v is a parallel field with respect to ∇.*

PROOF. For $r, s \in \mathfrak{X}M$, $z \in TM$, we have

$$\mathfrak{L}(u^*)L_z(r^*, s^*) = \mathfrak{L}(u)g_{\pi z}(r, s) ,$$

$$\mathfrak{L}(u^*)L_z(r^*, s^\circ) = -g_{\pi z}(R(u, r)z, s)\,,$$

$$\mathfrak{L}(u^*)L_z(r^\circ, s^\circ) = (\nabla_u g)_{\pi z}(r, s)\,.$$

And similarly

$$\mathfrak{L}(v^\circ)L_z(r^*, s^*) = \mathfrak{L}(v^\circ)L_z(r^\circ, s^\circ) = 0\,,$$

$$\mathfrak{L}(v^\circ)L_z(r^*, s^\circ) = g_{\pi z}(\nabla_r v, s)\,. \qquad \text{Q. E. D.}$$

From this, we know that the horizontal lift u^* or vertical lift v° of $u, v \in \mathfrak{X}M$, or their linear combination with real coefficients, cannot be an infinitesimal (non-isometric) conformal transformation.

REMARK 3. Let K and $\bar{K}$ be two connection maps, then there exists a (1,2) tensor U on M such that

$$\bar{K}_z X = K_z X + U_{\pi z}(\pi X, z)$$

for $X \in \mathfrak{X}TM$, $z \in TM$. Hence if we fix one connection map K and corresponding almost complex structure J, then $\bar{J}$ is related as follows

$$\pi_z(\bar{J}X) = \pi_z(JX) + \eta_{\pi z}(U(\pi X, z))\xi_{\pi z}\,,$$

$$\bar{K}_z(\bar{J}X) = K_z(JX) + \phi_{\pi z} \cdot U(\pi X, z)\,.$$

REMARK 4. Though we have handled J in the most simplified form, one may deal with the following somewhat intricated J's

$$(*) \qquad \begin{aligned} \pi JX &= \varepsilon_1 \phi\pi X + \{a\eta KX + b\eta\pi X\}\xi\,, \\ KJX &= \varepsilon_2 \phi KX - \{b\eta KX + a^{-1}(1+b^2)\eta\pi X\}\xi\,, \end{aligned}$$

ε_1, ε_2 being 1 or -1, a non-vanishing scalar and b arbitary scalar. Or

$$\pi JX = \varepsilon_1 \phi KX + \{a\eta KX + b\eta\pi X\}\xi\,,$$

$$KJX = \varepsilon_2 \phi\pi X - \{b\eta KX + a^{-1}(1+b^2)\eta\pi X\}\xi\,,$$

where $\varepsilon_1 \cdot \varepsilon_2 = 1$, and $a \neq 0$ everywhere on TM.

In the next section on principal fiber bundle, we shall adopt the style (*). (Cf. Proposition 4.2.)

3. Almost complex structures in principal fiber bundles over an almost contact manifold.

We consider an arbitrary connection w in P, $P = P(M, G, \pi)$ denoting the principal fiber bundle with group G and projection π. Analogously we use the notation u^*, A° for the horizontal lift of $u \in \mathfrak{X}M$ and fundamental vector field with respect to $A \in \mathfrak{G}$, $\mathfrak{G}$ being the Lie algebra of G. Then we have (see K. Nomizu [**10**])

$$[A^{\circ}, u^{*}]=0\,,$$

$$[A^{\circ}, B^{\circ}]=([A, B])^{\circ}\,,$$

$$w([u^{*}, v^{*}])=-\Omega(u^{*}, v^{*})$$

for $A, B\in\mathfrak{G}$ and $u, v\in\mathfrak{X}M$, where Ω stands for the curvature form of the connection. An odd dimensional (connected) Lie group G has many left invariant almost contact structures, from which we choose one $(\bar{\phi}, \bar{\xi}, \bar{\eta})$. If $\dim G=1$, we understand that $\bar{\phi}$ is a trivial operator. By making use of a connection form w on P and the structure tensors (ϕ, ξ, η) of M, one may see intuitively that P has a number of almost complex structures. Namely $J(w, a, b)$ is defined as follows

