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On the continuability of holomorphic functions on complex manifolds

By Hirotaka FUJIMOTO and Kenkiti KASAHARA

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§ 1. Introduction.

1. If K is a relatively compact domain in the space C^n of n complex variables $z_1, \cdots, z_n$ where $n \geqq 2$ and the boundary ∂K of K is connected, then every function f which is defined and holomorphic in a neighborhood of ∂K can be continued holomorphically into the whole K. This is well-known as a Hartogs-Osgood's theorem and one of the most remarkable facts which distinguish the theory of functions of several complex variables from that of one complex variable. The essential part of their proof ([**10**]) is to show the following two:

1) (Local continuability.) If we assume that U is a neighborhood of a point p on the sphere $\{v=\lambda\}$ where $v=\sum_{i=1}^{n}|z_i|^2$ and $\lambda \geqq 0$, then any holomorphic function in $U \cap \{v>\lambda\}$ is holomorphically continuable into a neighborhood of p.

2) (Global continuability.) If Λ is the set of non-negative real numbers λ such that f can be continued holomorphically into $K \cap \{v>\lambda\}$, then $\min \Lambda$ must be zero.

But unfortunately their discussions on 2) were incomplete. After a long while A. B. Brown completed the proof ([**3**]).

2. In this paper we investigate a similar theorem on a complex manifold. Main results are as follows:

Let X and Y be complex mainfolds and τ be a holomorphic mapping of X into Y whose rank is r at every point in X. Assume that there exists a strongly $(r-1)$-convex function v on Y (Def. 5). By the generalized Hartogs theorem of continuity essentially due to H. Behnke and F. Sommer ([**2**]), we have

(a) *Every holomorphic function in* $U \cap \{v \circ \tau > v \circ \tau(p)\}$, *where* U *is a neighborhood of a point* p *in* X, *admits a holomorphic continuation in a neighborhood of* p *which is possibly many-valued* (§ 3).

(b) *If we assume furthermore that* Y *is purely r-dimensional, then the continuation in* (a) *is single-valued* (§ 4).

About the global continuability, using some properties of real analytic functions (§ 6) we have

(c) *In addition to the assumptions in* (b), *we assume that* τ *is proper and* $\{\alpha < v < \beta\} \Subset Y$ *for any real numbers* $\alpha < \beta$. *If* K *is a compact set in* X *and* D *is an open set in* X *containing* K *such that* $D-K$ *is connected, then any holomorphic function in* $D-K$ *can be continued holomorphically into the whole* D (§§ 5~7).

Since a Stein manifold X with $\dim_p X \geqq 2$ for any p in X satisfies the assumptions of (c), the Hartogs-Osgood theorem is true on it. If τ is a proper holomorphic mapping of a complex manifold X into a purely r-dimensional Stein manifold Y where $r \geqq 2$ and the rank of τ is r at every point in X, then X satisfies the assumptions of (c) and the Hartogs-Osgood theorem holds on X.

3. In this paper we discuss the above results for $\mathfrak{F}$-valued holomorphic functions, where $\mathfrak{F}$ is a Hausdorff topological vector space over C which is locally convex and complete. Consequently we have some generalizations of the Hartogs-Osgood theorem as follows:

(d) *Let* X *be a complex manifold satisfying the assumptions of* (c) *and* Z *be an arbitrary complex manifold. Let* K *and* D *be the same as in* (c). *Then every holomorphic functions in* $(D-K)\times Z$ *can be continued holomorphically into* $D\times Z$ (§ 8).

(e) *In* (c), *we assume that* $f(p, t)$ *is a* k*-times continuously differentiable family of holomorphic functions in* $D-K$ *with parameters* t *in a differentiable manifold of class* C^k. *Then the continued functions* $g(p, t)$ *in* D *by* (c) *constitute also a* k*-times continuously differentiable family of holomorphic functions in* D (§ 8).

§ 2. Vector-valued holomorphic functions.

1. Let $\mathfrak{F}$ be a Hausdorff topological vector space over the complex number field C which is locally convex and complete.

DEFINITION 1. A multiple series $\sum_{\nu_1\cdots\nu_n} a_{\nu_1\cdots\nu_n}$ in $\mathfrak{F}$ is called absolutely convergent if $\sum_{\nu_1\cdots\nu_n} \|a_{\nu_1\cdots\nu_n}\|$ is convergent for any continuous semi-norm $\|\ \|$ on $\mathfrak{F}$.

We have

LEMMA 1. *Let* $\{a_{\nu_1\cdots\nu_n}\}$ *be a multiple sequence in* $\mathfrak{F}$ *and* $\rho_1, \cdots, \rho_n$ *be a system of positive real numbers. If the set* $\{\|a_{\nu_1\cdots\nu_n}\| \rho_1^{\nu_1}\cdots\rho_n^{\nu_n}\}$ *is bounded for any continuous semi-norm* $\|\ \|$ *on* $\mathfrak{F}$, *then the series* $\sum_{\nu_1\cdots\nu_n} a_{\nu_1\cdots\nu_n} z_1^{\nu_1}\cdots z_n^{\nu_n}$ *is absolutely and uniformly convergent on any compact set in* $\{|z_1| < \rho_1, \cdots, |z_n| < \rho_n\}$.

2. Let f be an $\mathfrak{F}$-valued function defined on an open set G in a complex manifold X.

DEFINITION 2. The function f is called holomorphic on G, if f is continuous and $u \circ f$ is holomorphic on G for any complex-valued continuous linear functional u on $\mathfrak{F}$.

In particular, a C-valued holomorphic function is nothing but an ordinary holomorphic function. In the following, for brevity, we call it a holomorphic function.

For every $\mathfrak{F}$-valued holomorphic function f which is defined on an open set containing the polycylinder $\{z; |z_i - a_i| \leqq \rho_i\}$, we have Cauchy's integral formula

$$f(z_1, \cdots, z_n) = \frac{1}{(2\pi i)^n} \int \cdots \int_{|\zeta_i - a_i| = \rho_i} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n \tag{1}$$

and hence a convergent power series expansion

$$f(z_1, \cdots, z_n) = \sum_{\nu_1 \cdots \nu_n} a_{\nu_1 \cdots \nu_n} (z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{\nu_n} \tag{2}$$

where

$$\begin{aligned} a_{\nu_1 \cdots \nu_n} &= \frac{1}{\nu_1 ! \cdots \nu_n !} \left(\frac{\partial^{\nu_1 + \cdots + \nu_n} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}} \right)_{z=a} \\ &= \frac{1}{(2\pi i)^n} \int \cdots \int_{|\zeta_i - a_i| = \rho_i} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - a_1)^{\nu_1 + 1} \cdots (\zeta_n - a_n)^{\nu_n + 1}} d\zeta_1 \cdots d\zeta_n \end{aligned} \tag{3}$$

Accordingly, we can verify many other function-theoretic properties for $\mathfrak{F}$-valued holomorphic functions, e. g. the theorem of identity, Weierstrass' double series theorem, Riemann's theorem on removable singularities and so on. We use some of them without proof in the following.

3. LEMMA 2. *Let $\| \ \|$ be a continuous semi-norm on $\mathfrak{F}$ and f an $\mathfrak{F}$-valued holomorphic function on an open set G. Then $v(p) = \|f(p)\|$ is a plurisubharmonic function on G.*

PROOF. For any holomorphic mapping $\tau := \tau(t)$ of a domain G' in C^1 into G, $f(\tau(t))$ is an $\mathfrak{F}$-valued holomorphic function on G'. By the formula (1) and the properties of $\| \ \|$, we have the inequality

$$\|(f \circ \tau)(t_0)\| \leqq \frac{1}{2\pi} \int_0^{2\pi} \|(f \circ \tau)(t_0 + \rho e^{i\theta})\| \, d\theta$$

where t_0 is a point in G' and ρ is a positive real number such that the set $\{t; |t_0 - t| \leqq \rho\}$ is contained in G'. This shows that the real-valued function $\|(f \circ \tau)(t)\|$ is subharmonic on G'. By definition, $v(p)$ is a plurisubharmonic function on G. q. e. d.

DEFINITION 3. Let M be an analytic set in an open set G in a complex manifold X. A real-valued function v on M is called plurisubharmonic if for any point p in M there exist a neighborhood V of p in X and a plurisubharmonic function $\tilde{v}$ in V such that $\tilde{v}$ coincides with v on $M \cap V$.

LEMMA 3. *Let M be a connected analytic set in a domain G in a complex* *manifold. A plurisubharmonic function v on M satisfies the maximum principle, that is, it cannot take the maximum at an interior point of M unless it is constant.* (*Cf. H. Grauert and R. Remmert* [5].)

PROOF. Suppose v takes its maximum at a point p in M. By the connectivity of M it is sufficient to show that v is constant in a neighborhood of p in M. Moreover, we may assume that M is irreducible and k-dimensional at p. As is well known, there exists a polycylinder $U:=\{|z_1|<\rho_1, \cdots, |z_n|<\rho_n\}$ referred to a suitable coordinates $z_1, \cdots, z_n$, with $p=(0)$, such that the canonical projection σ of $M\cap U$ onto $U^k:=\{|z_1|<\rho_1, \cdots, |z_k|<\rho_k\}$ is a proper and nowhere degenerate mapping with $\sigma^{-1}\circ\sigma(p)\cap M\cap U=\{p\}$. We take a point q in $M\cap U$ and the complex line L in U^k joining the origin with $\sigma(q)$. The analytic set $\sigma^{-1}(L)$ is of dimension 1 and any irreducible component of it contains the point p. We take its normalization $\tau:\tilde{L}\to\sigma^{-1}(L)$. The function $v\circ\tau$ is subharmonic on the Riemann surface $\tilde{L}$ and takes its maximum at the points in the set $\tau^{-1}(p)$ which intersects with any connected component of $\tilde{L}$. By the maximum principle for subharmonic functions, $v\circ\tau$ is constant on $\tilde{L}$ and, in particular, $v(p)=v(q)$. Hence, v is identically equal to $v(p)$ on $M\cap U$. This completes the proof. q. e. d.

Now, we have

PROPOSITION 1. *Let G be an open subset of a complex manifold and M a connected analytic set in G. For any $\mathcal{F}$-valued holomorphic function f in G and any continuous semi-norm $\| \ \|$ on $\mathcal{F}$, the restriction of $v(p):=\|f(p)\|$ to M satisfies the maximum principle on M.*

§ 3. Hartogs' theorem of continuity.

1. In this section, we give local boundary conditions for the continuation of $\mathcal{F}$-valued holomorphic functions. For a moment, we restrict ourselves to subsets in C^n. For a fixed system of coordinates in C^n, we define the distance $\rho(z, z')$ of two points $z=(z_1, \cdots, z_n)$ and $z'=(z'_1, \cdots, z'_n)$ by the equality $\rho(z, z'):=\max\{|z_i-z'_i|; 1\leqq i\leqq n\}$. By this distance, we can define the distance $\rho(A, B)$ of two sets A and B, the distance $\rho(p, A)$ of a point p from a set A, ε-neighborhood $S(A, \varepsilon)$ of a set A and so on.

The following proposition was proved by H. Cartan and P. Thullen [4]. Since we are considering here $\mathcal{F}$-valued holomorphic functions, we sketch the proof.

PROPOSITION 2. *Let G be an open set in C^n and K a compact subset of it. Suppose that a point p_0 in G satisfies the condition*

$$\|f(p_0)\|\leqq\max\{\|f(p)\|; p\in K\} \tag{4}$$

for any $\mathcal{F}$-valued holomorphic function f on G and any continuous semi-norm

$\| \ \|$ *on* $\mathfrak{F}$. *Then, for any* $\mathfrak{F}$*-valued holomorphic function* f *on* G, *there exists an* $\mathfrak{F}$*-valued holomorphic function* g *in* $S(p_0, \rho)$ *which is equal to* f *in a neighborhood of* p_0 *where* $\rho = \rho(K, \partial G)$.

