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On certain arithmetical Dirichlet series

By Yasutaka IHARA

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Introduction.

Let $\mathfrak{Q}(x)=\mathfrak{Q}(x_1,\cdots,x_r)$ be a *positive definite* quadratic form with rational integral coefficients. Any homogeneous polynomial $f(x)=f(x_1,\cdots,x_r)$ in r variables will be called a spherical function with respect to $\mathfrak{Q}$ if $\Delta f=0$, Δ being the Laplacian with respect to the metric $\mathfrak{Q}$. For each positive integer n, let G_n be the totality of all $r\times r$ matrices g with rational integral coordinates satisfying $\mathfrak{Q}(xg)=n\mathfrak{Q}(x)$.

We shall consider such spherical functions f as satisfying

(a) $\qquad f(xg)=f(x) \qquad$ for any $g\in G_1$

(b) $\qquad$ for each $n=1,2,\cdots$, $\sum_{g\in G_n} f(xg)$ is a constant multiple of $f(x)$.

For each spherical function f satisfying (a) (b), we denote by $\tau_f(n)$ $(n=1,2,\cdots)$ the constant multiplicator in (b) divided by the cardinal number of G_1, and put

$$Z_f(s)=\sum_{n=1}^{\infty}\tau_f(n)n^{-s}.$$

Under a certain condition for $\mathfrak{Q}$, $Z_f(s)$ has an Euler-product expression, i.e. it is the product of

$$E_f^{(p)}(x)=\sum_{n=0}^{\infty}\tau_f(p^n)x^n\,; \quad x=p^{-s}$$

for all primes p, and it is often the case that $E_f^{(p)}(x)$ are rational functions of x. We are interested in the zeros and poles of such $E_f^{(p)}(x)$, since we expect "Riemann conjectures" about them. Many questions arise, but before attacking them, we wish to develop the arguments and find out our functions more explicitly for the simplest non-trivial cases. One of which is the case of $r=5$[1]. But the isomorphism between $O(5)$ and $Sp(4)$ suggests that we shall

1) A certain reformulation of several works of Eichler [1], [2], [3], [4] shows that in the case $r=3$, our $Z_f(s)$ are essentially the same as the Dirichlet series corresponding (by Mellin transformation) to modular forms, hence we cannot obtain anything new out of the case $r=3$. The case $r=4$ is reduced to that of $r=3$.

obtain essentially the same functions $Z_f(s)$ if we consider two dimensional *positive definite* quaternion-hermitian spaces instead.

So, the purpose of this brief note is:

(i) To develop the arguments explicitly in the latter cases. We will find that, the functions f that provide us with actually interesting $Z_f(s)$ are those with the property that the so-called "ϑ-series" vanish identically.

$$\vartheta_f(\tau)=\sum_{(x)} f(x)e^{2\pi i\tau Q(x)}\equiv 0\,.$$

But as for the non-trivial *properties* of such $Z_f(s)$, we can only conjecture one or two of them. The precise form of the rational functions $E_f^{(p)}(x)$ will be determined. When the ϑ-series does not vanish, our $Z_f(s)$ can be expressible by the product of Dirichlet series which are already well-known.

The spherical functions f whose ϑ-series vanish, namely those with the property that the sum of the values of f at the integral points which are equidistant from the original point always vanishes, will be called of type C. (Since they play a role analogous to that of modular cusp forms in the theory of Hecke operators.) We will find a lot of them even under the conditions (a), (b).

(ii) To give some numerical examples of the rational functions $E_f^{(p)}(x)$, which suggest a certain conjecture about the absolute values of their poles when f *is of type C*. (Zeros are, in our case, rather trivial.)

The first two sections are preliminaries. In the first section, we write down explicitly the space of spherical functions with respect to our quaternion-hermitian spaces, and in the second section, we apply the local theory of quaternion-hermitian spaces (cf. Shimura [**6**]) to obtain what is necessary for our purposes. The main statements are in the third section, and numerical examples and conjectures, in the final section.