$$\pi_{p}(JX)=\phi_{\pi p}\pi_{p}X+\{a(p)\cdot\bar{\eta}_{e}(w_{p}X)+b(p)\cdot\eta_{\pi p}(\pi_{p}X)\}\xi_{\pi p}\,, \tag{3.1}$$

$$w_{p}(JX)=\bar{\phi}_{e}(w_{p}X)-\{b(p)\cdot\bar{\eta}_{e}(w_{p}X)+a^{-1}(p)(1+b^{2}(p))\eta_{\pi p}(\pi_{p}X)\}\bar{\xi}_{e}\,, \tag{3.2}$$

for any $X\in\mathfrak{X}P$ and $p\in P$, where a and b are arbitrary scalar fields such that a does not vanish on whole P. The most standard is, of course, $J(w, 1, 0)$. We investigate the integrability conditions of $J(w, a, b)$ for constant a, b. For a fundamental vector field A° and a horizontal lift u^{*} we have by definition

$$JA^{\circ}=(\bar{\phi}A)^{\circ}+a\alpha\xi^{*}-b\alpha\bar{\xi}^{\circ}\,, \tag{3.3}$$

$$Ju^{*}=(\phi u)^{*}+b\lambda\xi^{*}-a^{-1}(1+b^{2})\lambda\bar{\xi}^{\circ}\,, \tag{3.4}$$

where we have put $\bar{\eta}(A)=\alpha$, $\eta_{\pi p}(u)=\lambda_{p}$.

A. *Integrability*. Similarly we put $\bar{\eta}(B)=\beta$ and $\eta_{\pi p}(v)=\mu_{p}$, and we can assume that λ, μ are constant so far as the computations below are concerned.

$$\begin{aligned}\pi N(A^{\circ}, B^{\circ})&=a\bar{\eta}([\bar{\phi}A, B)+[A, \bar{\phi}B])\xi\\&\quad-ab\bar{\eta}(\alpha[\bar{\xi}, B]+\beta[A, \bar{\xi}])\xi\\&=a\bar{S}_{1}(A, B)\xi+ab\{\alpha\bar{S}_{3}(B)-\beta\bar{S}_{3}(A)\}\xi\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}wN(A^{\circ}, B^{\circ})&=\bar{S}(A, B)+b\bar{S}_{1}(A, B)\bar{\xi}+b\{\alpha\bar{S}_{2}(B)-\beta\bar{S}_{2}(A)\}\\&\quad-b^{2}\{\alpha\bar{S}_{3}(B)-\beta\bar{S}_{3}(A)\}\bar{\xi}\,,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\pi N(A^{\circ}, u^{*})&=-a\alpha S_{2}(u)-(1+b^{2})\lambda\bar{S}_{3}(A)\xi-ab\alpha S_{3}(u)\xi\\&\quad-a^{2}\alpha\bar{\eta}\Omega(\xi^{*}, u^{*})\cdot\xi\,,\end{aligned} \tag{3.7}$$

$$\begin{aligned}wN(A^{\circ}, u^{*})&=-a^{-1}(1+b^{2})\lambda\bar{S}_{2}(A)+a^{-1}(1+b^{2})b\lambda\bar{S}_{3}(A)\bar{\xi}\\&\quad+\alpha(1+b^{2})S_{3}(u)\bar{\xi}+a\alpha\Omega(\xi^{*}, (\phi u)^{*})\\&\quad-a\alpha\bar{\phi}\Omega(\xi^{*}, u^{*})+ab\alpha\bar{\eta}\Omega(\xi^{*}, u^{*})\cdot\bar{\xi}\,,\end{aligned} \tag{3.8}$$

$$\pi N(u^*, v^*) = S(u, v) - bS_1(u, v)\xi + b\{\mu S_2(u) - \lambda S_2(v)\}$$
$$- b^2\{\lambda S_3(v) - \mu S_3(u)\}\xi$$
$$- a\bar{\eta}\{\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*)\}\xi$$
$$- ab\bar{\eta}\{\Omega(\lambda\xi^*, v^*) + \Omega(u^*, \mu\xi^*)\}\xi, \tag{3.9}$$

$$wN(u^*, v^*) = -\Omega(u^*, v^*) + \Omega((\phi u)^*, (\phi v)^*)$$
$$+ b\{\Omega(\lambda\xi^*, (\phi v)^*) + \Omega((\phi u)^*, \mu\xi^*)\}$$
$$- \bar{\phi}\{\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*)\}$$
$$- b\bar{\phi}\{\Omega(\lambda\xi^*, v^*) + \Omega(u^*, \mu\xi^*)\}$$
$$+ b\bar{\eta}\{\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*) + b\Omega(\lambda\xi^*, v^*)$$
$$+ b\Omega(u^*, \mu\xi^*)\}\bar{\xi} + a^{-1}(1+b^2)S_1(u, v)\bar{\xi}$$
$$+ a^{-1}(1+b^2)b\{\lambda S_3(v) - \mu S_3(u)\}\bar{\xi}. \tag{3.10}$$