PROOF. The function f has a convergent power series expansion (2) with center at $p_0 = (a_1, \cdots, a_n)$. Then for a sufficiently small positive real number ε, we have

$$(\rho-\varepsilon)^{\nu_1+\cdots+\nu_n} \| a_{\nu_1\cdots\nu_n} \| = \frac{(\rho-\varepsilon)^{\nu_1+\cdots+\nu_n}}{\nu_1! \cdots \nu_n!} \left\| \left(\frac{\partial^{\nu_1+\cdots+\nu_n} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}} \right)_a \right\| \qquad \text{(from (3))}$$

$$\leqq \sup \left\{ \frac{(\rho-\varepsilon)^{\nu_1+\cdots+\nu_n}}{\nu_1! \cdots \nu_n!} \left\| \left(\frac{\partial^{\nu_1+\cdots+\nu_n} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}} \right)_q \right\| ; \ q \in K \right\} \quad \text{(from the hypothesis)}$$

$$\leqq \sup \{ \|f(q)\| ; \ q \in S(K, \rho-\varepsilon) \} \qquad \text{(from (3))}.$$

Hence $\{(\rho-\varepsilon)^{\nu_1+\cdots+\nu_n} \| a_{\nu_1\cdots\nu_n} \| \}$ is bounded for any continuous semi-norm on $\mathfrak{F}$. Then apply Lemma 1. q. e. d.

2. Let G be an open subset of a complex manifold X and p be a point in X. We consider a family of analytic sets

$$M_t : \varphi_i(z, t) = 0 , \qquad 1 \leqq i \leqq s$$

in a coordinate neighborhood U of p, where $z = (z_1, \cdots, z_n)$ is a fixed system of local coordinates in U, $t = (t_1, \cdots, t_m)$ is a system of parameters defined in a neighborhood of the origin 0 in C^m and every $\varphi_i(z, t)$ is a continuous family of holomorphic functions, that is, continuous with respect to variables (z, t) and holomorphic with respect to variables z.

DEFINITION 4. We call the set G to be analytically concave at p if there exist a coordinate neighborhood U of p and a family of analytic sets M_t in U as above with the following properties:

1) $M_0 \ni p$ and $M_0 - \{p\} \subset G$.

2) There exists a sequence of parameters $t^{(\nu)} = (t_1^{(\nu)}, \cdots, t_m^{(\nu)})$ converging to the origin 0 and satisfying the condition $M_{t^{(\nu)}} \cap U \subset G \cap U$.

3) M_0 and all $M_{t^{(\nu)}}$ have no isolated irreducible component in G.

4) Any neighborhood of p intersects with infinitely many $M_{t^{(\nu)}}$.

Now, we have the following generalized Hartogs' theorem of continuity, essentially due to H. Behnke and F. Sommer [**2**].

THEOREM 1. *Let* G *be an open subset of a complex manifold* X *and* p *a point in* X *such that* $\dim_p X \geqq 2$ *and* G *is analytically concave at* p. *Then there exists a connected neighborhood* W *of* p *such that every* $\mathfrak{F}$*-valued holomorphic function on* G *admits a holomorphic continuation in* W, *which is possibly many-valued.*

PROOF. By assumption, there exists a family of analytic sets M_t in a coordinate neighborhood U of p with the properties of Definition 4. To apply Proposition 2, we regard the neighborhood U as an open subset of C^n. Take a polycylinder $U' := S(p, \rho)$ such that $S(p, 2\rho) \subset U$. Then, for almost all ν $\rho_\nu := \rho(M_{t^{(\nu)}} \cap \partial U', U \cap \partial G)$ is bounded by a positive real number ε from below. In fact, if a subsequence $\{\rho_{\nu_\mu}\}$ of $\{\rho_\nu\}$ converges to zero, we can take a sequence of points $z^{(\mu)}$ in $\partial U' \cap M_{t^{(\nu_\mu)}}$ such that the sequence $\rho(z^{(\mu)}, \partial G)$ converges to zero. By the compactness of $\partial U'$, we may assume $\{z^{(\mu)}\}$ converges to a point $z^{(0)}$ in $\partial U'$, which must be contained in ∂G. The continuity of $\varphi_i(z, t)$ implies $\varphi_i(z^{(0)}, 0) = 0$ $(1 \leqq i \leqq s)$ and hence $z^{(0)} \in M_0 \cap \partial U'$. This contradicts $\partial U' \cap M_0 \cap \partial G = \phi$.

By the condition 4), we can take ν with $\rho_\nu > \rho(U' \cap M_{t^{(\nu)}}, p)$. Putting $K = M_{t^{(\nu)}} \cap \partial U'$, we take a point p_0 in $U' \cap M_{t^{(\nu)}}$ such that $\rho(p_0, p) < \rho_\nu$. By the condition 3) and Proposition 1, the relation (4) in Proposition 2 holds for any $\mathcal{F}$-valued holomorphic function on G and continuous semi-norm on $\mathcal{F}$. Let W be a neighborhood of p such that $W \subset S(p_0, \rho_\nu)$. Proposition 2 implies that, for any $\mathcal{F}$-valued holomorphic function f on G, there exists an $\mathcal{F}$-valued holomorphic function g on W which is equal to f in a non-empty open subset of $G \cap W$. This proves Theorem 1. q. e. d.

REMARK. In the above proof, the point p_0 can be chosen to be arbitrarily near to p. Thus, if we put $W := S\left(p, \frac{\varepsilon}{2}\right)$, for any small neighborhood V of p we can find an $\mathcal{F}$-valued holomorphic function in W which is equal to f on a non-empty open subset of $V \cap W \cap G$.

3. Now, we take a real-valued function v of class C^2 on an n-dimensional complex manifold X.

DEFINITION 5. The function v is called strongly s-convex at a point p in X if the hermitian matrix $\left(\left(\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\right)_p\right)$ has at least $n-s+1$ positive proper values, where $z_1, \cdots, z_n$ is a suitable system of coordinates in a neighborhood of p.

DEFINITION 5'. An open subset G of X is called strongly s-convex at a boundary point p of G, if $G \cap U = \{v < 0\} \cap U$ for a neighborhood U of p and a strongly s-convex function v at p defined on U. And G is called strongly s-convex if G is strongly s-convex at every boundary point of G.

PROPOSITION 3. *Let τ be a holomorphic mapping of a connected complex manifold X of dimension at least two into another complex manifold Y with the rank $r = r_\tau(p) := \operatorname{codim}_p \tau^{-1} \circ \tau(p)$ at a point p in X and v be a real-valued function on Y which is strongly $(r-1)$-convex at $\tau(p)$. Then the open subset $X_\lambda = \{v \circ \tau > \lambda := v \circ \tau(p)\}$ of X is analytically concave at p.*

PROOF. For every mapping τ with the rank r at p we can take a system of local coordinates $z_1, \cdots, z_n$ in a neighborhood U of p such that $\tau^{-1} \circ \tau(p) \cap U \cap \{z_{r+1} = \cdots = z_n = 0\} = \{p\}$. And every strongly s-convex function v is strongly plurisubharmonic on the submanifold $L' := \{w_1 = \cdots = w_{s-1} = 0\}$ in a neighborhood of $q:=\tau(p)$ in Y for a suitable system of coordinates $w_1, \cdots, w_m$, with $q=(0)$. Then, v has the expansion

$$v(w)=v(q)+\operatorname{Re}\varphi+\sum_{s\leqq j,k\leqq m}\left(\frac{\partial^2 v}{\partial w_j\partial \bar{w}_k}\right)_q w_j\bar{w}_k+o(|w|^2) \tag{5}$$

on L', where $\operatorname{Re}\varphi$ denotes the real part of the polynomial φ of degree at most two: $\varphi=2\times\left(\sum_{s\leqq j\leqq m}\left(\frac{\partial v}{\partial w_j}\right)_q w_j\right)+\sum_{s\leqq j,k\leqq m}\left(\frac{\partial^2 v}{\partial w_j\partial w_k}\right)_q w_jw_k$ and $o(|w|^2)$ is Landau's symbol. Now, we take a family of analytic sets

$$M_t\,;\ \tau_1-t_1=\cdots=\tau_{s-1}-t_{s-1}=z_{r+1}-t_s=\cdots=z_n-t_{n-r+s-1}=\varphi'-t_{n-r+s}=0 \tag{6}$$

with complex parameters $t=(t_1, \cdots, t_{n-r+s})$, where τ_i is the i-th component of τ with respect to the above coordinates $w_1, \cdots, w_m$, $\varphi'=\varphi\circ\tau$ if $\varphi\not\equiv 0$ and $\varphi'=z_{i_0}$ with $z_{i_0}\not\equiv 0$ on $\{\tau_1=\cdots=\tau_{s-1}=z_{r+1}=\cdots=z_n=0\}$ if $\varphi\equiv 0$. Since each M_t is the common zeros of $n-r+s$ holomorphic functions, it has no irreducible component of dimension smaller than $r-s$ unless it is empty. In particular, for a strongly $(r-1)$-convex function v, M_t has no isolated irreducible component. To complete the proof it is sufficient to show the above family of analytic sets satisfies the conditions 1), 2) and 4) of Definition 4. To this end, we use the expansion (5). Since the hermitian matrix $\left(\left(\frac{\partial^2 v}{\partial w_j\partial \bar{w}_k}\right)_q\right)_{s\leqq j,k\leqq m}$ is positive definite, we see $v>\lambda$ on the set $\{\operatorname{Re}\varphi\geqq 0\}\cap L'$ except the point q in a sufficiently small neighborhood of q. Therefore the set $M_0-\{p\}$ is contained in X_λ. If we take $\left(0, \cdots, 0, \frac{1}{\nu}\right)$ as $t^{(\nu)}$, a sequence of analytic sets $M_{t^{(\nu)}}$ satisfies the conditions 2) and 4). This completes the proof. q. e. d.

REMARK. As easily seen in the above proof, for a holomorphic mapping τ of X into Y with the rank r at p and a strongly s-convex function v on Y, there exists an $(r-s)$-dimensional locally analytic set at p contained in the set $X_\lambda:=\{v\circ\tau>\lambda\}$ except the point p.

COROLLARY. *Under the same notations and conditions, every $\mathfrak{F}$-valued holomorphic function on X admits a holomorphic continuation in a neighborhood of p, which is possibly many-valued.*

This is an immediate consequence of Theorem 1 and Proposition 3.

§ 4. Boundary conditions for the single-valued continuability.

1. In the previous section, we showed any $\mathfrak{F}$-valued holomorphic function f on an open set G is holomorphically continuable to a neighborhood W of a boundary point of G under some boundary conditions. For these results, however, we must pay attention to the possible many-valuedness of the continued function, that is, the continued function in Theorem 1 is not necessarily equal to the original function on the whole $G\cap W$. In this section, we state on some sufficient boundary conditions for the single-valued continuability of $\mathfrak{F}$-valued holomorphic functions.

As a consequence of Theorem 1 we have first the following

PROPOSITION 4. *Let G be an open set in a complex manifold X, which is analytically concave at a point p in X. Suppose there exists a fundamental system $\mathfrak{U}$ of neighborhoods of p such that $U\cap G$ is connected for every U in $\mathfrak{U}$. Then there exists a connected neighborhood V of p such that every $\mathfrak{F}$-valued holomorphic function in G admits a single-valued holomorphic continuation to the whole V.*

PROOF. As a desired neighborhood of p, it is sufficient to take a set V in $\mathfrak{U}$ which is contained in the set W defined in the remark for Theorem 1. Because for any $\mathfrak{F}$-valued holomorphic function f in G we have an $\mathfrak{F}$-valued holomorphic function g on W which is equal to f on a non-empty open subset of $V\cap G$ and so g is necessarily equal to f on the whole domain $V\cap G$ by the theorem of identity.