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Throughout the following, let D be a definite quaternion algebra over the field of rational numbers Q with discriminant d_D whose class number is one. Put $D\otimes_Q R=K$ (the real quaternion field). For any $x\in K$, $n(x)=x\bar{x}$ denotes the reduced norm of x.

Let (K, K) be the two dimensional left K-vector space with hermitian metric:

$$\mathfrak{Q}(x,y)=n(x)+n(y) \qquad (x,y)\in(K,K)\,.$$

It contains the left D-vector space (D,D) with the metric $\mathfrak{Q}$.

$M_2(K)$, $M_2(D)$ etc. will mean the set of 2×2 matrices over K, D, etc.

§ 1 Geometrical preliminaries.

Put $G_K^1=\{g\in M_2(K)\,;\ {}^t\bar{g}g=1\}$.

Then,
$$(x, y)\rightarrow(x, y)g\,, \qquad g\in G_K^1$$
give all the K-linear transformations of (K, K) which leave $\mathfrak{Q}$ invariant. Since G_K^1 operates on (K, K) from the right, it operates also on the projective line:
$$L_K=K^{\times}\backslash(K, K)-\{(0, 0)\}$$
and it can be easily seen that the latter operation is transitive. L_K can be regarded as a compact symmetric Riemannian space of rank one with respect to its unique G_K^1-invariant metric (up to constant).

Now, $(x, y)\in(K, K)$ can be regarded as the variable over the 8 dimensional vector space over R. For each $k=0, 1, 2,\cdots$, we denote by $\mathfrak{M}_k$ the R-vector space of homogeneous real valued polynomial functions $f(x, y)$ of degree $2k$ of this variable, such that
$$f(zx, zy)=n(z)^k f(x, y) \qquad z\in K$$
and satisfying
$$\Delta f=0$$
where Δ denotes the ordinary Laplacian in the metric space (K, K) with respect to $\mathfrak{Q}$. Then the family
$$\{f(x, y)\,\mathfrak{Q}(x, y)^{-k}\,;\ f\in\mathfrak{M}_k\,,\ k=0, 1, 2,\cdots\}$$
coincides with that of all spherical functions on L_K. Accordingly, $\mathfrak{M}_k$ can be regarded as representation spaces of G_K^1; namely, $g\in G_K^1$ operates on $\mathfrak{M}_k$ as
$$\mathfrak{M}_k\ni f(x, y)\rightarrow f((x, y)g)\in\mathfrak{M}_k\,.$$

This representation is equivalent to the irreducible representation of $USp(4)\cong G_K^1$ corresponding to the Young diagram:

1	2	. . . .	k
1	2	. . . .	k

and
$$\dim\mathfrak{M}_k=\frac{1}{6}(k+1)(k+2)(2k+3)\,.$$

§2 Arithmetical preliminaries.

(A) Let G_D be the group of all similitudes of the metric space (D, D) which operates on (D, D) from the right: namely,

$$G_D=\{g\in M_2(D)\,;\ {}^t\bar{g}g=\text{scalar}\}\,.$$

For $g\in G_D$, the scalar ${}^t\bar{g}g$ is denoted by $n(g)$.

Let O be a maximal order of D. We regard (O,O) as a lattice in our metric space (D,D) and assume once for all that the G_D-genus containing (O,O) consists of only one class[2], so that, roughly speaking, the structure of G_D with respect to (O,O) is the direct product of their local extensions. More precisely, if we put

$$G_o=\{g\in G_D\,;\ (O,O)g\subset(O,O)\}=G_D\cap M_2(O)$$

$$G_\Gamma=\{g\in G_D\,;\ (O,O)g=(O,O)\}=\left\{\begin{pmatrix}\gamma & 0\\ 0 & \gamma'\end{pmatrix},\begin{pmatrix}0 & \gamma\\ \gamma' & 0\end{pmatrix}\,;\ \gamma,\gamma'\in\Gamma\right\}\quad\text{(finite)}\,,$$

where Γ denotes once for all the unit group of O, and if $n=p_1^{a_1}\cdots p_r^{a_r}$ is the prime factor decomposition of n, every $g\in G$ with $n(g)=n$ can be written uniquely in the form

$$g=\gamma g_1g_2\cdots g_r\,,\tag{1}$$

where g_i are given representatives of the left coset space $G_\Gamma\backslash G_o$,

$$n(g_i)=p_i^{a_i}\quad(1\leqq i\leqq r)\quad\text{and}\quad\gamma\in G_\Gamma\,.$$