We suppose that S and $\bar{S}$ vanish, and J is integrable, then we have from (3.7)

$$\bar{\eta}\Omega(\xi^*, v^*) = 0. \tag{3.7'}$$

From (3.8), (3.9) and (3.7)′, we get

$$\Omega(\xi^*, (\phi v)^*) - \bar{\phi}\Omega(\xi^*, v^*) = 0, \tag{3.8'}$$

$$\bar{\eta}\{\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*)\} = 0. \tag{3.9'}$$

By (3.7)′, (3.8)′, (3.9)′ and (3.10), we obtain

$$\Omega(u^*, v^*) - \Omega((\phi u)^*, (\phi v)^*) + \bar{\phi}\{\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*)\} = 0. \tag{3.11}$$

Conversely if $S = \bar{S} = 0$ and (3.11) hold, then J is integrable.

PROPOSITION 3.1. *The principal fiber bundle $P(M, G, \pi)$ over an almost contact manifold M, on whose structure group G of odd dimension we fix a left invariant almost contact structure, admits an almost complex structure $J(w, a, b)$ depending upon a connection form w and scalar fields a $(\neq 0)$ and b on P. When a and b are constant and both almost contact structures are normal, $J(w, a, b)$ is integrable if and only if Ω satisfies* (3.11). *In particular, if Ω is of type* (1,1), *then J is integrable.*

To prove the last statement, we note that the condition

$$\Omega(JX, Y) + \Omega(X, JY) = 0 \tag{3.12}$$

for $X, Y \in \mathfrak{X}P$ is equivalent to

$$\Omega((\phi u)^*, v^*) + \Omega(u^*, (\phi v)^*) = 0. \tag{3.13}$$

Because, if we set $X = u^*$, $Y = A^\circ$ in (3.12) we have $\Omega(u^*, \xi^*) = 0$. Next we put $X=u^*$, $Y=v^*$, then (3.13) follows. Converse may be verified also.

REMARK 1. If $\dim G=1$ and the almost contact structure of M is normal, then J for constant $a\neq 0$, b is integrable if and only if Ω is of type (1,1) ([**11**]).

REMARK 2. Any reductive, for example compact, odd dimensional Lie group has a left invariant normal almost contact structure ([**6**]).

PROPOSITION 3.2. *Suppose that the almost contact structure of M satisfies $S_1=0$. Then $J(w, a, 0)$, a being non-zero constant, is integrable if and only if $S=\bar{S}=0$ and* (3.11) *are satisfied.*

PROOF. *Sufficiency* is contained in Proposition 3.1. *Necessity*: As $b=0$, $\bar{S}=0$ follows from (3.6). Hence by (3.7) and $\eta\cdot S_2(u)=S_1(u, \xi)=0$ we get $\bar{\eta}\Omega(\xi, u^*)=0$. Furthermore from (3.10) we get (3.11). If we replace u of (3.11) by ϕu and operate $\bar{\eta}$, we have (3.9)$'$. Finally from (3.9) we have $S=0$.

REMARK 3. If the almost contact structure is derived from a contact structure, $S_1=0$ holds always ([**16**]).

