J. Math. Soc. Japan
Vol. 17, No. 1, 1965

Extension of certain subfields to coefficient fields in commutative algebras

By J. N. MORDESON and B. VINOGRADE*)

(Received June 22, 1964)

Introduction.

Let A be a commutative algebra with identity over a subfield K. Let N be a maximal ideal of A and let g be the natural K-homomorphism of A onto A/N (K and gK identified). Denote A/N by F_0. Then, consistent with the usual meaning of the term coefficient field, we define a K-coefficient field as a subfield F of A such that $F \supseteqq K$ and $gF = F_0$.

The existence of coefficient fields for complete local algebras is assured by well known results [**3**, p. 106], but as simple examples show, the existence of K-coefficient fields is not a consequence. In Theorem 1, we give a necessary and sufficient condition for the stepwise extension of suitable subfields of A to K-coefficient fields when K has characteristic $p \neq 0$. These suitable subfields are situated in $A^{p^e} = \{a^{p^e} \mid a \in A\}$, e a positive integer, analogous to the way a K-coefficient field would be situated in A. This result applies of course to quasi-local algebras. In Theorem 2, we note an extension to the case of arbitrary characteristic of a result in [**2**] which can also be obtained by a modification of the proof of Corollary 2 in [**4**, p. 280], namely, the existence of a K-coefficient field when A is quasi-local, N is nil and F_0 has a separating transcendence basis over K. This theorem reduces the case of any quasi-local algebra with N nil to the case to which Theorem 1 applies.

1. By a counterimage $M \subseteqq A$ of a set $M_0 \subseteqq F_0$, we mean a set M such that $gM = M_0$ and $g \mid M$ is one-one. Unless otherwise specified, e always denotes a fixed positive integer. Let $M^{p^e} = \{m^{p^e} \mid m \in M\}$, and similarly for other prime powers of sets appearing hereafter. By the symbol $E(M)$ we mean the set of all polynomials in elements from M with coefficients from a field E.

LEMMA 1. *Suppose there exists a field $E \subseteqq A$ with the same identity as A such that $gE = F_0^{p^e}$. Then a counterimage $M \subseteqq A$ of a p-basis M_0 of F_0,* [**3**, p. 107], *is such that $M^{p^e} \subseteqq E$ if and only if $E(M)$ is a field. If such an M exists, $gE(M) = F_0$.*

PROOF. Suppose $M^{p^e} \subseteqq E$. Well order M and put $M_j = \{m_\alpha \mid \alpha < j\}$ for an ordinal j. Suppose $E(M_j)$ is a field for some ordinal j. Now m_j satisfies

*) This paper was supported in part by NSF grant G-23418.

$x^{p^e}-d=0$, $d\in E(M_j)$. If this polynomial is reducible over $E(M_j)$, then there exists a finite subset $B_0\subseteqq gM_j$ of r elements say and a positive integer f such that $gm_j^{p^{e-f}}\in F^{p^e}(B_0)$. But this contradicts the degree relation $[F_0^{p^e}(B_0, gm_j): F_0^{p^e}]=p^{e(r+1)}$. The case for m_1 is similar and since the union of an ascending sequence of fields under inclusion is a field, it follows by transfinite induction that $E(M)$ is a field.

On the other hand, suppose $E(M)$ is a field. For each $m\in M$, there exist $b\in E$, $n\in N$ such that $m^{p^e}=b+n$. However, $n=0$ since $E(M)\cap N=(0)$.

Finally, if M exists, $gE(M)=F_0$ since $F_0=F_0^{p^e}(M_0)$. Q. E. D.

LEMMA 2. *If there exists a field $E\subseteqq A^{p^e}$ with the same identity as A and such that $gE=F_0^{p^e}$, then E can be extended to a field F such that $gF=F_0$.*

PROOF. Clearly, $N^{p^e}\subseteqq A^{p^e}\cap N$. Let $a^{p^e}\in A^{p^e}\cap N$. Then $g(a)^{p^e}=0$ which implies $g(a)=0$ since F_0 is a field. Thus $a\in N$ and $a^{p^e}\in N^{p^e}$. Hence, $A^{p^e}\cap N=N^{p^e}$. Now, let M_0 be a p-basis of F_0 and M a counterimage of M_0. Since, by hypothesis and the preceding remark, $A^{p^e}=E+N^{p^e}$, $m^{p^e}=b+n^{p^e}$ for all $m\in M$, where $b\in E$. Thus, the subset $\{m-n\}=M'$ of A is a counterimage of M_0 such that $M'^{p^e}\subseteqq E$. The result now follows from Lemma 1. Q. E. D.