(B) We mean by G_Γ-invariant cycle a *formal* sum of any finite set of points of (K,K) which is G_Γ-invariant as a whole. For any G_Γ-invariant cycle X and any $n=1,2,\cdots$, we put

$$T(n)X=\sum_{\substack{g\in G_\Gamma\backslash G_o\\ n(g)=n}}Xg$$

$$S(n)X=\sum_{\substack{\lambda\in o/\Gamma\\ n(\lambda)=n}}\lambda X\qquad\text{(formal sums)}\,.$$

It is clear that $T(n)X$, $S(n)X$ are again G_Γ-invariant cycles, and that by (1), we have

$$T(mn)=T(m)T(n)\qquad\textit{for }(m,n)=1\,.\tag{2}$$

(C) The determination of the multiplication laws between $T(p^r)$ and $T(p^s)$ for fixed p is reduced to the determination of the structure of the so-called "local double-coset rings". Let us denote by suffix p the local extension, (e. g. $(G_\Gamma)_p$ is the group of all similitudes of the metric space (D_p,D_p) which leave (O_p,O_p) invariant). There is a canonical bijection between two left-coset spaces.

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2) i. e. if L is an O-lattice such that $L_p=(O_p,O_p)g_p$, $g_p\in(G_D)_p$ for all primes p, then $L=(O,O)\,g$, $g\in G_D$, (where the suffix p denote local extensions). This assumption is for the sake of simplicity. If it is not satisfied, we should consider the direct sum of a finite number of (D,D) instead of one.

$$G_\Gamma \backslash \{g \in G_o\,;\ n(g)=\text{power of } p\} \approx (G_\Gamma)_p \backslash (G_o)_p\,.$$

Consider the double-coset rings[3] formed with respect to the above two coset spaces. Since any $(G_\Gamma)_p$-double-coset corresponds to a finite "sum" of G_Γ-double-cosets by this bijection, the double-coset ring with respect to $(G_o)_p$, $(G_\Gamma)_p$, which we denote by $\mathcal{H}((G_o)_p, (G_\Gamma)_p)$, can be regarded as the subring of the other double-coset ring. By this way, the determination of the multiplication laws between $T(p^r)$, $T(p^s)$ can be reduced to that of $\mathcal{H}((G_o)_p,(G_\Gamma)_p)$. In particular, the commutativity of $\mathcal{H}((G_o)_p, (G_\Gamma)_p)$ (essentially Shimura [6]) implies that $T(p^r)$, $T(p^s)$ commutes with each other. From this, and (2), we have

$$T(m)T(n)=T(n)T(m) \qquad \text{for any } m, n. \tag{3}$$

(D) Moreover, if p is a prime such that $p \nmid d_D$, our double-coset ring, $\mathcal{H}((G_o)_p, (G_\Gamma)_p)$ is isomorphic with the double-coset ring with respect to the semi-group of all similitude symplectic matrices of rank two over Z_p, and its obvious maximal compact subgroup. This enable us to apply a result of Shimura [7] from which we get the following symbolical formulae.

$$\sum_{n=0}^{\infty} T(p^n)t^n=(1-p^2R(p)t^2)\{1-T(p)t+(T(p)^2-T(p^2)-p^2R(p))t^2 \\ -p^3R(p)T(p)t^3+p^6R(p)^2t^4\}^{-1} \tag{4}$$

where t is a variable and $R(p)X=\sum_i px_i$ for any G_Γ-invariant cycle $X=\sum_i x_i$. (px is the scalar multiplication of a vector x, and the summation is the *formal* sum.)