B. *The actions of A° and u^* to J.*

PROPOSITION 3.3. *Let A°, $A\in\mathfrak{G}$, be a fundamental vector field and consider $J(w, a, b)$ for constants $a\neq 0$ and b. A° is almost analytic with respect to J if and only if A leaves $\bar{\xi}$ and $\bar{\phi}$ invariant.*

PROOF. All we have to check is the value of $\mathfrak{L}(A^\circ)J\cdot B^\circ$ and $\mathfrak{L}(A^\circ)J\cdot u^*$ at a point $p\in P$,

$$\pi_p(\mathfrak{L}(A^\circ)J\cdot B^\circ)=a\{\mathfrak{L}(A)\bar{\eta}\cdot B\}\xi_{\pi p}\,, \tag{3.14}$$

$$w\mathfrak{L}(A^\circ)J\cdot B^\circ=\mathfrak{L}(A)\bar{\phi}\cdot B-b\{\mathfrak{L}(A)\bar{\eta}\cdot B\}\bar{\xi}-b\bar{\eta}(B)\cdot\mathfrak{L}(A)\bar{\xi}\,, \tag{3.15}$$

$$\pi\mathfrak{L}(A^\circ)J\cdot u^*=0\,, \tag{3.16}$$

$$w_p(\mathfrak{L}(A^\circ)J\cdot u^*)=-a^{-1}(1+b^2)\eta_{\pi p}(u)\cdot\mathfrak{L}(A)\bar{\xi}\,. \tag{3.17}$$

REMARK 4. If $\dim G=1$, any fundamental vector field is of the form $\alpha\bar{\xi}^\circ$ for constant α and is almost analytic with respect to J in Proposition 3.3.

PROPOSITION 3.4. *In P we suppose that the curvature form satisfies*

$$\Omega(u^*, v^*)+\bar{\phi}\Omega(u^*, (\phi v)^*)=0 \tag{3.18}$$

for $u, v\in\mathfrak{X}M$. Then the necessary and sufficient condition that the lift x^*, $x\in\mathfrak{X}M$, is almost analytic with respect to $J(w, a, b)$ for constants $a\neq 0$ and b, is that x leaves ξ and ϕ invariant.

PROOF. This is a consequence of the following computations

$$\pi\mathfrak{L}(x^*)J\cdot A^\circ=a\bar{\eta}(A)[x, \xi]\,, \tag{3.19}$$

$$w\mathfrak{L}(x^*)J\cdot A^\circ=-a\bar{\eta}(A)\Omega(x^*, \xi^*)\,, \tag{3.20}$$

$$\pi\mathfrak{L}(x^*)J\cdot u^* = \mathfrak{L}(x)\phi\cdot u + b\eta(u)[x,\ \xi] + a\bar{\eta}\Omega(x^*,\ u^*)\cdot\xi + b(\mathfrak{L}(x)\eta\cdot u)\xi\,, \tag{3.21}$$

$$w_p(\mathfrak{L}(x^*)J\cdot u^*) = a^{-1}(1+b^2)(\mathfrak{L}(x)\eta_{\pi p}\cdot u)\bar{\xi} - \Omega(x^*,\ (\phi u)^*) + \bar{\phi}\Omega(x^*,\ u^*) - b\eta_{\pi p}(u)\Omega(x^*,\ \xi^*) - b\bar{\eta}\Omega(x^*,\ u^*)\cdot\bar{\xi}\,. \tag{3.22}$$

REMARK 5. The condition (3.11) is weaker than (3.18). Hence, if the almost contact structures of M and G are normal and the curvature form of the connection satisfies (3.18), then the lift x^* of a vector field x on M which leaves ξ and ϕ invariant, is an analytic vector field. Consequently $Jx^* = (\phi x)^* + b\eta(x)\xi^* - a^{-1}(1+b^2)\eta(x)\bar{\xi}^\circ$ is also analytic. As for the analytic fundamental vector field A°, similarly $(\bar{\phi}A)^\circ + a\bar{\eta}(A)\xi^* - b\bar{\eta}(A)\bar{\xi}^\circ$ is analytic, but in this case each component is itself analytic.

REMARK 6. If the holonomy algebra is odd dimensional, then (3.18) means that the curvature form Ω vanishes.

C. *Almost hermitian metrics.* These almost complex structures have almost hermitian metrics, however, for brevity we restrict ourselves to $J(w,\ a,\ 0)$ for non-zero scalar a. Then natural metric $L = L(w,\ a,\ g,\ \bar{g},\ k)$ may be

$$L_p(X,\ Y) = k(p)g_{\pi p}(\pi_p X,\ \pi_p Y) + a^2(p)k(p)\bar{g}_e(w_p X,\ w_p Y)\,,$$

g and $\bar{g}$ denoting the associated metrics of $(\phi,\ \xi,\ \eta)$ and $(\bar{\phi},\ \bar{\xi},\ \bar{\eta})$ respectively and k a positive scalar on P. Now we have the fundamental 2-form