A similar result has already been noted by Bray [**1**]. Closely related results for complete local rings can be found in [**3**, Ex., p. 112].

THEOREM 1. *Suppose that for some positive integer i there exists a field E such that $F_0^{p^i}\cap K\subseteqq E\subseteqq A^{p^i}$ and $gE=F_0^{p^i}$. Then E can be extended to a field E' such that $F_0^{p^{i-1}}\cap K\subseteqq E'\subseteqq A^{p^{i-1}}$ and $gE'=F_0^{p^{i-1}}$ if and only if* (i) *$g(A^{p^{i-1}}\cap K)=F_0^{p^{i-1}}\cap K$ and* (ii) *$E(A^{p^{i-1}}\cap K)$ is a field.*

PROOF. Suppose (i) and (ii) hold. Choose a p-basis $G_0\cup M_0$ of $F_0^{p^{i-1}}$ over $F_0^{p^i}$ so that $F_0^{p^i}(G_0)=F_0^{p^i}(F_0^{p^{i-1}}\cap K)$. Identifying $A^{p^{i-1}}$ with A and A^{p^i} with A^{p^e}, we have by Lemma 2 that there exist counterimages G and M of G_0 and M_0 respectively in $A^{p^{i-1}}$ such that $E(G\cup M)$ is a field and $gE(G\cup M)=F_0^{p^{i-1}}$. By (i), $g(A^{p^{i-1}}\cap K)=F_0^{p^{i-1}}\cap K$, and thus G can be chosen so that $E(G)\subseteqq E(A^{p^{i-1}}\cap K)$. It remains to be shown that $F_0^{p^{i-1}}\cap K\subseteqq E(G\cup M)$. Now g induces an isomorphism of $E(G)$ onto $F_0^{p^i}(G_0)=F_0^{p^i}(F_0^{p^{i-1}}\cap K)$. By (ii), $g|E(A^{p^{i-1}}\cap K)$ is one-one. Thus $E(G)$ and $E(A^{p^{i-1}}\cap K)$ are E-isomorphic. Since $E(A^{p^{i-1}}\cap K)$ is pure inseparable over E, it is not E-isomorphic to a proper subfield. Hence, $E(G)=E(A^{p^{i-1}}\cap K)$. Therefore, $F_0^{p^{i-1}}\cap K\subseteqq E(G\cup M)$.

On the other hand, suppose E can be extended to the field E'. (i) Since $E'\supseteqq F_0^{p^{i-1}}\cap K$, it follows that $A^{p^{i-1}}\cap K=F_0^{p^{i-1}}\cap K$. ($A^{p^{i-1}}\cap K$ is always in $F_0^{p^{i-1}}\cap K$.) (ii) Since $E'\supseteqq E(A^{p^{i-1}}\cap K)$, the latter is a field and is in fact isomorphic to $F_0^{p^i}(F_0^{p^{i-1}}\cap K)$. Q. E. D.

COROLLARY 1. *Suppose there exists a field E such that $A^{p^e}\cap K\subseteqq E\subseteqq A^{p^e}$ and $gE=F_0^{p^e}$. Then E can be extended to a K-coefficient field if $A^{p^i}\cap K=F_0^{p^i}\cap K$ and, in addition, either one of the following conditions holds for*

$i=1, \cdots, e$:

(i) *the pair* $F_0^{p^i}$, $F_0^{p^{i-1}} \cap K$ *is linearly disjoint over* $F_0^{p^i} \cap K$; *or*

(ii) *there exists a subset* G_{0i} *of a p-basis of* $F_0^{p^{i-1}}$ *over* $F_0^{p^i}$ *such that* $(F_0^{p^i} \cap K)(G_{0i}) = F_0^{p^{i-1}} \cap K$.