2. To give an important example satisfying the conditions in Proposition 4, we need some preparations.

DEFINITION 6. Take an open set G in a complex manifold X. The set G is called Rothstein s-convex at a boundary point p of G if and only if any locally analytic set of dimension at least s at p contains an exterior point of G. And G is called Rothstein s-convex if G is Rothstein s-convex at every boundary point of G.

The following proposition was given by W. Rothstein ([**13**], p. 117) in the case v has the condition of regularity.

PROPOSITION 5. *An open set G in a complex manifold which is strongly s-convex at a boundary point p is Rothstein s-convex at p.*

Therefore, a strongly s-convex set is Rothstein s-convex.

PROOF. By Definition 5′, G is represented in a neighborhood U of p as $G\cap U=\{v<0\}\cap U$ with a strongly s-convex function v in U. Then v may be considered as a strongly plurisubharmonic function on the submanifold $L:=\{z_1=\cdots=z_{s-1}=0\}$ of U for a suitable coordinates $z_1,\cdots,z_n$. Take an arbitrary analytic set M in U satisfying $\dim_p M\geqq s$. Since $M\cap L$ is an at least 1-dimensional analytic subset, Lemma 3 implies that the non constant function v on $M\cap L$ has a positive value at a point sufficiently near to p. This shows that M contains an exterior point of G. q. e. d.

3. Now, we take a noetherian local ring A. The homological codimension of A is by definition the number of elements in a maximal A-sequence, that is, a maximal sequence $\varphi_1,\cdots,\varphi_r$ in the maximal ideal $\mathfrak{m}$ with the property that, for every ideal $\mathfrak{a}_i=(\varphi_1,\cdots,\varphi_i)A$, $(i=1,\cdots,r)$, $\mathfrak{a}_0=(0)$, the residue class of φ_i modulo $\mathfrak{a}_{i-1}$ is not a zero divisor in the ring $A/\mathfrak{a}_{i-1}$. As is well known, the ring $A_p(M)$ of all holomorphic functions on an analytic set M at a point p is a noetherian local ring. According to [**1**], p. 198, we denote the homological codimension of $A_p(M)$ by $\operatorname{dih}_p M$. Easily we obtain $\dim_p M \geqq \operatorname{dih}_p M$. If an analytic subset M of a complex manifold X is a complete intersection at p, that is, the common zeros of $\operatorname{codim}_p M$ holomorphic functions in a neighborhood of p, we have $\dim_p M = \operatorname{dih}_p M$.

PROPOSITION 6. *Let M be an analytic set in a neighborhood of a point p in C^n. Suppose* $\operatorname{dih}_p M \geqq 2$. *Then, there exists a fundamental system $\mathfrak{U}$ of neighborhoods of p such that $U \cap (M - \{p\})$ is connected for every U in $\mathfrak{U}$.* (R. Hartshorne [**7**].)

PROOF. We take a sufficiently small neighborhood U such that any irreducible component M_i of $M \cap U$ contains p. Then the set $M_i - \{p\}$ is connected for each M_i. If $U \cap (M - \{p\})$ is not connected the family M_i divides into two disjoint classes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $M_i \cap M_j$ contains p isolatedly for every $M_i \in \mathfrak{M}_1$, $M_j \in \mathfrak{M}_2$. This shows that the noetherian local ring $A_p(M)$ has the following ring-theoretic properties. For any irredundant primary decomposition of the zero ideal of $A_p(M)$

$$(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_s$$

the family $\{\mathfrak{q}_i\}$ divides into two disjoint classes Q_1 and Q_2 such that the radical of $\mathfrak{q}_i \cup \mathfrak{q}_j$ is equal to the maximal ideal for every $\mathfrak{q}_i \in Q_1$ and $\mathfrak{q}_j \in Q_2$. Now, we can apply the proof of R. Hatshorne [**7**] Proposition 2.1 to our case. We omit the details.

4. LEMMA 4. *Take a family of analytic sets*

$$M_t\,;\ \varphi_1 - t_1 = \cdots = \varphi_{n-k} - t_{n-k} = 0$$

in a neighborhood of the origin in C^n with parameters $t = (t_1, \cdots, t_{n-k})$, where $0 < k < n$. Suppose M_0 is of dimension k at the origin. Then there exist two polycylinders U, V ($U \Subset V$) with centers at the origin and a positive number ε such that (i) *each irreducible component of $U \cap M_0$ contains the origin and* (ii) *each connected component of the set $L := \bigcup_{|t_j| < \varepsilon} M_t \cap (U - \bar{V})$ intersects with M_0.*

PROOF. We consider the analytic set $M^* = \{\varphi_1 - w_1 = \cdots = \varphi_{n-k} - w_{n-k} = 0\}$ in a neighborhood of the origin in $C^{2n-k} = C^n \times C^{n-k}$, where $w_1, \cdots, w_{n-k}$ is a system of coordinates in C^{n-k}. Since $M_0 = M^* \cap \{w_1 = \cdots = w_{n-k} = 0\}$ is of dimension k at the origin, Remmert-Stein's Einbettungssatz ([**12**]) implies that for a suitable coordinates $z_1, \cdots, z_n$ in a neighborhood of the origin, there exists a polycylinder $U \times W := \{|z_i| < \rho_i\,;\ 1 \leqq i \leqq n\} \times \{|w_j| < \varepsilon_j\,;\ 1 \leqq j \leqq n-k\}$ with the following properties: the canonical projection σ of $M^* \cap (U \times W)$ onto the set $U^k \times W := \{|z_i| < \rho_i\,;\ 1 \leqq i \leqq k\} \times W$ is proper and nowhere degenerate and M_0 satisfies the condition (i) for this U. Moreover, we can take another polycylinder $V:=\{|z_i|<\rho_i';\ 1\leqq i\leqq n\}$ with $\rho_i'<\rho_i$ $(1\leqq i\leqq n)$ and sufficiently small polycylinder $W':=\{|w_j|<\varepsilon;\ 1\leqq j\leqq n-k\}$ satisfying the condition $\sigma^{-1}(\bar{V}^k\times W')=M^*\cap(\bar{V}\times W')$. Then σ is also a proper and nowhere degenerate mapping of the analytic set $L^*:=((U-\bar{V})\times W')\cap M^*$ onto the domain $D:=(U^k-\bar{V}^k)\times W'$. Hence σ is open and closed on L^*. Since each connected component C^* of L^* is open and closed in L^*, its image $\sigma(C^*)$ is open and closed, and therefore covers the domain D. This shows each C^* intersects with M_0. Now, we take a point p in L and the point p^* in L^* corresponding to p. The connected component of L containing p includes the image of the connected component of L^* containing p^*, which intersects with M_0. Thus Lemma 4 has been proved.

LEMMA 5. *Let φ be the function* 1) 0, 2) z, 3) z^2 *or* 4) zw *in the coordinates z, w of C^2, N be a thin analytic set in a neighborhood V of the origin and κ be a positive constant. For the function $v=\mathrm{Re}\,\varphi+\kappa(|z|^2+|w|^2)$, there exists a fundamental system of neighborhoods U of the origin such that any curve c in $G:=U-N$ with the end points $c(0)$, $c(1)$ in G_0 is homotopic in G to a curve contained in G_0, where G_0 denotes the set $G\cap\{v<0\}$.*

PROOF. For the case 1), it is sufficient to take an arbitrary connected neighborhood U of the origin.

For the other cases, we take a sufficiently small neighborhood $U:=U'\cap\{\mathrm{Re}\,\varphi>-\alpha\}$ for a polycylinder $U'=\{|z|<\rho, |w|<\rho'\}$ and a positive number α such that $U_0=U\cap\{v>0\}$ is connected and $U\cap\{v\leqq 0\}\cap\{\varphi=\beta\}$ is compact in $U\cap\{\varphi=\beta\}$ for each β with $\mathrm{Re}\,\beta>-\alpha$. In fact, this is possible. In any case, the set $\{w: v(a, w)>0\}$ is the exterior of a disc in the w-plane for every a. Making U sufficiently small, we see the fiber $U_0^{(a)}=U\cap\{z=a\}\cap\{v>0\}$ is a non-empty connected set for an arbitrarily fixed a with $|a|<\rho$ in case 4) and furthermore $\mathrm{Re}\,\varphi(a)>-\alpha$ in case 2) or 3) where α is sufficiently small. From this it follows easily that U_0 is connected. Then, it is sufficient to take α satisfying $\alpha<\min(\kappa\rho^2, \kappa\rho'^2)$.

Under these circumstances we shall show the set U satisfies the conditions in Lemma 5. By the same argument as above for a sufficiently small polycylinder U', we see $U_0'=U'\cap\{v>0\}$ is connected. If N is of dimension 0 at the origin, it is sufficient to take a polycylinder such that U_0' is connected. Suppose N to be of dimension 1. Then N may be assumed to be defined as follows;

$$N:\ \ \chi(z, w)=0 \tag{7}$$

where $\chi(z, w)$ is holomorphic in a neighborhood of $\bar{U}$. Without loss of generality, we may assume the given curve $c(t)=(z(t), w(t))$ has the properties $z(t)w(t)\neq 0$, whence we restrict ourselves to the set $U\cap\{zw\neq 0\}$. For the case 4), we take new coordinates $x=zw$ and $y=w$ in $U\cap\{zw\neq 0\}$. For the case 2) and 3), we take the original coordinates $x=z$ and $y=w$ in U. By the connectivity of G_0 the curve $c(t)=(x(t), y(t))$ may be assumed to satisfy the conditions that $c(0)$ and $c(1)$ are equal to a point p in U with $\operatorname{Re}\varphi(p)>0$ and N can be represented as $y=\psi(x)$ in a neighborhood of each point of $N\cap\{x=x(t)\}$. Take the set $\mathfrak{T}$ of parameters τ in the interval $[0, 1]$ such that there exists a curve $c_\tau(t)=(x_\tau(t), y_\tau(t))$ homotopic to $c(t)$ in $G\cap\{zw\neq 0\}$ with the properties (i) for some sequence $\tau_0=0,\ \tau_1, \cdots, \tau_{2r}=\tau$ $c_\tau(t)\in G_0$ if $\tau_{2j}\leqq t\leqq\tau_{2j-1}$ $(1\leqq j\leqq r)$ and $x_\tau(t)=x(\tau)$ if $\tau_{2j+1}\leqq t\leqq\tau_{2j}$ $(0\leqq j\leqq r)$, and (ii) $c_\tau(t)=c(t)$ if $t\geqq\tau$. And let τ_0 be the least upper bound of $\mathfrak{T}$. Suppose $\tau_0<1$. From the assumption of $c(t)$, N has only a finite number of points over $x=x(t)$ on $\bar{U}$ which depend continuously on the parameter t. We denote by $y_1, \cdots, y_s$ the points of $\bar{U}\cap\{x=x(\tau_0)\}\cap N$. We consider them as points in the y-plane. Then we can take a neighborhood $T=\{t\,;\,|t-\tau_0|<\varepsilon\}$ of τ_0 and sufficiently small open discs D_i with centers y_i $(1\leqq i\leqq s)$ such that (i) $\bar{D}_i\cap\bar{D}_j=\phi$, (ii) if $(x(t), y)\in\bar{U}$, $t\in T$ and $y\notin\bar{D}_i$, then $(x(t), y)\notin N$ and (iii) if a point $(x(t), y)\in U\cap\{v\leqq 0\}$ for some $t\in T$ and $y\in\bar{D}_i$, then $(x(t), y)\in U$ for each $t\in T$ and $y\in\bar{D}_i$. By the definition of τ_0 there exists a curve $c_\tau(t)=(x_\tau(t), y_\tau(t))$ with $\tau\in T\cap\mathfrak{T}$. After a suitable deformation and parametrization c_τ may be assumed to satisfy the conditions $y_\tau(t)\notin\bar{D}_i$ for each $\tau_{2j-1}\leqq t\leqq\tau_{2j}$, $(x(t'), y_\tau(t))\in G$ for t', $t\in T$ and $(x(t), y(\tau_j))\in G_0$ for each $t\in T$, where τ_j denote the numbers defined as the condition (ii) for $\tau\in\mathfrak{T}$.