$$\Xi(X,\ Y) = L(X,JY) = -L(JX,\ Y)\,.$$

In the case of $\dim G = 1$, Y. Ogawa proved in [**11**] that, the almost hermitian structure $\{J(w,1,0),\ L(w,1,g,1,k)\}$ of P defined by an integrable connection w and an almost contact structure associated with a contact structure η, is almost kählerian if and only if $d\log k = -w$ i.e. $dk = -kw$. From the last condition one sees that, if P is almost kählerian, 1-dimensional Lie group G is not a toroidal group but a group of real numbers, since $\mathfrak{L}(\bar{\xi}^\circ)\log k = -1$. Further, each fiber over $x \in M$ has a unique point $\tilde{x}$ satisfying $k(\tilde{x}) = 1$. Then $\theta : x \rightarrow \tilde{x}$ is a horizontal cross section and $\theta(M)$ can be considered as a contact Riemannian manifold by $\theta^{-1*}\eta$ and $\theta^{-1*}g$. Thus, $P \approx M\times R$ and $\theta(M) \approx (M,\ 0)$, $k(x,\ t) = e^{-t}$. Consequently, we have $dt = w$ and by Y. Tashiro's result [**18**] $\theta(M)$ is a totally umbilical hypersurface of P.

Now, we clarify the Y. Ogawa's result.

PROPOSITION 3.5. *If the almost hermitian structure* $(J(w,1,0),\ L(w,1,g,1,k))$ *of* $P(\dim G = 1)$ *is defined by the integrable connection* w. *Then from the two of the next conditions, the remaining one follows*

(i) η: *contact form*,

(ii) (J, L): *almost kählerian*,

(iii) $dk = -kw$.

We have only to prove (ii), (iii)→(i). For u, v such that $\eta(u)$ and $\eta(v)$ are constant, we have

$$(d\Xi)(u^*, v^*, \bar{\xi}^\circ) = u^*k \cdot \eta(v) - v^*k \cdot \eta(u) + \bar{\xi}^\circ k \cdot g(u, \phi v) - kg([u, v], \xi).$$

By $u^*k = i(u^*)dk = 0$, $\bar{\xi}^\circ k = -k$ and $\eta([u, v]) = -d\eta(u, v)$, we get

$$g(u, \phi v) = d\eta(u, v).$$

This means that η is a contact form on M.

4. Almost contact structures of M induced from an almost complex structure of $P(M, G)$, ($\dim G = 1$).

In the principal fiber bundle P with a 1-dimensional Lie group G, let A° and w stand for the fundamental vector field and connection form such that $w(A^\circ) = 1$, considering w as scalar valued. We assume that P has an almost complex structure J which is invariant under the operation of G. Define $\xi = \pi J A^\circ$, then ξ is a vector field on M which does not vanish everywhere on M. Therefore it is possible to find a 1-form η such that $\eta(\xi) = 1$. Finally we define ϕ as follows

$$\phi u = \pi J(u^* - \eta(u)\xi^*) - \eta\pi J(u^* - \eta(u)\xi^*) \cdot \xi \tag{4.1}$$

for $u \in \mathfrak{X}M$, where u^* and ξ^* are the lifts with respect to w and $\xi^* = JA^\circ - w(JA^\circ)A^\circ$. Then ϕ satisfies $\phi\xi = 0$, $\eta\phi = 0$ and

$$(\phi u)^* = J(u^* - \eta(u)\xi^*) - wJ(u^* - \eta(u)\xi^*)A^\circ - \eta\pi J(u^* - \eta(u)\xi^*) \cdot \xi^*. \tag{4.2}$$

Replacing u of (4.1) by ϕu and using (4.2) we have

$$\phi^2 u = -u + \eta(u)\xi.$$

This proves

PROPOSITION 4.1. *Let G be a 1-dimensional Lie group. If J is a G-invariant almost complex structure in P, then M has an almost contact structure ϕ, ξ, η defined above.*

From (4.2) it follows that

$$Ju^* = (\phi u)^* + \eta\pi J u^* \cdot \xi^* + wJu^* \cdot A^\circ. \tag{4.3}$$

And as $JA^\circ = \xi^* + w(JA^\circ) \cdot A^\circ$, if $\eta\pi J u^*$ and wJu^* are proportional to $\eta(u)$, we have

$$\eta\pi Ju^* = -w(JA^\circ)\eta(u)\,,$$

$$wJu^* = -(1+w^2(JA^\circ))\eta(u)\,.$$

Therefore

PROPOSITION 4.2. *Notations being as above, if $\eta\pi Ju^*$ and wJu^* are proportional to $\eta(u)$. The relation between J and ϕ, ξ, η is $J(w, 1, -w(JA^\circ))$. Moreover if wJA° is constant, then we can refer the integrability condition to Proposition* 3.1, 3.2.