PROOF. (i) Suppose E_i is a field such that $A^{p^i} \cap K \subseteqq E_i \subseteqq A^{p^i}$ and $gE_i = F_0^{p^i}$. Since $gE_i(A^{p^{i-1}} \cap K) = F_0^{p^i}(F_0^{p^{i-1}} \cap K)$, it follows by application of the universal mapping theorem for tensor products that $g \mid E_i(A^{p^{i-1}} \cap K)$ is one-one. Thus, $E_i(A^{p^{i-1}} \cap K)$ is a field and from Theorem 1 it follows by induction that E can be extended to a K-coefficient field.

(ii) Suppose there exists a field E_i such that $A^{p^i} \cap K \subseteqq E_i \subseteqq A^{p^i}$ and $gE_i = F_0^{p^i}$. Since $(F_0^{p^i} \cap K)(G_{0i}) = F_0^{p^{i-1}} \cap K$, $E_i(G_{0i}) = E_i(A^{p^{i-1}} \cap K)$. Thus, $E_i(A^{p^{i-1}} \cap K)$ is a field. Q. E. D.

COROLLARY 2. *Suppose there exists a field E such that* $A^{p^e} \cap K \subseteqq E \subseteqq A^{p^e}$ *and* $gE = F_0^{p^e}$. *Then E can be extended to a K-coefficient field if either one of the following conditions holds:*

(i) F_0 *is separable over K; or*

(ii) *A has no nilpotent elements.*

PROOF. (i) Let $a_0^{p^i} \in F_0^{p^i} \cap K$ for some positive integer i. Then $a_0 \in K$ since F_0 has no pure inseparable elements over K. Thus, $a_0^{p^i} \in K^{p^i}$ and $a_0^{p^i} \in A^{p^i} \cap K$. Hence, $K^{p^i} \supseteqq F_0^{p^i} \cap K$ and $A^{p^i} \cap K \supseteqq F_0^{p^i} \cap K$. Thus, $K^{p^i} = F_0^{p^i} \cap K$ and $A^{p^i} \cap K = F_0^{p^i} \cap K$. Furthermore, $F_0^{p^{i-1}} \cap K$ is pure inseparable over $F_0^{p^i} \cap K$ and $F_0^{p^i}$ is separable over $F_0^{p^i} \cap K = K^{p^i}$. Hence the conditions in Corollary 1 (i) hold.

(ii) By Lemma 2, E can be extended to a field F such that $gF = F_0$. If $K \nsubseteq F$, then there exists $k \in K$, $k \notin F$ and $a \in F$ such that $ga = k$, $a - k \neq 0$ and $(a-k)^{p^e} = 0$ (since $F \supseteqq E \supseteqq K^{p^e}$), which is impossible by hypothesis. Q. E. D.

Let $g_i = g \mid A^{p^i}(A^{p^{i-1}} \cap K)$ and let (N^{p^i}) be the ideal in $A^{p^i}(A^{p^{i-1}} \cap K)$ generated by N^{p^i}.

REMARK 1. If A has a K-coefficient field F, then

(i) there exists a field E such that $A^{p^e} \cap K \subseteqq E \subseteqq A^{p^e}$ and $gE = F_0^{p^e}$;

(ii) $A^{p^i} \cap K = F_0^{p^i} \cap K$ for all positive integers i;

(iii) $\operatorname{Ker} g_i = (N^{p^i})$ for all positive integers i.

PROOF. (i) Take $E = F^{p^e}$ for any e.

(ii) Given $a_0 \in F_0$ such that $a_0^{p^i} \in K$, there exists $a \in F$ such that $ga = a_0$ and such that $a^{p^i} \in K$. Hence $A^{p^i} \cap K \supseteqq F_0^{p^i} \cap K$. (The inclusion $A^{p^i} \cap K \subseteqq F_0^{p^i} \cap K$ always holds.)

(iii) $F^{p^i}(A^{p^{i-1}} \cap K)$ and $F_0^{p^i}(F_0^{p^{i-1}} \cap K)$ are naturally isomorphic and thus there exists a homomorphism h_i of $A^{p^i}(A^{p^{i-1}} \cap K)$ onto $F^{p^i}(A^{p^i} \cap K)$ such that $\operatorname{Ker} h_i = \operatorname{Ker} g_i$ and $h_i \sum_j a_j^{p^i} k_j = \sum_j b_j k_j$, where $k_j \in A^{p^{i-1}} \cap K$, $b_j \in F^{p^i}$ and $h_i a_j^{p^i} = b_j$. Let $c = \sum a_j^{p^i} k_j \in \operatorname{Ker} h_i$. Then $0 = h_i c = h_i \sum (b_j + n_j^{p^i}) k_j = h_i \sum b_j k_j = \sum b_j k_j$ where $n_j^{p^i} \in N^{p^i}$. Thus, $c = \sum n_j^{p^i} k_j \in (N^{p^i})$. Hence $\operatorname{Ker} g_i \subseteqq (N^{p^i})$. Clearly,

$(N^{p^i}) \subseteqq \operatorname{Ker} g_i$. Q. E. D.