Now, we can define a new curve $c_{\tau'}$ homotopic to $c(t)$ with $\tau_0<\tau'<\tau_0+\varepsilon$ as follows. We join the curve segment $c_\tau(t)$ $(0\leqq t\leqq\tau_1)$ with the segments $(x(t), y_\tau(\tau_1))$ $(\tau\leqq t\leqq\tau')$, $(x(\tau'), y_\tau(t))$ $(\tau_1\leqq t\leqq\tau_2)$, $(x(t), y_\tau(\tau_2))$ $(\tau'\geqq t\geqq\tau$, decreasingly), $(x_\tau(t), y_\tau(t))$ $(\tau_2\leqq t\leqq\tau_3), \cdots, (x_\tau(\tau'), y_\tau(t))$ $(\tau_{2r-1}\leqq t\leqq\tau')$ and $c(t)=c_\tau(t)$ $(\tau'\leqq t\leqq 1)$, and parametrize the obtained curve so as to satisfy $\tau'\in\mathfrak{T}$. This contradicts the definition of τ_0. Hence, we have $\tau_0=1$. By the same arguments, we have also $1\in\mathfrak{T}$. Then, the curve c_1 satisfies obviously the condition in Lemma 5.

5. The purpose of this section is to prove the following

THEOREM 2. *Let τ be a holomorphic mapping of a complex manifold X into another r-dimensional complex manifold Y with the rank r at a point p and v be a real-valued function on Y which is strongly $(r-1)$-convex at $\tau(p)$. We denote by X_λ the set $\{v\circ\tau>\lambda:=v\circ\tau(p)\}$. Then, there exists a fundamental system $\mathfrak{U}$ of neighborhoods of p such that every $\mathfrak{F}$-valued holomorphic function in $U\cap X_\lambda$ admits a single-valued holomorphic continuation to U for any U in $\mathfrak{U}$.*

PROOF. Since, by Proposition 3, X_λ is analytically concave at p, it is sufficient to show the existence of a fundamental system of neighborhoods of p which satisfies the condition of Proposition 4.

The function v on Y has the expansion (5) on a submanifold $L'=\{w_1=\cdots=w_{r-2}=0\}$ of Y in a neighborhood of $\tau(p)$, where the polynomial φ may be assumed to be 1) 0, 2) w_r, 3) w_r^2, or 4) $w_{r-1}w_r$ for a suitable system of local coordinates $w_1, \cdots, w_r$ at $\tau(p)$ with $\tau(p)=(0)$.

We take again a family of analytic sets

$$M_t\,;\ \tau_1-t_1=\cdots=\tau_{r-2}-t_{r-2}=z_{r+1}-t_{r-1}=\cdots=z_n-t_{n-2}=\varphi'-t_{n-1}=0$$

defined in the proof of Proposition 3. By the assumption $r=\dim Y$, the inverse image of the analytic set $\{w_1=\cdots=w_{r-2}=0\}$ by $\tau|\{z_{r+1}=\cdots=z_n=0\}$ is of dimension two and hence M_0 is of dimension one. Accordingly, we can take polycylinders U, V and positive number ε as in Lemma 4. Then, we may assume $v\circ\tau>\lambda$ on $M_t\cap(U-\bar{V})$ and $v\circ\tau$ is plurisubharmonic on M_t for $|t_j|<\varepsilon$, because the proper values of a matrix depend continuously on its components. Under these circumstances, we finish the proof if we can take a connected set M' such that $(M_0-\{p\})\cap U\subset M'\subset P_\lambda:=P\cap\{v\circ\tau>\lambda\}$ for the neighborhood $P:=\bigcup_{|t_j|<\varepsilon} M_t\cap U$ of p.

To see this, we take a connected component C of P_λ and a point q in C, where q is contained in some $M_t\cap U$. Then C includes the connected component C' of $M_t\cap U$ containing q, which intersects with $U-\bar{V}$. Otherwise, C' is contained in $\bar{V}\cap M_t$ and v has the maximum at a point q' in $\bar{C}'$, which cannot be an interior point of C' by the maximum principle of v. On the other hand, q' belongs to $M_t\cap U$ and has a connected neighborhood on $M_t\cap U$. This is a contradiction. According to Lemma 4, any connected component of the subset $\bigcup_{|t_j|<\varepsilon} M_t\cap(U-\bar{V})$ of P_λ intersects with M_0. Hence, C intersects with M_0 and eventually the fixed connected subset M' of p. This shows P_λ itself is connected.

Now, we show the existence of the connected set M' with the desired properties. We take a new real-valued function $v'=\lambda+\mathrm{Re}\,\varphi+\kappa(|w_{r-1}|^2+|w_r|^2)$ for a positive constant κ such that $v'\leqq v$ holds in a neighborhood of $\tau(p)$ in L'. On the other hand, the mapping τ is nowhere degenerate on $L=\{z_{r+1}=\cdots=z_n=0\}$ and therefore the restriction of τ to the analytic set $L^*:=\{\tau_1=\cdots=\tau_{r-2}=0\}\cap L$ is also nowhere degenerate. As is well known, there exist a neighborhood U^* of p in L^*, U' of $\tau(p)$ in L' and a thin analytic set N in U' such that $U^*-\tau^{-1}(N)$ is an unramified and unlimited covering space over $U'-N$ with the covering mapping τ. Then, regarding the above coordinate neighborhood of $\tau(p)$ in L' as an open subset of C^2, we may assume the neighborhood U' satisfies the condition of Lemma 5 for the above function v' and the analytic set N. Moreover, each irreducible component U_i^* of U^* may be assumed to contain p and to be irreducible at p.

We shall show the open set $U_i^\dagger:=\tau^{-1}(U'\cap\{v'>\lambda\})\cap U^*$ is connected.

To this end, we take two arbitrary points q_1, q_2 in $U_\lambda^* \cap (U_i^* - \tau^{-1}(N))$ and a curve c' in $U_i^* - \tau^{-1}(N)$ joining q_1 to q_2 by the connectivity of $U_i^* - \tau^{-1}(N)$. According to Lemma 5 the τ-image $c = \tau c'$ of c' is homotopic in $U' - N$ to a curve c'' in $U' \cap \{v' > \lambda\} - N$. Then there exists a unique curve c^* in $U_\lambda^* \cap (U_i^* - \tau^{-1}(N))$ with the initial point q_1 such that $\tau c^* = c''$. Obviously, c^* has the terminal point q_2. This shows $U_\lambda^* \cap (U_i^* - \tau^{-1}(N))$ is arcwise connected, and hence $U_\lambda^* \cap U_i^*$ is connected. On the other hand, since the analytic set L^* is a complete intersection at p, Proposition 6 implies that, for any two irreducible components, say, U_0^* and U_1^*, there exist a chain of irreducible components $U_1^*, U_2^*, \cdots, U_s^* = U_0^*$ such that $\dim_p U_i^* \cap U_{i+1}^* \geqq 1$ for every $i = 1, \cdots, s-1$. Then, $U_i^* \cap U_{i+1}^* \cap U_\lambda^*$ is not empty because the image of $U_i^* \cap U_{i+1}^*$ is a 1-dimensional analytic set in U' through $\tau(p)$ and therefore contains at least one point where $v' > \lambda$, by Proposition 5. Thus, we have shown the connectivity of U_λ^*.

We put $M' := (U \cap (M_0 - \{p\})) \cup U_\lambda^*$. It is easy to show M' has the desired properties. Theorem 2 has been proved completely. q. e. d.

REMARK. If the mapping τ in Theorem 2 has the rank $r \geqq 3$ at p and v is a strongly plurisubharmonic function, then Theorem 2 is true without the hypothesis $r = \dim Y$. For, in this case the connected analytic set M_0 in the proof of Proposition 3 is a complete intersection at p and of dimension at least two. Proposition 5 implies the set $M' := M_0 - \{p\}$ is connected. The set M' satisfies the same condition as in the proof of Theorem 2.

§ 5. Sufficient conditions for the global continuability.

1. Let X be a complex manifold. We shall say that a pair (D, K) is *an H-problem on X* if K is a compact subset of X and D is an open subset of X containing K such that $D - K$ is connected. An H-problem (D, K) is said to be *solvable* if for an arbitrary $\mathfrak{F}$ every $\mathfrak{F}$-valued holomorphic function f in $D - K$ can be holomorphically continuable to the whole D, that is, we can find an $\mathfrak{F}$-valued holomorphic function g in D satisfying $g = f$ on $D - K$. In this section we shall give some conditions under which H-problems are solvable.

DEFINITION 7. A real-valued continuous function v on X is called admissible on X if it satisfies the following two conditions:

1) v satisfies the maximum principle, that is, for any $p \in X$ we can find points p_ν $(\nu = 1, 2, \cdots)$ satisfying $v(p_\nu) > v(p)$ and $\lim_{\nu \to \infty} p_\nu = p$.

2) Any point p in X has a fundamental system $\mathfrak{U}$ of connected neighborhoods of p such that for every U in $\mathfrak{U}$ any $\mathfrak{F}$-valued holomorphic function in $U \cap \{v > v(p)\}$ admits a single-valued holomorphic continuation to the whole U.

From 2) we have easily that $U \cap \{v > v(p)\}$ is connected if U belongs to $\mathfrak{U}$.

Examples of admissible functions are easily obtained from Theorem 2 and its remark. (See n°. 4. 4) and 5) of this section.)

2. LEMMA 6. *Let X be a paracompact complex manifold and B an open subset of X. Let Γ be a subset of the boundary ∂B of B and $\{U_\lambda ; \lambda \in \Lambda\}$ an open covering of Γ. Assume that for each $\lambda \in \Lambda$ there exists an $\mathcal{F}$-valued holomorphic function f_λ in U_λ satisfying $f_\lambda = f_{\lambda'}$ on $U_\lambda \cap U_{\lambda'} \cap B$ for any $\lambda, \lambda' \in \Lambda$. Then we can find an open covering $\{V_\nu ; \nu \in N\}$ of Γ and a mapping λ of N into Λ such that $V_\nu \subset U_{\lambda(\nu)}$ for each $\nu \in N$ and $g_\nu = g_{\nu'}$ on $V_\nu \cap V_{\nu'}$ for any $\nu, \nu' \in N$ where $g_\nu = f_{\lambda(\nu)} | V_\nu$.*

PROOF. Since X is paracompact, we can take an open covering $\{V'_\nu ; \nu \in N\}$ of Γ such that $V'_\nu \cap V'_{\nu'}$ is either empty or connected for any $\nu, \nu' \in N$ and there exists a mapping λ of N into Λ satisfying $\bar{V}'_\nu \subset U_{\lambda(\nu)}$ for any $\nu \in N$, where $N = \{1, 2, 3, \cdots\}$.

We put $V_1 = V'_1$. Assume $\nu \geqq 2$. Let $\{\mu_1^{(\nu)}, \cdots, \mu_{l_\nu}^{(\nu)}\}$ be the set of all indices μ such that $1 \leqq \mu < \nu$, $V'_\mu \cap V'_\nu \neq \phi$ and the connected component of $U_{\lambda(\mu)} \cap U_{\lambda(\nu)}$ containing $\overline{V'_\mu \cap V'_\nu}$ does not intersect with B. We denote by V_ν the set $V'_\nu - \bigcup_{j=1}^{l_\nu} \overline{V'_{\mu_j^{(\nu)}} \cap V'_\nu}$. If $p \in \Gamma$, $p \notin V'_1 \cup \cdots \cup V'_{\nu-1}$ and $p \in V'_\nu$ then p can not belong to $\bar{V}'_{\mu_j^{(\nu)}}$ and we have $p \in V_\nu$. Thus we have easily $\Gamma \subset \bigcup_{\nu=1}^{\infty} V_\nu$. If $V_\nu \cap V_{\nu'} \neq \phi$, then $V'_\nu \cap V'_{\nu'}$ is not empty and the connected component of $U_{\lambda(\nu)} \cap U_{\lambda(\nu')}$ containing it intersects with B. Therefore we have $g_\nu = g_{\nu'}$ on $V_\nu \cap V_{\nu'}$. q. e. d.