5. An almost complex structure of TM depending only on an almost contact structure of M.

For a coordinate neighborhood U of M with coordinates $(s^i : i=1, 2, \cdots, 2n+1)$, we may take $\pi^{-1}(U) \approx U \times E$ as a coordinate neighborhood of TM, E denoting $(2n+1)$-dimensional Euclidean space, with coordinates (s^i, z^j), where z^j are components of a tangent vector with respect to the natural frame. If $U(s^i) \cap U'(s'^i)$ is non-empty, the coordinate transformation of $(\pi^{-1}U) \cap \pi^{-1}(U')$ is

$$s'^i = s'^i(s^1, s^2, \cdots, s^{2n+1})\,,$$

$$z'^j = \frac{\partial s'^j}{\partial s^i} z^i\,.$$

And we have the matrix

$$T_U^{U'} = \begin{pmatrix} \dfrac{\partial s'^i}{\partial s^j} & 0 \\ \dfrac{\partial^2 s'^i}{\partial s^j \partial s^k} z^k & \dfrac{\partial s'^i}{\partial s^j} \end{pmatrix}. \tag{5.1}$$

Let $X=(X^i, X^{m+i})$, $m=2n+1$, be a vector field on TM, then we have

$$(\pi_z X)^i = X^i\,,$$

$$(K_z X)^j = X^{m+j} + z^r \Gamma^j_{rt} X^t\,,$$

where Γ^j_{rt} is the connection coefficients. By the definition of J in § 2

$$[\pi_z(JX)]^i = \phi^i_j X^j + \eta_j (X^{m+j} + z^r \Gamma^j_{rt} X^t)\xi^i\,,$$

$$[K_z(JX)]^i = \phi^i_j (X^{m+j} + z^r \Gamma^j_{rt} X^t) - \eta_j X^j \xi^i\,.$$

From this we have the local expression of J

$$\begin{aligned} J^i_j &= \phi^i_j + z^r \eta_l \Gamma^l_{rj} \xi^i\,, \\ J^i_{m+j} &= \xi^i \eta_j\,, \\ J^{m+i}_j &= \phi^i_l \Gamma^l_{rj} z^r - z^p z^q \xi^t \eta_r \Gamma^r_{pj} \Gamma^i_{qt} - \phi^t_j \Gamma^i_{rt} z^r - \xi^i \eta_j\,, \end{aligned} \tag{5.2}$$

$$J^{m+i}_{m+j}=\phi^i_j-z^r\Gamma^i_{rt}\xi^t\eta_j\,.$$

We want to get an almost complex structure which is independent of the linear connection. For this purpose, suppose tentatively that the linear connection Γ is symmetric (ϕ, ξ, η)-connection (such a connection does not always exist, cf. [**16**]), then (5.2) turns

$$J^i_j=\phi^i_j+z^r\xi^i\frac{\partial\eta_j}{\partial s^r}\,,$$

$$J^i_{m+j}=\xi^i\eta_j\,,$$

$$J^{m+i}_j=z^r\frac{\partial\phi^i_j}{\partial s^r}+z^tz^r\frac{\partial\xi^i}{\partial s^t}\frac{\partial\eta_j}{\partial s^r}-\xi^i\eta_j\,,$$

$$J^{m+i}_{m+j}=\phi^i_j+z^r\frac{\partial\xi^i}{\partial s^r}\eta_j\,,$$

because, for example, $\xi^t\Gamma^i_{qt}=\Gamma^i_{tq}\xi^t=-\dfrac{\partial\xi^i}{\partial s^q}$. These suggest us the possibility of definition of Φ_U in $\pi^{-1}(U)$ by

$$\Phi_U=\begin{pmatrix}\phi^i_j+z^r\xi^i\dfrac{\partial\eta_j}{\partial s^r} & \xi^i\eta_j\\ z^r\dfrac{\partial\phi^i_j}{\partial s^r}+z^tz^r\dfrac{\partial\xi^i}{\partial s^t}\dfrac{\partial\eta_j}{\partial s^r}-\xi^i\eta_j & \phi^i_j+z^r\dfrac{\partial\xi^i}{\partial s^r}\eta_j\end{pmatrix}. \tag{5.3}$$