If the pairs A^{p^i}, $A^{p^{i-1}} \cap K$ are linearly disjoint over $A^{p^i} \cap K$, and if $\operatorname{Ker} g_i = (N^{p^i})$, then the pairs $F_0^{p^i}$, $F_0^{p^{i-1}} \cap K$ are linearly disjoint over $F_0^{p^i} \cap K$ by application of elementary properties of tensor products. Thus, under these conditions, A has a K-coefficient field by Corollary 1 (i) when the field E in Corollary 1 exists.

REMARK 2. If (i) $N^{p^e} = (0)$, or (ii) $F_0^{p^e} \subseteqq K$ and $A^{p^e} \cap K = F_0^{p^e} \cap K$, or (iii) $N^{p^e} = N^{p^{e+1}}$, $A^{p^e} \cap K$ is a field and F_0 is pure inseparable over K, or (iv) the ideal R of all nilpotent elements of A is such that $R^{p^e} = (0)$, $A^{p^i} \cap K = F_0^{p^i} \cap K$ for all positive integers i, and F_0 is pure inseparable over K, then there exists a field E such that $A^{p^e} \cap K \subseteqq E \subseteqq A^{p^e}$ and $gE = F_0^{p^e}$. (Here we allow $e = 0$.)

PROOF. (i) $N^{p^e} = (0)$ implies A is quasi-local which then implies A^{p^e} is a field.

(ii) Since $F_0^{p^e} \subseteqq K$ and $A^{p^e} \cap K = F_0^{p^e} \cap K$ we can take $E = A^{p^e} \cap K$.

(iii) By Zorn's lemma there exists a maximal field E in A^{p^e} and containing $A^{p^e} \cap K$. If $gE \subset F_0^{p^e}$ (strict inclusion), then there exists $a_0 \in F_0^{p^e}$, $a_0 \notin gE$, $a \in A^{p^e}$, $a \notin E$ such that $ga = a_0$. Since F_0 is pure inseparable over K, there exists a smallest positive integer f such that $a_0^{p^f} \in gE$. Thus, $a^{p^f} = b + n^{p^e}$ where $b \in E$ and $n^{p^e} \in N^{p^e}$. Since $N^{p^e} = N^{p^{e+1}}$, there exists $n_1 \in N$ such that $n_1^{p^{e+f}} = n^{p^e}$. Thus, $E(a - n_1^{p^e})$ is a field which contradicts the maximality of E.

(iv) Let E be a maximal field such that $A^{p^e} \cap K \subseteqq E \subseteqq A^{p^e}$. If $gE \subset F_0^{p^e}$, then there exists $a_0 \in F_0^{p^e}$, $a_0 \notin gE$, $a \in A^{p^e}$, $a \notin E$ such that $ga = a_0$. Since F_0 is pure inseparable over K, there exists a smallest positive integer f and a positive integer h such that $a_0^{p^f} \in gE$ and $a_0^{p^{f+h}} \in K$. Thus, $a^{p^f} = b + n^{p^e}$ where $b \in E$ and $n^{p^e} \in N^{p^e}$. By hypothesis, a can be chosen so that $a^{p^{f+h}} \in K$. Thus, $a^{p^{f+h}} = b^{p^h} + n^{p^{e+h}}$ and since $b^{p^h} \in K$, $n^{p^{e+h}} = 0$. Since $R^{p^e} = (0)$, A^{p^e} has no nilpotent elements. Hence, $n^{p^e} = 0$. Therefore, $E(a)$ is a field which contradicts the maximality of E. Q. E. D.

In particular, when $e = 0$ in (iii) and (iv), then E is a K-coefficient field in A. The conditions in (iv) with $e = 0$ contrast with those of Theorem 2 below.