On the other hand, if p is a prime such that $p \mid d_D$, analogous calculations show that

$$\sum_{n=0}^{\infty} T(p^n)t^n=(1+p^2R'(p)t)\{1+(p^2R'(p)-T(p))t+p^3R'(p^2)t^2\}^{-1} \tag{5}$$

where $R'(p)X=Xu$, $n(u)=p$ for any G_Γ-invariant cycle X, which is well-defined because for such p, elements u of D of norm p are contained in O and they belong to the same (left and right) Γ-coset.

(E) Put
$$X_n=\{(x, y)\in(O, O)\,;\ Q(x, y)=n\}$$
$$Y_n=\{(x, y)\in X_n\,;\ xO+yO=O\} \qquad n=1, 2, \cdots.$$

X_n, Y_n can be regarded as G_Γ-invariant cycles and by our first assumption that the class number of D is one, we get

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3) Or "Hecke ring." It is a free Z-module generated by all double-cosets supplied with "convolution product."

(6) $$X_n = \sum_{d \mid n} S(n/d) Y_d$$

(7) $$T(n)X_1 = \sum_{d \mid n} \sigma_D(n/d) S(n/d) Y_d\,,$$

where $\sigma_D(n)$ denotes the sum of divisors of n which are relatively prime to d_D.

§ 3 Spherical eigenfunctions and Dirichlet series.

(A) Put now

$$\begin{aligned}\mathfrak{M}_k^\Gamma &= \{ f \in \mathfrak{M}_k\,;\ f((x,y)\gamma) = f(x,y)\ \ \forall \gamma \in G_\Gamma \} \\ &= \{ f \in \mathfrak{M}_k\,;\ f(\gamma_1 x \gamma_2, \gamma_3 y \gamma_4) = f(x,y) = f(y,x)\ \ \forall \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma \}\,.\end{aligned}$$

We define linear endomorphisms $T_k(n)$ $n=1,2,\cdots$ of $\mathfrak{M}_k^\Gamma$ as

$$T_k(n):\ \ \mathfrak{M}_k^\Gamma \ni f(x,y) \to \sum_{\substack{g \in G_\Gamma \backslash G_O \\ n(g)=n}} f((x,y)g) \in \mathfrak{M}_k^\Gamma\,.$$

Then, it is clear that for any G_Γ-invariant cycle $X = \sum_i x_i$ and $f \in \mathfrak{M}_k^\Gamma$, we have

(8) $$f(T(n)X) = T_k(n) f(X)$$

(9) $$f(S(n)X) = \sigma_D(n) n^k f(X)$$

where $f(X) \underset{\text{def}}{=} \sum_i f(x_i)$.

So, by §2, (3), we have

(10) $$T_k(m)T_k(n) = T_k(n)T_k(m) \qquad \text{for any } m, n\,.$$

Moreover, if we denote by (f, h) for $f, h \in \mathfrak{M}_k$ the G_K^1-invariant positive symmetric bilinear form in $\mathfrak{M}_k$ (unique up to constant), we have

(11) $$(T_k(n)f, h) = (f, T_k(n)h) \qquad \text{for any } f, h \in \mathfrak{M}_k^\Gamma \quad n=1,2,\cdots\,.$$

Therefore $\mathfrak{M}_k^\Gamma$ is spanned by functions each of which is an eigenfunction for all $T_k(n)$, $n=1,2,\cdots$, and the eigenvalues of $T_k(n)$ are real numbers. (Moreover, we can deduce without any difficulty from the definitions of $\mathfrak{M}_k^\Gamma$ and of $T_k(n)$ that the eigenvalues are totally real algebraic numbers.)