We say merely that an $\mathcal{F}$-valued holomorphic function is defined and holomorphic in a set if it is defined and holomorphic in a neighborhood of the set. For two $\mathcal{F}$-valued holomorphic functions f and g, '$f = g$ at a point p' means that $f = g$ in a neighborhood of p.

LEMMA 7. *Let D, B and G be open subsets of a paracompact complex manifold X satisfying $X \supset D \Supset B$ and $X \Supset G$. Let us assume that $-\infty \leqq \alpha < \beta$ and v is an admissible function on X satisfying the followings:*

1) *Each point p in $\partial G \cap \{v > \alpha\}$ has a neighborhood $U(p)$ satisfying $U(p) \cap G = U(p) \cap B$.*

2) *Each point p in $\partial G \cap \{\alpha < v \leqq \beta\}$ belongs to the set $\overline{\{v > v(p)\} \cap (X - \bar{G})}$.*

Let f be an $\mathcal{F}$-valued holomorphic function in $D - B$ and g_β an $\mathcal{F}$-valued holomorphic function in $(D \cup G) \cap \{v > \beta\}$. Assume that $(D \cup G) \cap \{v > \beta\} \neq \phi$ and $f = g_\beta$ at each point of $(D - G) \cap \{v > \beta\}$.

Then we can uniquely find an $\mathcal{F}$-valued holomorphic function g_α in $(D \cup G) \cap \{v > \alpha\}$ such that $f = g_\alpha$ at each point of $(D - G) \cap \{v > \alpha\}$.

PROOF. Let Λ be the set of all λ such that $\alpha \leqq \lambda \leqq \beta$ and there exists an $\mathcal{F}$-valued holomorphic function g_λ in $(D \cup G) \cap \{v > \lambda\}$ satisfying $f = g_\lambda$ at each point of $(D-G)\cap\{v>\lambda\}$. We have $\beta\in\Lambda$ and so $\Lambda\neq\phi$.

Let C be a connected component of $(D\cup G)\cap\{v>\lambda\}$ where $\lambda\in\Lambda$. If $C\subset G$, we can take easily a sequence of points p_ν in C and a point p in $\bar{G}$ satisfying $\lim_{\nu\to\infty} p_\nu=p$ and $v(p)=\sup v(C)$. Since v satisfies the maximum principle we have $p\notin G\cup D$, whence $p\in\partial G$. This is contrary to the assumption $\partial G\subset\partial B\subset D$. Therefore C is not contained in G, and we have $C\cap(D-G)\cap\{v>\lambda\}\neq\phi$. This shows the uniqueness of g_λ.

Suppose that $\lambda_0=\inf\Lambda$ would be greater than α. Because of the uniqueness of g_λ, λ_0 belongs to Λ and g_{λ_0} exists. Since v is admissible, g_{λ_0} has a single-valued holomorphic continuation in a neighborhood of each point in $\bar{G}\cap\{v=\lambda_0\}$. If $p\in\partial G\cap\{v=\lambda_0\}$, f is already given in a subset of a neighborhood of p. From the assumption 2) each neighborhood of p intersects with $\{v>\lambda_0\}\cap(D-\bar{G})$, and therefore the continuation of g_{λ_0} coincides with f in a subset of a neighborhood of p. By Lemma 6, there exists a neighborhood W of $\bar{G}\cap\{v=\lambda_0\}$ contained in $D\cup G$ and an $\mathcal{F}$-valued holomorphic function $\tilde{g}_{\lambda_0}$ in W satisfying $g_{\lambda_0}=\tilde{g}_{\lambda_0}$ on $W\cap\{v>\lambda_0\}$. Furthermore there exists a neighborhood W' of $\partial G\cap\{v=\lambda_0\}$ contained in W where $\tilde{g}_{\lambda_0}=f$. Then we can take λ_1 such that $\lambda_1<\lambda_0$, $\bar{G}\cap\{\lambda_1<v\leqq\lambda_0\}\subset W$ and $\partial G\cap\{\lambda_1<v\leqq\lambda_0\}\subset W'$. The function g_{λ_1} such that $g_{\lambda_1}=f$ in $(D-G)\cap\{v>\lambda_1\}$, $g_{\lambda_1}=g_{\lambda_0}$ in $(D\cup G)\cap\{v>\lambda_0\}$ and $g_{\lambda_1}=\tilde{g}_{\lambda_0}$ in $\bar{G}\cap\{\lambda_1<v\leqq\lambda_0\}$ is well-defined and asserts $\lambda_1\in\Lambda$. This is contrary to the definition of λ_0, and implies $\alpha=\inf\Lambda$. As previously stated, we have $\inf\Lambda=\min\Lambda$ which concludes the proof.

3. As a corollary of Lemma 7 we have

PROPOSITION 7. *Let X be a paracompact complex manifold and (D,K) an H-problem on X. Let us assume that there exist an admissible function v on X and an open set B such that $D\Supset B\supset K$ and each point p of ∂B belongs to the set $\overline{\{v>v(p)\}\cap(X-\bar{B})}$. Then the H-problem (D,K) is solvable.*

PROOF. We put $\beta=\sup v(B)$. Let f be an $\mathcal{F}$-valued holomorphic function in $D-K$. Putting $G=B$, $g_\beta=f$ and $\alpha=-\infty$, we can apply Lemma 7 and obtain an $\mathcal{F}$-valued holomorphic function $g_{-\infty}$ in D which coincides with f on $D-B$. Since $D-K$ is connected we have $g_{-\infty}=f$ on $D-K$. q. e. d.

In Proposition 7 the assumption about the boundary ∂B depends on v. We want to give a sufficient condition which is idependent on v.

DEFINITION 8. Let v be an admissible function on a complex manifold X. We shall say that v is of degree r if for every point p in X there exist a neighborhood U of p and an analytic set M in U satisfying $\dim_p M\geqq r$ and $M-\{p\}\subset U\cap\{v>v(p)\}$.

Examples of admissible functions of degree r are obtained by the remark of Proposition 3 and Theorem 2. (See n°. 4. 4) and 5) of this section.)

THEOREM 3. *Let X be a paracompact complex manifold and (D, K) an H-problem on X. Let us assume that there exist an admissible function v of degree r and a Rothstein r-convex set B satisfying $D \Supset B \supset K$, where $r \geqq 1$. Then the H-problem (D, K) is solvable.*

PROOF. By the assumptions on v and B, we have easily that each point p in ∂B belongs to $\overline{\{v > v(p)\} \cap (X-\bar{B})}$. By Proposition 7 this proves our theorem.

In virtue of Proposition 5 we have

COROLLARY. *We assume that X, (D, K) and v are the same as in Theorem 3. Let us assume that there exists a strongly r-convex set B satisfying $D \Supset B \supset K$. Then the H-problem (D, K) is solvable.*

4. We are now going to study some conditions on a complex manifold under which every H-problem is always solvable.

THEOREM 4. *Let X be a complex manifold. Let us assume that there exists a real analytic admissible function v on X such that the set $\{\alpha < v < \beta\}$ is relatively compact in X for any real numbers $\alpha < \beta$. Then every H-problem on X is solvable.*

The following two sections are employed in the proof of this theorem. Here, we shall give some examples of complex manifolds satisfying the assumptions of Theorem 4. For brevity, we assume that X and Y are always purely dimensional complex manifolds in the following examples:

1) *An n-dimensional Stein manifold X where $n \geqq 2$.*

In fact, we can regard X as a closed submanifold of C^{2n+1} $(z_1, \cdots, z_{2n+1})$ by the Remmert-Narasimhan's imbedding theorem ([**9**]). The function $v = \sum_{i=1}^{2n+1} |z_i|^2$ satisfies the assumptions of Theorem 4 on X.

2) *A product $X \times Y$ of an n-dimensional Stein manifold X and a compact complex manifold Y where $n \geqq 2$.*

3) *A complex manifold X such that there exists a proper holomorphic mapping τ of X into an r-dimensional Stein manifold Y where $r \geqq 2$ satisfying either $r_\tau(p) = r$ for any $p \in X$ or $r_\tau(p) \geqq 3$ for any $p \in X$.*

2) and 3) are corollaries of the following 4) and 5).

4) *Let τ be a holomorphic mapping of a complex manifold X into an r-dimensional complex manifold Y where $r \geqq 2$. We assume $r_\tau(p) = r$ for any $p \in X$. If v is a strongly s-convex function on Y where $1 \leqq s \leqq r-1$, then $v \circ \tau$ is an admissible function of degree $r-s$ on X by the remark of Proposition 3 and Theorem 2. If τ is proper and v is a real analytic strongly $(r-1)$-convex function on Y such that $\{\alpha < v < \beta\} \Subset Y$ for any real numbers $\alpha < \beta$, then X and $v \circ \tau$ satisfy the assumptions of Theorem 4.*

5) *Let τ be a holomorphic mapping of a complex manifold X into a complex manifold Y and v a strongly plurisubharmonic function on Y. If $r_\tau(p) \geqq 3$*

for any $p\in X$, then $v\circ\tau$ is an admissible function on X in view of the remark of Theorem 2. *If $r_\tau(p)=r\geqq 3$ for any $p\in X$, then $v\circ\tau$ is an admissible function of degree $r-1$ by the remark of Proposition* 3. *If $r_\tau(p)\geqq 3$ for any $p\in X$, τ is proper and v is a real analytic function satisfying $\{\alpha<v<\beta\}\Subset Y$ for any real numbers $\alpha<\beta$, then X and $v\circ\tau$ satisfy the assumptions of Theorem* 4.

5. By the Whitney-Shiga approximation theorem we can weaken the assumption of the real analyticity of v in the examples 4) and 5).

THEOREM 5. *Let τ be a proper holomorphic mapping of a complex manifold X into a purely r-dimensional complex manifold Y where $r\geqq 2$. We assume one of the followings:*

1) *$r_\tau(p)=r$ for any $p\in X$ and there exists a strongly $(r-1)$-convex function v on Y.*

2) *$r_\tau(p)\geqq 3$ for any $p\in X$ and there exists a strongly plurisubharmonic function v on Y.*

In either case we assume $\{\alpha<v<\beta\}\Subset Y$ for any real numbers $\alpha<\beta$.

Then every H-problem on X is always solvable.

PROOF. By the assumptions, X and Y are paracompact. In view of the following theorem we can easily find a real analytic function v on Y which approximates v and satisfies the conditions of the example 4) or 5).

THEOREM. (H. Whitney [**15**] and K. Shiga [**14**].) *Let X and Y be real analytic manifolds where X is paracompact. Let τ be a C^r-mapping of X into Y where $1\leqq r<\infty$. We give a locally finite covering $\{U_\iota;\iota\in I\}$ of X and a covering $\{W_\lambda;\lambda\in\Lambda\}$ of Y such that each U_ι and each W_λ are coordinate neighborhoods and there exists a mapping λ of I into Λ satisfying $\tau(U_\iota)\Subset W_{\lambda(\iota)}$ for any $\iota\in I$. Moreover we give an open covering $\{V_\iota;\iota\in I\}$ of X satisfying $V_\iota\Subset U_\iota$ for any $\iota\in I$ and a positive number ε_ι for each $\iota\in I$.*

Then there exists a real analytic mapping σ of X into Y satisfying the followings:

(1) *$\sigma(U_\iota)\subset W_{\lambda(\iota)}$ for any $\iota\in I$.*

(2) *$\|\sigma-\tau\|_{V_\iota}<\varepsilon_\iota$, $\left\|\dfrac{\partial^\nu\sigma}{\partial x^\nu}-\dfrac{\partial^\nu\tau}{\partial x^\nu}\right\|_{V_\iota}<\varepsilon_\iota$ for $1\leqq\nu\leqq r$ and any $\iota\in I$. (Regarding σ and τ as mappings of U_ι into $W_{\lambda(\iota)}$ and using local coordinates of U_ι and $W_{\lambda(\iota)}$, we denote by $\|\ \|_{V_\iota}$ the supremum in V_ι and by $\dfrac{\partial^\nu\sigma}{\partial x^\nu}$ the partial derivative of order ν.)*

§ 6. Some properties of real analytic functions.