2. Let A be quasi-local with unique maximal ideal N and let the characteristic of F_0 be arbitrary.

THEOREM 2. *If N is nil and F_0 has a separating transcendence basis over K, then A has a K-coefficient field.*

PROOF. Let B_0 be a separating transcendence basis of F_0 over K and $B \subseteqq A$ a counterimage of B_0. Then $K[B] \cap N = (0)$, otherwise the algebraic independence of B_0 over K is contradicted. Since A is quasi-local, $K(B) \subseteqq A$. Let F be a maximal field $\supseteqq K(B)$. If $gF \subset F_0$ then F_0 is algebraic over gF and there exists $a_0 \in F_0$, $a_0 \notin gF$, $a \in A$, $a \notin F$ such that $ga = a_0$. Since N is nil, A is algebraic over F and it follows that $F(a)$ is quasi-local with unique

maximal ideal $F(a)\cap N$. Since $\operatorname{Ker} g|F(a)=F(a)\cap N=$ radical of $F(a)$ and $F(a)$ is finite dimensional over F and $gF(a_0)$ is separable over gF, it follows that there exists a field $F^*\supset F$ such that $gF^*=gF(a_0)$ by Wedderburn's Principal Theorem. This contradicts the maximality of F. Hence $gF=F_0$.
Q. E. D.

We also see that if F_0 is arbitrary over K, there exists a field F, $K\subsetneqq F\subsetneqq A$, such that F_0 is pure inseparable over gF.

Appendix

The following is an example of a quasi-local algebra which is such that $A^{p^i}\cap K=F_0^{p^i}\cap K$ for all positive integers i and yet A does not have a K-coefficient field.

Let $K=J_p(s, t, u, v)(s^{1/p}, t^{1/p})$ where $J_p=GF[p]$ and s, t, u, v are independent indeterminates over J_p. Let

$$F_0=J_p(s^{1/p}, t^{1/p}, u^{1/p}, v^{1/p})(u^{1/p^2}, v^{1/p^2})(s^{1/p^2}u^{1/p^2}+t^{1/p^2}v^{1/p^2})\,.$$

Then

$$F_0^p=J_p(s, t, u, v)(u^{1/p}, v^{1/p})(s^{1/p}u^{1/p}+t^{1/p}v^{1/p})$$

and

$$F_0^p\cap K=J_p(s, t, u, v)\,.$$

Let $K_0=F_0^p\cap K$ and consider the tensor product $A=F_0\times K$ over K_0. Let g be the homomorphism of $F_0\times K$ onto F_0 such that $gF_0\times 1=F_0$ and $g1\times K=K$. Identify $1\times K$ and K and let $N=\operatorname{Ker} g$.

For any positive integer i, $A^{p^i}\cap K\supseteqq A^{p^i}\cap(F_0^p\cap K)$ and since A has a K_0-coefficient field, namely $F_0\times 1$, $A^{p^i}\cap(F_0^p\cap K)=F_0^{p^i}\cap(F_0^p\cap K)$ by Remark 1 following Corollary 1. Thus, $A^{p^i}\cap K\supseteqq F_0^{p^i}\cap(F_0^p\cap K)=F_0^{p^i}\cap K$. Hence, $A^{p^i}\cap K=F_0^{p^i}\cap K$.

Since $K^p\subsetneqq K_0$, $N^p=(0)$ and thus $(N^p)\subsetneqq A^p(K)$ is the zero ideal. Now $A^p(K)=(F_0^p\times 1)(K)=F_0^p\times K$ and since F_0^p, K are not linearly disjoint over K_0, $\operatorname{Ker} g|A^p(K)\neq(0)$. Hence, $\operatorname{Ker} g|A^p(K)\neq(N^p)$ and thus, by Remark 1, A cannot have a K-coefficient field.

Creighton University
Iowa State University

References

[1] H. Bray, Embedding subfields of quasi-local rings in residue fields, unpublished Ph. D. thesis. Ames, Iowa, Library, Iowa State Univ., 1962.
[2] S. J. Bryant and J. L. Zemmer, A note on completely primary rings, Proc. Amer. Math. Soc., 8 (1957), 140-141.
[3] M. Nagata, Local rings, Interscience Pub., Vol. 13, 1962.
[4] O. Zariski and P. Samuel, Commutative algebra, vol. 2, D. Van Nostrand, Princeton, N. J., 1960.