(B) Let $f \in \mathfrak{M}_k^\Gamma$ be an eigenfunction of $T_k(n)$ for all $n=1,2,\cdots$, and put

$$T_k(n) f = \tau_f(n) f \qquad n=1,2,\cdots\,,$$

$$Z_f(s) = \sum_{n=1}^{\infty} \tau_f(n) n^{-s}\,.$$

When $k=0$, $\mathfrak{M}_0^\Gamma = \mathfrak{M}_0 =$ constant, and so,

$$T_0(n) = \frac{|\, g \in G\,;\ n(g) = n \,|}{|\, G_\Gamma \,|} \qquad \text{(scalar)}\,,$$

where, in general, $|S|$ denotes the cardinal number of the finite set S. To determine the exact values of $T_0(n)$, we need only to calculate them when n is a prime number or the square of a prime number because of (2), (4), (5), (and (8)). The result is

$$\begin{aligned} T_0(p)&=(1+p)(1+p^2) && \text{for } p \nmid d_D, \\ &=1+p^2+p^3 && \text{for } p \mid d_D, \\ T_0(p^2)&=p(p+1)(p^2+1)^2+1 && \text{for } p \nmid d_D, \\ &=p^2(p+1)(p^3+1)+1 && \text{for } p \mid d_D. \end{aligned}$$

From this follows

$$Z_{\text{const.}}(s)=\sum_{n=1}^{\infty}T_0(n)n^{-s}=\zeta(s)\,\zeta(s-2)\,\zeta(s-3)\,\zeta(2s-2)^{-1}\prod_{p\mid d_D}\frac{1-p^{4-2s}}{1-p^{2-2s}}, \tag{12}$$

where $\zeta(s)$ is the zeta function of Riemann.

Returning to the general case, choose $(x_0, y_0)\in(K,K)$ such that $|f(x_0,y_0)| = \operatorname*{Max}_{\mathfrak{O}(x,y)=\mathfrak{O}(x_0,y_0)} |f(x,y)|=1$. Then, we have

$$|\tau_f(n)| \leqq \frac{1}{|f(x_0,y_0)|}\sum_{\substack{g\in Gr\backslash Go \\ n(g)=n}}|f((x_0,y_0)g)| \leqq n^k T_0(n).$$

Since $\sum_{n=1}^{\infty}T_0(n)n^{-s}$ is absolutely convergent when $R(s)>4$, so is $Z_f(s)=\sum_{n=1}^{\infty}\tau_f(n)n^{-s}$ when $R(s)>k+4$. The Euler-product expression for our $Z_f(s)$ is the direct consequence of (4), (5), (and (8)).

(C) $f, Z_f(s)$ being as in §3 (B), let us suppose now that

$$f(X_1)\neq 0.$$

Then our $Z_f(s)$ breaks up into the product of Dirichlet series which are already known. To see this, we apply the equality

$$\sum_{m=0}^{\infty}\left(\frac{y^{m+1}-1}{y-1}\right)^2 x^m=\frac{1+xy}{(1-x)(1-xy)(1-xy^2)}. \tag{13}$$

Put $x=p^{-s}$, $y=p$ in (13), and multiply over all p such that $p\nmid d_D$. Then we obtain

$$\sum_{n=1}^{\infty}\sigma_D(n)^2n^{-s}=\zeta(s)\,\zeta(s-1)^2\,\zeta(s-2)\,\zeta(2s-2)^{-1}\prod_{p\mid d_D}\frac{(1-p^{-s})(1-p^{1-s})^2(1-p^{2-s})}{(1-p^{2-2s})}. \tag{14}$$

Since $f(X_1)\neq 0$, $\tau_f(n)=f(X_1)^{-1}f(T(n)X_1)$ and applying (7), we obtain

$$Z_f(s)=f(X_1)^{-1}\sum_{n=1}^{\infty}f(T(n)X_1)n^{-s}=f(X_1)^{-1}\sum_{n=1}^{\infty}n^{-s}\Big(\sum_{d\mid n}\sigma_D(n/d)^2(n/d)^k f(Y_d)\Big).$$