1. In order to prove Theorem 4 we need some properties of real analytic functions. Let U be a neighborhood of the origin 0 in the space $\mathfrak{R}^m$ of m real variables $x_1,\cdots,x_m$. We write merely x instead of $(x_1,\cdots,x_m)$. We denote by $\mathcal{A}$ the set of real-valued functions which are real analytic in U and vanish at 0.

PROPOSITION 8. *Let* φ *be a function in* $\mathcal{A}$ *and* N *be a set* $\left\{x \in U; \frac{\partial \varphi}{\partial x_1} = \cdots = \frac{\partial \varphi}{\partial x_m} = 0\right\}$. *We assume* $0 \in N$. *Then there exists a neighborhood* V *of* 0 *contained in* U *such that the restriction of* φ *to* $N \cap V$ *is identically zero.*

PROOF. If we regard the variables $x_1, \cdots, x_m$ as complex variables, the set N is an analytic set. We may assume $\dim_0 N > 0$. The number of irreducible components of N at 0 is finite. Taking local coordinates $u_1, \cdots, u_t$ at an ordinary point of an irreducible component, we have

$$\frac{\partial \varphi}{\partial u_j} = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} \frac{\partial x_i}{\partial u_j} \equiv 0 \qquad (j = 1, \cdots, t).$$

From these equations, φ must be identically constant on the coordinate neighborhood of the ordinary point and therefore on the whole component. Because of $\varphi(0) = 0$, we have $\varphi \equiv 0$ on $N \cap V$ where V is a sufficiently small neighborhood of 0. Restricting variables to real numbers, we conclude the proof.

A real number α is called *a stationary value of a differentiable function* φ if $\alpha = \varphi(p)$ and each derivative $\frac{\partial \varphi}{\partial x_i}$ vanishes at p. We have

COROLLARY. *A real-valued real analytic function on a real analytic compact manifold has only finitely many stationary values.*

2. If there is no possibility of misunderstanding, we shall say merely that a curve lies in a set even if the curve lies in the set with the exception of the end points.

In general the set of zeros of a real analytic function is very complicated. It is not always locally connected. We give

PROPOSITION 9. *Let* $\varphi_1, \cdots, \varphi_s$ *be functions in* $\mathcal{A}$. *Then there exists a neighborhood* V *of* 0 *contained in* U *satisfying the following: For an arbitrary decomposition* $\{\nu_1, \cdots, \nu_l\} \cup \{\mu_1, \cdots, \mu_{s-l}\}$ *of the set* $\{1, \cdots, s\}$ $(0 \leqq 1 \leqq s)$, *if* $G = \{x \in V; \varphi_{\nu_j} > 0, \varphi_{\mu_k} < 0, (j = 1, \cdots, l;\ k = 1, \cdots, s-l)\}$ *is not empty, any point of* G *can be joined to* 0 *by a curve lying in* G.

PROOF. We give a proof by induction on m. We assume that the proposition is valid if $\varphi_1, \cdots, \varphi_s$ are functions of $m-1$ variables, and show that it holds for functions of m variables. We can see at once that it is true for functions of one variable.

After suitable transformations of coordinates, by a Späth's theorem we may assume that $\varphi_1, \cdots, \varphi_s$ are distinguished polynomials of x_m in a small neighborhood U' of 0 contained in U. The product $\varphi_1 \cdots \varphi_s$ can be decomposed into the product $\phi_1^{r_1}\cdots\phi_t^{r_t}$ of (real) irreducible factors where $\phi_i\neq\phi_j$ if $i\neq j$. We denote by ϕ the product $\phi_1\cdots\phi_t$. Since ϕ has no multiple factors, the discriminant δ of ϕ can not be identically zero. We take a neighborhood $V=V'\times\{|x_m|<\rho\}$ of 0 contained in U' satisfying the followings: V' is a neighborhood of the origin of the space of $m-1$ variables $x_1,\cdots,x_{m-1}$, and any point of $V'\cap\{\delta\neq 0\}$ can be joined to the origin by a curve lying in $V'\cap\{\delta\neq 0\}$. Every solution x_m of the equation $\phi(x_1,\cdots,x_{m-1},x_m)=0$ satisfies $|x_m|<\rho$ for any $(x_1,\cdots,x_{m-1})\in V'$.

Let $(x_1^{(0)},\cdots,x_m^{(0)})$ be an arbitrary point of $G\cap V$. Since G is open, we may assume $\delta(x_1^{(0)},\cdots,x_{m-1}^{(0)})\neq 0$. In $V'\cap\{\delta\neq 0\}$ there exists a curve $c':x_i=x_i(t)$ where $x_i(0)=x_i^{(0)}$, $x_i(1)=0$, $0\leqq t\leqq 1$ and $i=1,\cdots,m-1$. We denote by $x_m^{(2)}(t)$, $\cdots$, $x_m^{(r-1)}(t)$ the real roots of the equation $\phi(x_1(t),\cdots,x_{m-1}(t),x_m)=0$ and assume $x_m^{(2)}(t)<\cdots<x_m^{(r-1)}(t)$ for $0\leqq t<1$. We can see easily that r does not depend on the parameter t and $x_m^{(\nu)}(t)$ is a continuous function of t, $(2\leqq\nu\leqq r-1)$. We put $x_m^{(1)}(t)=-\rho$ and $x_m^{(r)}(t)=\rho$ for $0\leqq t\leqq 1$. We take ν satisfying $x_m^{(\nu)}(0)<x_m^0<x_m^{(\nu+1)}(0)$ and a continuous function $x_m(t)$ for $0\leqq t\leqq 1$ such that $x_m^{(\nu)}(t)<x_m(t)<x_m^{(\nu+1)}(t)$ for $0\leqq t<1$ and $x_m(1)=0$. Thus we obtain the curve $c:x_i=x_i(t)$ where $0\leqq t\leqq 1$ and $i=1,\cdots,m$, which proves our assertion.

3. PROPOSITION 10. *Let $\varphi_1,\cdots,\varphi_s$ be functions in $\mathcal{A}$. We assume that the Jacobian matrix $\left(\frac{\partial\varphi_\nu}{\partial x_i}\right)_{\substack{1\leqq\nu\leqq s\\1\leqq i\leqq m}}$ has rank s at 0. We put $\varphi_0=\sum_{i=1}^m(x_i-a_i)^2-\rho^2$. Then we can take a neighborhood V of 0 contained in U which satisfies the following: For any positive number ρ with the exception of at most one there exists a set N_ρ of the first category in $\mathcal{R}^m$ such that for any $(a_1,\cdots,a_m)\notin N_\rho$ the Jacobian matrix $\left(\frac{\partial\varphi_\nu}{\partial x_i}\right)_{\substack{0\leqq\nu\leqq s\\1\leqq i\leqq m}}$ has rank $s+1$ at each point of the set $M=\{x\in V;\ \varphi_0=\cdots=\varphi_s=0\}$.*

PROOF. We may assume that $s<m$ and the simultaneous equations $\varphi_1=0_1,\cdots,\varphi_s=0$ have a solution $x_j=\phi_j(x_{s+1},\cdots,x_m)$ $(j=1,\cdots,s)$ which vanishes at the origin and is real analytic in a neighborhood of the origin. Substituting these equations in φ_0, we obtain

$$\varphi_0=\sum_{j=1}^s(\phi_j-a_j)^2+\sum_{k=s+1}^m(x_k-a_k)^2-\rho^2$$

and

$$\frac{1}{2}\frac{\partial\varphi_0}{\partial x_k}=\sum_{j=1}^s(\phi_j-a_j)\frac{\partial\phi_j}{\partial x_k}+x_k-a_k\qquad(k=s+1,\cdots,m).$$

We denote by $\tilde{N}_\rho$ a subset of $V\times\mathcal{R}^m$ defined by the relations

$$\left\{\begin{array}{l} \sum_{j=1}^{s}(\phi_j-a_j)^2+\sum_{k=s+1}^{m}\left\{\sum_{j=1}^{s}(\phi_j-a_j)\dfrac{\partial\phi_j}{\partial x_k}\right\}^2=\rho^2\,, \\ x_j=\phi_j(x_{s+1},\cdots,x_m)\,,\quad a_k=x_k+\sum_{j=1}^{s}(\phi_j-a_j)\dfrac{\partial\phi_j}{\partial x_k}\,, \\ (x_1,\cdots,x_m)\in V \text{ and } (a_1,\cdots,a_m)\in\mathfrak{R}^m \qquad (1\leqq j\leqq s\,;\ s+1\leqq k\leqq m)\,. \end{array}\right. \tag{8}$$

In order that the rank of the Jacobian matrix $\left(\dfrac{\partial\varphi_\nu}{\partial x_i}\right)_{\substack{0\leqq\nu\leqq s\\1\leqq i\leqq m}}$ is less than $s+1$ at a point $x\in M$, it is necessary and sufficient that the point (x, a) belongs to $\tilde{N}_\rho$. Let N_ρ be the image of $\tilde{N}_\rho$ under the natural projection of $V\times\mathfrak{R}^m$ onto $\mathfrak{R}^m$. To complete the proof, it is sufficient to show that N_ρ is of the first category in $\mathfrak{R}^m$.

If the left side of the first equation of (8) satisfied by $\tilde{N}_\rho$ is identically equal to a constant ρ_0, such ρ_0 is the exception. We assume $\rho\neq\rho_0$.

If we regard the variables x, a as complex variables, the set $\tilde{N}_\rho$ is either empty or a purely $(m-1)$-dimensional analytic set. By a Remmert's theorem ([**11**]) the image of $\tilde{N}_\rho$ under the holomorphic mapping is the countable union of at most $(m-1)$-dimensional locally analytic sets. The intersection of a locally analytic set of dimension at most $m-1$ and the real plane $\{\operatorname{Im} a_1=\cdots=\operatorname{Im} a_m=0\}$ is contained in the set of zeros of a real analytic function which is not identically zero, and so it is nowhere dense in $\mathfrak{R}^m$. Since a countable union of nowhere dense sets in $\mathfrak{R}^m$ is of the first category, N_ρ is of the first category in $\mathfrak{R}^m$.

§7. Proof of Theorem 4.

1. Let X be a complex manifold, K a compact subset of X and D an open neighborhood of K.

Take a point p in K and a coordinate neighborhood $U(p)$ containing p. Let $B'(p)$ and $B''(p)$ be concentric open balls with centers at p in the local coordinates such that $B'(p)\Subset B''(p)\Subset D$. Since K is compact, we can find finitely many points $p_1', \cdots, p_m'$ in K satisfying $\bigcup_{i=1}^{m} B'(p_i')\supset K$. From Proposition 10 we can take open balls $B_1, \cdots, B_m$ inductively which satisfy the followings: (i) $B'(p_i')\subset B_i\subset B''(p_i')$, (ii) for any subset $\{i_1, \cdots, i_l\}$ of $\{1, \cdots, m\}$ the set $S_{i_1}\cap\cdots\cap S_{i_l}$ is either empty or a regular surface, where $S_i=\partial B_i$ $(i=1, \cdots, m)$.

Denoting by B the set $\bigcup_{i=1}^{m} B_i$, we have $K\subset B\Subset D$. Let v be a real analytic admissible function on X such that $\{\inf v(K)-\varepsilon<v<\sup v(K)+\varepsilon\}\Subset X$ where ε is a positive number. Taking B sufficiently near to K, we may assume $\left\{\inf v(B)-\dfrac{1}{2}\varepsilon<v<\sup v(B)+\dfrac{1}{2}\varepsilon\right\}\Subset X$. Under these assumptions, we study some topological properties of B.