And as the following relation is demonstrated directly

$$T^{U'}_U\Phi_UT^U_{U'}=\Phi_{U'} \tag{5.4}$$

in $(\pi^{-1}U)\cap(\pi^{-1}U')$, we see that Φ is a tensor field on TM. Further, $\Phi\Phi X=-X$ for any $X\in\mathfrak{X}TM$ is also verified. Thus we obtain

PROPOSITION 5.1. *The tangent bundle of an almost contact manifold M has an almost complex structure Φ depending on the almost contact structure of M. The necessary and sufficient condition that Φ is integrable is normality of the almost contact structure.*

Proof of the last statement follows from the next lemma. Namely if $S=0$, we see that Φ is integrable. Conversely if Φ is integrable, by (5.5) and (5.8) we get $S=0$.

LEMMA. *Let Ψ be the Nijenhuis tensor for Φ, and N^i_{jk}, N_{jk}, N^i_j and N_j be the components of S, S_1, S_2 and S_3 respectively (see* [**16**]*). Then the followings are valid.*

$$\Psi^i_{jk}=N^i_{jk}+2\xi^i\eta_{[j}N_{k]}+z^t(2\partial_t\eta_{[k}N^i_{j]}-\xi^i\partial_tN_{jk}+2z^s\xi^iN_{[j}\partial_{|s|}\eta_{k]})\,, \tag{5.5}$$

$$\Psi^i_{j\,m+k}=\eta_kN^i_j+\xi^iN_{kj}+z^t\xi^i(\partial_tN_j\eta_k-N_k\partial_t\eta_j)\,, \tag{5.6}$$

(5.7) $$\Psi^{m+i}_{j\,k} = \xi^i N_{jk} + 2\eta_{[j}N^i_{k]} + z^t(\partial_t N^i_{jk} - 2N_{[j}\eta_{k]}\partial_t\xi^i + 2\xi^i\partial^t\eta_{[j}N_{k]} - z^p\partial_p\xi^i\partial_t N_{jk} + 2z^p z^q\partial_p\xi^i\partial_q\eta_{[k}\partial_{|t|}N_{j]} - 2z^p\partial_p\eta_{[j}\partial_{|p|}N^i_{k]}) ,$$

(5.8) $$\Psi^i_{m+j\; m+k} = 2\xi^i\eta_{[k}N_{j]} ,$$

(5.9) $$\Psi^{m+i}_{m+j\; k} = N^i_{jk} + \xi^i\eta_j N_k + z^t(\partial_t\eta_k N^i_j - \eta_j\partial_t N^i_k - \partial_t\xi^i N_{jk} - z^p\partial_n\xi^i\eta_j\partial_t N_k + z^p\partial_p\xi^i\partial_t\eta_k N_j) ,$$

(5.10) $$\Psi^{m+i}_{m+j\; m+k} = 2N_{[j}^{\;i}\eta_{k]} + 2z^t\partial_t\xi^i\eta_{[k}N_{j]} ,$$

where $2\eta_{[j}N_{k]} = \eta_j N_k - \eta_k N_j$, *etc., and* $\partial_r = \partial/\partial s^r$.

PROOF. Because of complexity and lengthiness of the calculations, we give here only for (5.5). We use the notations $\partial_{m+r} = \partial/\partial z^r$. Then

$$\partial_{[r}\Phi^i_{j]} = \partial_{[r}\phi^i_{j]} + z^t\partial_t\eta_{[j}\partial_{r]}\xi^i + \xi^i z^t\partial_t(\partial_{[r}\eta_{j]}) .$$