So, if we put $\phi_f(s)=f(X_1)^{-1}\sum_{n=1}^{\infty}f(X_n)n^{-s}$, $\phi_f'(s)=f(Y_1)^{-1}\sum_{n=1}^{\infty}f(Y_n)n^{-s}$, and remark that $X_1=Y_1$, we obtain

$$Z_f(s)=\sum_{n=1}^{\infty}\sigma_D(n)^2 n^{k-s}\,\phi_f{}'(s)\,. \tag{15}$$

On the other hand, we have by (6),

$$\phi_f(s)=\sum_{n=1}^{\infty}\sigma_D(n)n^{k-s}\,\phi_f{}'(s)=\zeta(s-k)\,\zeta(s-k-1)\,\phi_f{}'(s)\prod_{p\mid d_D}\frac{1-p^{k-s}}{1-p^{k+1-s}}\,. \tag{16}$$

Therefore, by (14), (15), (16), we obtain finally, *if* $f(X_1)\neq 0$

$$Z_f(s)=\zeta(s-k-1)\,\zeta(s-k-2)\,\zeta(2s-2k-2)^{-1}\,\phi_k(s)\,, \tag{17}$$

up to finite number of factors which are rational functions of p^{-s} for $p\mid d_D$.

$\phi_f(s)$ is nothing but the Dirichlet series which correspond to the ϑ-series

$$\vartheta_f(\tau)=f(X_1)^{-1}\{f(0,0)+\sum_{n=1}^{\infty}f(X_n)\,e^{2\pi i\tau n}\}$$

by Mellin transformation, and we can verify without difficulty by applying the result of Schönberg (cf. the complete work of E. Hecke, p. 855) that the above $\vartheta_f(\tau)$ is a modular form of weight $2k+4$ which belongs to the following Fuchsian group,

$$\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,Z)\,;\quad c\equiv 0 \pmod{d_D}\right\}.$$

REMARK: The equality (12) for $k=0$ is the special case of (17).

(D) What about $Z_f(s)$ if $f(X_1)=0$ then? Since f is a simultaneous eigenfunction for $T_k(n)$; $n=1,2,\cdots$, we have

$$f(T(n)X_1)=T_k(n)f(X_1)=0 \qquad n=1,2,\cdots.$$

So, applying (6) and (7)[4], we obtain

$$f(X_n)=0 \qquad n=1,2,\cdots. \tag{18}$$

In other words, for such eigenfunctions f, $\vartheta_f(\tau)$ vanishes identically (we have also $f(0,0)=0$ since $k=0$ is not the case now), and so, our $Z_f(s)$—— for which the "integral similitudes" of norm n play essential roles——cannot be expressed by means of known Dirichlet series such as $\phi_f(s)$ for which the "integral vectors" of norm n are essential.

(E) The above arguments suggest that our $Z_f(s)$ has essentially different properties according to whether

$$f(X_1)=0 \qquad \text{or} \qquad \neq 0\,.$$

We shall call a function in $\mathfrak{M}_k^{\Gamma}$ as of type C (resp. type E). if it is a linear combination of the eigenfunctions f such that $f(X_1)=0$, (resp. $\neq 0$). They are

4) Eliminate Y_d from (6) and (7).

orthogonal to each other according to (11). By § 3 (D), it is obvious that if any $f\in\mathfrak{M}_k^\Gamma$ (which is not necessarily an eigenfunction) is of type C, the simultaneous equation (18) holds for it. We can prove that (18) is also sufficient for (any) $f\in\mathfrak{M}_k^\Gamma$ to be of type C[5].

To see that there exists actually $0\neq f\in\mathfrak{M}_k^\Gamma$ which satisfies (18), let us consider the linear map

$$\mathfrak{M}_k^\Gamma\ni f\to \vartheta_f(\tau)\sum_{(x,y)\in(o,o)} f(x,y)\, e^{2\pi i\tau\mathfrak{Q}(x,y)}$$

of $\mathfrak{M}_k^\Gamma$ into the space of all modular forms of weight $2k+4$ belonging to the Fuchsian group given in §3 (C). Since the dimension of $\mathfrak{M}_k^\Gamma$ increases as $k\to\infty$ at the order of k^3, while the dimension of the latter space increases at the order of k, we can conclude that if k is large enough, most functions $f\in\mathfrak{M}_k^\Gamma$ satisfy (18), i. e. they are of type C.