We denote by $X_{\alpha,\beta}$ the set $\{\alpha<v<\beta\}$ where α and β are real numbers. We put $\rho_0=\sup v(B)$. Because of Proposition 8 and its corollary, the set of stationary values of v and all $v|S_{i_1}\cap\cdots\cap S_{i_\nu}$ and of all values of v at discrete $S_{i_1}\cap\cdots\cap S_{i_\nu}$ which are between $\inf v(B)-\frac{1}{2}\varepsilon$ and ρ_0 consists of finitely many elements. We denote all of them by $\rho_s<\rho_{s-1}<\cdots<\rho_1$. In case that $\{\alpha<v<\rho_0\}\Subset X$ for any α satisfying $-\infty<\alpha<\rho_0$ the set of those which are less than ρ_0 consists of at most countable elements and is denoted by ρ_i where $\cdots<\rho_{s+1}<\rho_s<\cdots<\rho_1$ and $\inf_i\{\rho_i\}=-\infty$ unless $\{\rho_i\}$ is a finite set.

2. Now, we have

(a) *Let U be a neighborhood of $p\in X-B$ and G one of the sets $(X-B)\cap\{v>v(p)\}$, $(X-B)\cap\{v<v(p)\}$, $\partial B\cap\{v>v(p)\}$ and $\partial B\cap\{v<v(p)\}$. Then p has neighborhoods W and V such that $W\subset V\subset U$ and any point of $W\cap G$ can be joined to p by a curve lying in $V\cap G$.*

PROOF. We may assume $p\in S_1\cap\cdots\cap S_l$ and $p\notin S_{l+1}\cup\cdots\cup S_m$ $(0\leqq l\leqq m)$. Taking a small neighborhood of p, we may disregard $S_{l+1},\cdots,S_m$. For any decomposition $\{i_1,\cdots,i_\nu\}\cup\{j_1,\cdots,j_{l-\nu}\}$ of $\{1,\cdots,l\}$, we can take local coordinates of the regular surface $S_{i_1}\cap\cdots\cap S_{i_\nu}$ at p. Using the local coordinates, we make the inequalities giving the exteriors of the closed balls $\bar{B}_{j_1},\cdots,\bar{B}_{j_{l-\nu}}$ and the set $\{v>v(p)\}$ (or $\{v<v(p)\}$). By Proposition 9 we obtain the neighborhood $V_{i_1\cdots i_\nu}$ as in the proposition. We denote by W the set $\cap V_{i_1\cdots i_\nu}$ and by V the set $\cup V_{i_1\cdots i_\nu}$ where $i_1,\cdots,i_\nu$ runs on all of subsets of $\{1,\cdots,l\}$, which asserts the proof.

(b) *Let p be a point in X satisfying $v(p)<\rho_0$ and $v(p)\neq\rho_i$ for all i. Denoting by G one of the sets $X-\bar{B}$, $X-B$, ∂B and B, we obtain a neighborhood U of p such that any two points of $G\cap U$ can be joined by a curve in $G\cap U$ on which v varies monotonously. (U does not depend on the choice of G.)*

PROOF. We may assume $p\in S_1\cap\cdots\cap S_l$ and $p\notin S_{l+1}\cup\cdots\cup S_m$, and disregard $S_{l+1},\cdots,S_m$. Let $x_1,\cdots,x_{2n}$ be local coordinates (real) of X at p centered at the origin. Using the local coordinates, we denote by $\varphi(x)$ the function $v-v(p)$ and by $\psi_\nu(x)=0$ the equation of S_ν where $\{\psi_\nu<0\}$ is in B $(\nu=1,\cdots,l)$. Since $v(p)\neq\rho_i$, we may assume that the equations $\psi_\nu=\alpha_\nu$ $(\nu=1,\cdots,l)$ and $\varphi=\rho$ have a solution

$$\begin{gathered}x_1=\chi_1(x_{l+2},\cdots,x_{2n},\alpha_1,\cdots,\alpha_l,\rho)\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{l+1}=\chi_{l+1}(x_{l+2},\cdots,x_{2n},\alpha_1,\cdots,\alpha_l,\rho)\,,\end{gathered}$$

where $\alpha_1,\cdots,\alpha_l$ and ρ have sufficiently small absolute values. For a positive number ε, χ_j is real analytic in the set

$$U=\{|x_{l+2}|<\varepsilon,\cdots,|x_{2n}|<\varepsilon,\ |\alpha_1|<\varepsilon,\cdots,|\alpha_l|<\varepsilon,\ |\rho|<\varepsilon\}$$

and $x_{l+2}, \cdots, x_{2n}, \alpha_1, \cdots, \alpha_l$ constitute local coordinates of X at p in the real analytic sense. The set U is a neighborhood of p and obviously has the property stated in the above.

(c) *We consider the set* X_{ρ_{i+1},ρ_i} *as the whole space. Let* G *be its open subset such that any point of* ∂G *has a neighborhood* U *satisfying* $U \cap \partial G = U \cap \partial B$. *Let* C *be a connected component of* G *and* ρ *a number satisfying* $\rho_{i+1} < \rho < \rho_i$. *Then the set* $C_\rho = C \cap \{v > \rho\}$ *is connected.*

PROOF. Let L be the set of curves in C joining q' and q'', where q' and q'' are two points in C_ρ. We put $\rho' = \sup_{l \in L} \{\inf v(l) ; l \in L\}$.

Suppose that $\rho' \leqq \rho$, that is, q' could not be joined to q'' by a curve in C_ρ. To any point p in the set $\bar{G} \cap \{v = \rho'\}$ we give a neighborhood $U'(p)$ as in (b), and furthermore assume $U'(p) \subset G$ if $p \in G$ and $U'(p) \cap \partial G = U'(p) \cap \partial B$ if $p \in \partial G$. Since $\bar{G} \cap \{v = \rho'\}$ is compact, we can take finitely many points $p_1, \cdots, p_{t'}$ in it satisfying $\bigcup_{\nu=1}^{t'} U'(p_\nu) \supset \bar{G} \cap \{v = \rho'\}$. We write U'_ν instead of $U'(p_\nu)$. Let $\{U''_1, \cdots, U''_t\}$ be an open covering of $\bar{G} \cap \{v = \rho'\}$ such that $U''_\nu \cap U''_\mu$ is either empty or connected for any μ, ν, and let τ be a mapping from $\{1, \cdots, t\}$ into $\{1, \cdots, t'\}$ satisfying $\bar{U}''_\nu \subset U'_{\tau(\nu)}$. We put $U_1 = U''_1$. Let $\mu_1^{(\nu)}, \cdots, \mu_{l_\nu}^{(\nu)}$ be the set of all indices μ such that $1 \leqq \mu \leqq \nu - 1$, $U''_\mu \cap U''_\nu \neq \phi$ and the connected component of $U'_{\tau(\mu)} \cap U'_{\tau(\nu)}$ containing $\overline{U''_\mu \cap U''_\nu}$ does not intersect with the set $G \cap \{v > \rho'\}$. We denote by U_ν the set $U''_\nu - \bigcup_{j=1}^{l_\nu} \overline{U''_{\mu_j^{(\nu)}} \cap U''_\nu}$, and want to show $\bigcup_{\nu=1}^{t} U_\nu \supset \bar{G} \cap \{v = \rho'\}$. In fact, suppose that a point p of $\bar{G}$ satisfies $v(p) = \rho'$ and $p \notin \bigcup_{\nu=1}^{t} U_\nu$. Then we can find ν such that $p \notin U''_1 \cup \cdots \cup U''_{\nu-1}$, $p \in U''_\nu$ and $2 \leqq \nu \leqq t$. Because of $p \notin \bigcup_{\nu=1}^{t} U_\nu$, p belongs to $\overline{U''_{\mu_j^{(\nu)}} \cap U''_\nu}$. From this fact, it follows that p is a point of ∂G and has a neighborhood $V(p)$ satisfying $V(p) \cap \{v > \rho'\} \subset \bar{G}^c$ and $V(p) \cap \partial G = V(p) \cap \partial B$. The function $v | \partial B \cap V(p)$ attains its maximum at p. Because of $v(p) \neq \rho_i$ $(i = 0, 1, 2, \cdots)$, we are led to a contradiction.

We can take $l \in L$ satisfying $l \cap \{v \leqq \rho'\} \subset U_1 \cup \cdots \cup U_t$. Since $U'_{\tau(\mu)} \cap U'_{\tau(\nu)} \cap G \cap \{v > \rho'\} \neq \phi$ if $U_\mu \cap U_\nu \neq \phi$ and U'_j satisfies the conclusion of (b), replacing $l \cap \left(\bigcup_{\nu=1}^{t} U_\nu \right)$ by arcs in $\{v > \rho'\}$ we have a curve in $G \cap \{v > \rho'\}$ joining q' and q''. This contradicts the definition of ρ'.

(d) *Let* C *be a connected component of* $(X - B) \cap \bar{X}_{\rho,\rho_i}$ *where* $\rho_{i+1} < \rho < \rho_i$. *For any* ρ' *satisfying* $\rho < \rho' < \rho_i$, *the set* $C_{\rho'} = C \cap \{v \geqq \rho'\}$ *is connected.*

PROOF. Suppose that $C_{\rho'} = O_1 \cup O_2$ where O_1 and O_2 are relatively open and disjoint. Let $C^{(\nu)}$ $(\nu = 1, 2, \cdots)$ be connected components of $(X - \bar{B}) \cap X_{\rho,\rho_i}$ contained in C. By (b) we have $C - \bigcup_\nu \bar{C}^{(\nu)} \subset \{v = \rho_i\}$ and $\bar{X}_{\rho,\rho'} \cap \bar{C}^{(\mu)} \cap \bar{C}^{(\nu)} = \phi$

if $\mu \neq \nu$. Since $C^{(\nu)} \cap \{v > \rho'\}$ is connected by (c), it is contained in either O_1 or O_2. We denote by O'_j $(j=1, 2)$ the union of O_j and all $\bar{X}_{\rho,\rho'} \cap \bar{C}^{(\nu)}$ satisfying $C^{(\nu)} \cap \{v > \rho'\} \subset O_j$. By (b) we can see easily that $C = O'_1 \cup O'_2$ where O'_1 and O'_2 are relatively open and disjoint. Since C is connected, either O'_1 or O'_2 must be empty.

(e) *We assume* $(X-B) \cap \{v=\rho_i\} = O^{(1)} \cup S^{(1)}$ *where* $O^{(1)}$ *and* $S^{(1)}$ *are relatively open and disjoint. The set* $O^{(2)}$ *consists of all points* p *in* $(X-B) \cap X_{\rho_{i+1},\rho_i}$ *satisfying that the connected component of* $(X-B) \cap \bar{X}_{v(p),\rho_i}$ *containing* p *intersects with* $O^{(1)}$. *We put* $S^{(2)} = (X-B) \cap X_{\rho_{i+1},\rho_i} - O^{(2)}$, $O = O^{(1)} \cup O^{(2)}$ *and* $S = S^{(1)} \cup S^{(2)}$. *Then* O *and* S *are relatively open.*

PROOF. We denote by $\alpha_\nu \uparrow \alpha$ $(\alpha_\nu \downarrow \alpha)$ if a sequence $\{\alpha_\nu\}$ is increasing (decreasing) and converges to α. Let us assume that $\lim_\nu p_\nu = p$ where p_ν and p are points in $(X-B) \cap \bar{X}_{\rho_{i+1},\rho_i}$.