And so

(5.11) $$2\Phi^r_k\partial_{[r}\Phi^i_{j]} = A^i_{jk} + B^i_{jk} ,$$

where

$$A^i_{jk} = 2\phi^r_k\partial_{[r}\phi^i_{j]} + \phi^r_k z^t\partial_t\eta_j\partial_r\xi^i - \phi^r_k z^t\partial_t\eta_r\partial_j\xi^i + 2\phi^r_k\xi^i z^t\partial_t(\partial_{[r}\eta_{j]}) ,$$

$$B^i_{jk} = 2\xi^r\partial_{[r}\phi^i_{j]}z^p\partial_p\eta_k + \xi^r\partial_r\xi^i\partial_r z^t\partial_t\eta_j z^p\partial_p\eta_k - z^t z^p\xi^r\partial_t\eta_r\partial_p\eta_k\partial_j\xi^i + 2z^p z^t\xi^i\partial_p\eta_k\xi^r\partial_t(\partial_{[r}\eta_{j]}) .$$

On the other hand,

$$2\partial_{[m+r}\Phi^i_{j]} = 2\xi^i\partial_{[r}\eta_{j]} - \partial_j\xi^i\eta_r ,$$

and hence

(5.12) $$2\Phi^{m+r}_k\partial_{[m+r}\Phi^i_{j]} = C^i_{jk} + D^i_{jk} + E^i_{jk} ,$$

where

$$C^i_{jk} = 2\xi^i z^t\partial_t\phi^r_k\partial_{[r}\eta_{j]} - \partial_j\xi^i\eta_r z^t\partial_t\phi^r_k ,$$

$$D^i_{jk} = 2\xi^i z^t z^p\partial_t\xi^r\partial_p\eta_k\partial_{[r}\eta_{j]} - z^t z^p\partial_t\eta_k\partial_j\xi^i\partial_p\xi^r\eta_r ,$$

$$E^i_{jk} = -2\xi^i\eta_k\xi^r\partial_{[r}\eta_{j]} + \eta_k\partial_j\xi^i .$$

As Ψ^i_{jk} is given by

$$\Psi^i_{jk} = 2A^{\;i}_{[jk]} + 2B^{\;i}_{[jk]} + 2C^{\;i}_{[jk]} + 2D^{\;i}_{[jk]} + 2E^{\;i}_{[jk]} ,$$

we simplify the right hand side. Writing, for example, the first term of A^i_{jk} by $A^i_{jk}(1)$, we have

(5.13) $$2A^{\;i}_{[jk]}(1) + 2E^{\;i}_{[jk]}(2) = N^i_{jk} .$$

Following relations may be verified also.

$$B^{\,i}_{[jk]}(2)=0\,, \tag{5.14}$$

$$B^{i}_{jk}(3)+D^{i}_{jk}(2)=0\,, \tag{5.15}$$

$$A^{i}_{jk}(3)+C^{\,i}_{jk}(2)=0\,, \tag{5.16}$$

$$E^{\,i}_{[jk]}(1)=\xi^{i}\eta_{[j}N_{k]}\,, \tag{5.17}$$

$$A^{\,i}_{[jk]}(2)+B^{\,i}_{[jk]}(1)=z^{t}\partial_{t}\eta_{[k}N^{i}_{j]}\,, \tag{5.18}$$

$$B^{\,i}_{[jk]}(4)+D^{\,i}_{[jk]}(1)=\xi^{i}z^{t}z^{p}\partial_{t}\eta_{[k}\partial_{|p|}N_{j]}\,, \tag{5.19}$$

$$2A^{\,i}_{[jk]}(4)+2C^{\,i}_{[jk]}(1)=2\xi^{i}z^{t}\partial_{t}[\phi^{\,r}_{[k}(\partial_{|r|}\eta_{j]}-\partial_{j}\eta_{r})]=-\xi^{i}z^{t}\partial_{t}N_{jk}\,. \tag{5.20}$$

Summing up (5.13)$\sim$(5.20), we have proved (5.5).

It has shown that an almost contact manifold has a symmetric (ϕ, ξ, η)-connection ∇ if and only if η is closed and the structure is normal ([**16**]). Therefore if we notice that, when J is defined by this connection ∇, J coincides with Φ, we get from Proposition 2.1 and 5.1 the following

COROLLARY. *Let M be a normal almost contact manifold such that η is closed, then any symmetic (ϕ, ξ, η)-connection satisfies the following*

$$R(u, v)-R(\phi u, \phi v)+\phi\cdot[R(\phi u, v)+R(u, \phi v)]=0$$

for any $u, v\in\mathfrak{X}M$.

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