We may roughly say that the spherical functions of type C (resp. type E) play a role analogous to that of "cusp forms" (resp. "Eisenstein series") in the theory of original Hecke operators. Thus we are interested in $Z_f(s)$ when f is of type C. We give some numerical examples of them in the final section.

(F) We give an additional remark that if k is odd, every $f\in\mathfrak{M}_k^\Gamma$ is of type C. The proof is as follows.

Since $$G_K^1/\pm 1\cong US_p(4)/\pm 1\cong SO(5, R)$$

we have $$L_K=K\cup\{\infty\}\cong \text{4-dimensional sphere-surface}$$

as (symmetric) Riemannian spaces, and we can check easily that x and $-\bar{x}^{-1}$ on L_K correspond to the antiposal points of the four dimensional sphere. From this, we conclude that if $f\in\mathfrak{M}_k^\Gamma$, we have

$$f(x,1)=(-1)^k n(x)^k f(-\bar{x}^{-1},1)\,. \tag{19}$$

By the definition of $\mathfrak{M}_k^\Gamma$, (19) implies for $f\in\mathfrak{M}_k^\Gamma$,

$$f(1,-\bar{x})=(-1)^k f(1,x)\,. \tag{20}$$

So, if k is odd, we have

$$f(1,0)=0 \qquad \text{for } f\in\mathfrak{M}_k^\Gamma\,. \tag{21}$$

Since $X_1=\{(\gamma,0),(0,\gamma)\,;\,\gamma\in\Gamma\}$ and since f is G_Γ-invariant, f takes the same value at all points of X_1, hence (21) is equivalent to

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5) For any generator T of the double coset ring $\bigotimes_p \mathcal{H}((G_o)_p, (G_\Gamma)_p)$ and G_Γ-invariant cycle X, we can define TX in the same way as $T(n)X$. It suffices to show that for any T, TX_1 is the sum of G_Γ-invariant cycles of the form $S_0(n)R(m)R'(l)Y_d$ where $S(n)=\sum_{d^2\mid n} S_0(n/d^2)R(d)$. The latter follows from the elementary arithmetic of quaternion algebra of class number one.

$$f(X_1)=0 \qquad \text{for} \qquad f\in\mathfrak{M}_k^\Gamma,\ k:\text{odd}. \tag{22}$$

This proves that if k is odd, any eigenfunction in $\mathfrak{M}_k^\Gamma$ is of type C, and consequently, any function in $\mathfrak{M}_k^\Gamma$ is of type C.

§ 4. Numerical examples.

Let D be such that $d_D=3$. Then, the class number of D is one, and if O is the unique maximal order (up to conjugates) of D, we can verify that the G_D-genus containing (O, O) consists of only one class, hence the necessary assumptions are satisfied.

If we denote by $\mathfrak{M}_k^\Gamma(C)$ the space of all functions (in $\mathfrak{M}_k^\Gamma$) of type C, the table of $\dim\mathfrak{M}_k^\Gamma$, $\dim\mathfrak{M}_k^\Gamma(C)$ for small k's is

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$\dim\mathfrak{M}_k^\Gamma$	1		1		2		3		6	1	7	1	13	4	$\cdots$
$\dim\mathfrak{M}_k^\Gamma(C)$									2	1	2	1	7	4	$\cdots$

where the blank columns stand for dimension zero. So, the smallest k for which we have a spherical function of type C is $k=8$.

Represent D, explicitly as

$$D=Q+Qi+Qj+Qk \qquad i^2=-1 \quad j^2=-3 \quad ij=-ji=k,$$

$$O=Z+Zi+Z\frac{1+j}{2}+Z\frac{i+k}{2}.$$

Denote $\quad D\ni x=x_1+x_2i+x_3j+x_4k$.

<u>$k=8$</u> $\quad \dim\mathfrak{M}_8^\Gamma=6$, $\dim\mathfrak{M}_8^\Gamma(C)=2$. A base of $\mathfrak{M}_8^\Gamma$ is given by

$$F(x,1)=t^8+1-36(t^7+t)+336(t^6+t^2)-1176(t^5+t^3)+1764t^4$$

$$G(x,1)=14\{2(-t^6+t^5+t^3-t^2)-9t^4+A(t^4+4t^3+6t^2+4t+1)-2B(t+1)^2\}+17C$$

$$H(x,1)=(5t^3-5At+B)(4t^2-9t+4)$$

$$I(x,1)=9t^4-15At^2+7Bt-C$$

$$J(x,1)=153t^4-150At^2+35A^2$$

$$K(x,1)=x_3x_4(x_1^2-x_3^2)(x_2^2-x_4^2)(x_1^2-x_2^2+3x_3^2-3x_4^2),$$

where $1/6\sum_{\gamma\in\Gamma/\pm1}(tr(\gamma x))^{2r}=t, A, B, C$ according to $r=1,2,3,4$ (respectively). (In particular, $t=n(x)$.)

A base of $\mathfrak{M}_8^\Gamma(C)$ is given by

$$f(x,1)=K(x,1)$$

$$g(x, 1)=\frac{2}{3} G(x, 1)+7H(x, 1)+9I(x, 1)-\frac{5}{18} J(x, 1).$$

The formulae (4), with numerical computations show that both f and g are eigenfunctions of the operators $T(2^r)$ $r=1, 2, \cdots$ with the same eigenvalues, hence the Euler-factor of $Z_f(s)$, $Z_g(s)$ coincides for $p=2$, denominator of which is given by (numerator is rather trivial)

$$\{1-12(-9+\sqrt{1489})2^{-s}+2^{19-2s}\}\{1-12(-9-\sqrt{1489})2^{-s}+2^{19-2s}\}.$$

$\underline{\underline{k=9}}$ $\dim \mathfrak{M}_9^\Gamma=\dim \mathfrak{M}_9^\Gamma(C)=1$, and its base is given by

$$f(x, 1)=x_1x_2(x_1-9x_3^2)(x_2^2-9x_4^2)(x_1^2-x_2^2+3x_3^2-3x_4^2)(1-n(x)).$$

The denominator of the Euler factor of $Z_f(s)$ for $p=2$ is given by

$$\{1-24(-69+\sqrt{1049})2^{-s}+2^{21-2s}\}\{1-24(-69-\sqrt{1049})2^{-s}+2^{21-2s}\}.$$

$\underline{\underline{k=11}}$ $\dim \mathfrak{M}_{11}^\Gamma=\dim \mathfrak{M}_{11}^\Gamma(C)=1$, and its base is given by

$$f(x,1)=x_1x_2(x_1^2-9x_3^2)(x_2^2-9x_4^2)(x_1^2-x_2^2+3x_3^2-3x_4^2)(1-n(x))(5-13n(x)+5n(x)^2).$$

The denominator of the Euler factor of $Z_f(s)$ for $p=2$ is given by

$$\{1-48(-69+\sqrt{22297})2^{-s}+2^{25-2s}\}\{1-48(-69-\sqrt{22297})2^{-s}+2^{25-2s}\}.$$

The above examples suggest the following conjecture. When $f\in\mathfrak{M}_k^\Gamma$ is an eigenfunction *of type C*, and when $p\nmid d_D$, the roots of the denominator of the p-Euler factor of $Z_f(s)$——which is a polynomial of $x=p^{-s}$ of degree four——have the absolute values $|x|=p^{-\left(k+\frac{3}{2}\right)}$?

REMARK. Similarity of trace-formula suggests some connections between our $Z_f(s)$, $f\in\mathfrak{M}_k^\Gamma$ and Dirichlet series given by "Hecke operator theory" in the space of Siegel modular forms of degree 2, *weight* $k+3$, but as yet, we cannot ascertain it.

University of Tokyo

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