(i) Suppose $p_\nu \in O$. We may assume $v(p_\nu) \uparrow v(p)$, or else we have easily $p \in O$ by (b). Denoting by C_ν the connected component of $(X-B) \cap X_{v(p_\nu),\rho_i}$ containing p, by (a) we have $p_\mu \in C_\nu$ for a sufficiently large μ and consequently $C_\nu \cap O^{(1)} \neq \phi$. Since $\{C_\nu\}$ is a decreasing sequence of connected compact sets, $C = \bigcap_\nu C_\nu$ is connected and compact. Taking $q_\nu \in C_\nu \cap O^{(1)}$, we find a point q in $C \cap O^{(1)}$. By the definition of the set O, we have $p \in O$. This implies that S is relatively open.

(ii) Suppose $p_\nu \in S$ and $v(p) < \rho_i$. We may assume $v(p_\nu) \downarrow v(p)$, otherwise we have easily $p \in S$ in view of (b). Denoting by C the connected component of $(X-B) \cap \bar{X}_{v(p),\rho_i}$ containing p and by U a neighborhood of p as in (b), we have $U \cap (X-B) \cap \{v \geqq v(p)\} \subset C$. We may assume $p_\nu \in U$. Since, by (b) and (c), $C \cap \{v \geqq v(p_\nu)\}$ is connected and contains p_ν, we have $C \cap \{v \geqq v(p_\nu)\} \cap O^{(1)} = \phi$. Consequently we have $C \cap O^{(1)} = \phi$ and hence $p \in S$.

(iii) Suppose $p_\nu \in S$ and $v(p) = \rho_i$. In view of (a) p_ν can be joined to p by a curve in $(X-B) \cap X_{\rho_{i+1},\rho_i}$ for a sufficiently large ν. By (i) and (ii) the curve is contained in S except the terminal point p. If p were in $O^{(1)}$, the point at which the restriction of v to the curve attained its greatest lower bound would belong to $O^{(2)}$. This is a contradiction. Thus we have $p \in S$.

3. When we study the number of intersecting points of ∂B and a curve in X, we may assume generally that there exists only finitely many points of ∂B on the curve and at each of them the curve runs from the interior of B to the exterior of B or conversely.

Assume $D-K$ to be connected and f to be an $\mathfrak{F}$-valued holomorphic function in $D-K$. By induction we shall continue f to K.

Now, we assume the following four facts $(i=0, 1, 2, \cdots; \rho_{-1} = +\infty)$:

(I) B_i^* is an open set such that $B \subset B_i^*$, $B_i^* \cap \{v \leqq \rho_i\} = B \cap \{v \leqq \rho_i\}$ and $B_i^* \Subset X$.

We shall say that p is *a singular point* if $p \in B_i^* - B$ and *an ordinary point* if $p \in (X - B_i^*) \cap \{v > \rho_i\}$. A singular (ordinary) point belonging to ∂B is called a singular (ordinary) boundary point.

(II) Any point in $p \in \partial B_i^* \cap \{v > \rho_i\}$ has a neighborhood U satisfying $U \cap B_i^* = U \cap B$.

(III) There exists an $\mathfrak{F}$-valued holomorphic function g_{ρ_i} in $(D \cup B_i^*) \cap \{v > \rho_i\}$ which coincides with f at each point of $(D - B_i^*) \cap \{v > \rho_i\}$.

Here, we say that g_{ρ_i} coincides with f at a point p if g is identically equal to f in a neighborhood of p. By the assumptions, g_{ρ_i} coincides with f at each ordinary boundary point. A singular boundary point at which g_{ρ_i} does not coincide with f is called *a singular boundary point of the first kind,* and at which g_{ρ_i} coincides with f is called *of the second kind.* For brevity we write $o.b.p.$, $s.b.p.1.$ and $s.b.p.2.$ instead of ordinary boundary point, singular boundary point of the first kind and singular boundary point of the second kind, respectively.

Let $\mathfrak{G}$ be a non-commutative group generated by three elements $\{o.b.p., s.b.p.1., s.b.p.2.\}$ satisfying the relations $(o.b.p.)^2 = (s.b.p.1.)^2 = (s.b.p.2.)^2 = e$ where e is the unit of $\mathfrak{G}$. Let c be a curve in X (satisfying the assumptions as in the beginning of this section n°. 3). The set of all points of $c \cap \partial B$ except end points is naturally considered as an elements $\tilde{c}$ of $\mathfrak{G}$, that is, if $c \cap \partial B$ except end points is $\{q_1, \cdots, q_k\}$ in the order on c then $\tilde{c} = q_1 \cdots q_k$ and if $c \cap \partial B$ except end points is empty then $\tilde{c} = e$. We shall say that the curve c is *cancelable* if $\tilde{c}$ becomes the unit of $\mathfrak{G}$.

(IV) Any $s.b.p.2.$ and any $o.b.p.$ in $X_{\rho_i, \rho_{i-1}}$ can never be joined by a cancelable curve lying in $X_{\rho_i, \rho_{i-1}}$.

For $i = 0$ the above four assumptions is true if we put $B_0^* = B$. For the purpose of induction, let us assume that B_i^* and g_{ρ_i} exist and satisfy all assumptions.

4. Now, we shall construct B_{i+1}^*. First we show

(f) $\overline{(B_i^* - B)} \cap \overline{(X - B_i^*) \cap \{v > \rho_i\}} \cap \{v = \rho_i\} = \phi$.

PROOF. Suppose that $\lim_\nu q'_\nu = \lim_\nu q''_\nu = q$ where q'_ν are singular points, q''_ν are ordinary points and $q \in (X - B) \cap \{v = \rho_i\}$.

(i) Suppose $q \in X - \bar{B}$. Then q has a neighborhood U contained in $X - \bar{B}$ such that $U \cap \{v > \rho_i\}$ is connected. Taking q'_ν and q''_μ in U, we find an arc c in $U \cap \{v > \rho_i\}$ joining q'_ν and q''_μ. Since $q'_\nu \in B_i^*$ and $q''_\mu \notin B_i^*$, there exists a point of ∂B_i^* on c and by assumption (II) it belongs to ∂B. This is a contradiction.

(ii) Suppose $q \in \partial B$. We take a neighborhood U of q such that f is holomorphic in U and $U \cap \{v > \rho_i\}$ is connected. As in (i), q'_ν in U can be joined to a point $q' \in \partial B_i^* \cap U \cap \{v > \rho_i\}$ by a curve in $B_i^* \cap U \cap \{v > \rho_i\}$. Since

$g_{\rho_i}=f$ at q' we have $g_{\rho_i}=f$ at any point on the curve. In virtue of the assumption (II), there exists a point of ∂B on the curve, which is an *s.b.p.* 2.. This contradicts the assumption (IV). q. e. d.

Similarly we can show that it never happens that $\lim_{\nu} q'_\nu = \lim_{\nu} q''_\nu = q$ where q'_ν are *s.b.p.* 1. and q''_ν are *s.b.p.* 2..

We decompose the set $(X-B)\cap\{v=\rho_i\}$ into the union $O^{(1)}\cup S^{(1)}$ of two disjoint sets as follows: We take a point p in $(X-B)\cap\{v=\rho_i\}$. If p is a limit point of ordinary (singular) points, then we assume $p\in O^{(1)}$ $(S^{(1)})$. When p has a neighborhood U satisfying $U\cap\{v>\rho_i\}\subset B$, p belongs to ∂B and g_{ρ_i} is holomorphically continued to a neighborhood of p. If $g_{\rho_i}\neq f$ at p we assume $p\in S^{(1)}$. If $g_{\rho_i}=f$ at p, we distinguish two cases; if p can be joined to an *s.b.p.* 2. in $X_{\rho_i,\rho_{i-1}}$ by a cancelable curve in $X_{\rho_i,\rho_{i-1}}$ we assume $p\in S^{(1)}$, and if not we assume $p\in O^{(1)}$.

(g) *$O^{(1)}$ and $S^{(1)}$ satisfy the assumptions of* (e).

PROOF. In view of (f) and the definitions of $O^{(1)}$ and $S^{(1)}$, it is trivial that $(X-B)\cap\{v=\rho_i\}=O^{(1)}\cup S^{(1)}$ and $O^{(1)}\cap S^{(1)}=\phi$. Assume $\lim_{\nu} p_\nu = p$ where p_ν and p are in $O^{(1)}\cup S^{(1)}$.

(i) If $p_\nu\in\overline{(X-B)\cap\{v>\rho_i\}}$ for infinitely many ν, according as p_ν belong to $O^{(1)}$ or $S^{(1)}$ p belongs to the same set in virtue of (f).

(ii) If p has a neighborhood U satisfying $U\cap\{v>\rho_i\}\subset B$, making U so small that f and the continuation of g_{ρ_i} are holomorphic in U and $U\cap\{v>\rho_i\}$ is connected, we have $p_\nu\in O^{(1)}$ or $p_\nu\in S^{(1)}$ according as $p\in O^{(1)}$ or $p\in S^{(1)}$, respectively.

(iii) Let us assume $p_\nu\notin\overline{(X-B)\cap\{v>\rho_i\}}$ but $p\in\overline{(X-B)\cap\{v>\rho_i\}}$. We take a neighborhood U of p such that f and the continuation of g_{ρ_i} are holomorphic in U and $U\cap\{v>\rho_i\}$ is connected. By (f) we may assume $(X-B)\cap U\cap\{v>\rho_i\}$ is contained in either B_i^*-B or $X-B_i^*$. Any p_ν in U can be joined to a point p' in $\partial B\cap U\cap\{v>\rho_i\}$ by a curve in $B\cap U\cap\{v>\rho_i\}$. We have easily $p_\nu\in O^{(1)}$ or $p_\nu\in S^{(1)}$ according as $p\in O^{(1)}$ or $p\in S^{(1)}$. q. e. d.

Now, we can define the set B_{i+1}^*. Applying (e) to $O^{(1)}$ and $S^{(1)}$, we have sets O and S as in (e). We define $B_{i+1}^*=B_i^*\cup S$. By this definition and (e), we can see easily that B_{i+1}^* satisfies the assumptions (I) and (II).

We shall show that the assumption (III) is true. In fact, at any point of $\overline{B_{i+1}^*}\cap\{v=\rho_i\}$ the continuation of g_{ρ_i} coincides with f. Using Lemma 6 we can easily construct an $\mathcal{F}$-valued holomorphic function $g_{\rho_i-\varepsilon}$ in $(D\cup B_{i+1}^*)\cap\{v>\rho_i-\varepsilon\}$ which coincides with f on $(D-B_{i+1}^*)\cap\{v>\rho_i-\varepsilon\}$ for a sufficiently small positive number ε. Applying Lemma 7 we get $g_{\rho_{i+1}}$, which satisfies the assumption (III).

Similarly, points of $(X-B)\cap\{v>\rho_{i+1}\}$ can be distinguished into singular points and ordinary points. The meanings of *o.b.p.*, *s.b.p.* 1. and *s.b.p.* 2.

are the same as before.

5. *We shall now prove the assumption* (IV). Let us assume that q_1 is an $s.b.p.2.$ in X_{ρ_{i+1},ρ_i} and q_{2k} is an $o.b.p.$ in X_{ρ_{i+1},ρ_i} such that they are joined by a cancelable curve c in X_{ρ_{i+1},ρ_i}. The curve c satisfies the assumptions stated at the beginning of this section n°. 3. As c is cancelable, the number of points of $c\cap\partial B$ is even. We denote then by $q_1, q_2, \cdots, q_{2k-1}, q_{2k}$ in the order.

Let $c_{j,j+1}$ be a part of c between q_j and q_{j+1}, where we assume that q_j and q_{j+1} do not belong to $c_{j,j+1}$ $(j=1,\cdots,2k-1)$. We have $c_{1,2}\subset B$. In fact, if $c_{1,2}\subset\bar{B}^c$ we have $c_{2k-1,2k}\subset\bar{B}^c$ and so q_{2k-1} is an $o.b.p.$. Since c is cancelable, we can find q_{2j} such that it is an $o.b.p.$ and $q_{2j+1}\cdots q_{2k-2}$ is the unit of the group $\mathfrak{G}$. By the same reason q_{2j-1} is also an $o.b.p.$. Repeating this process, we are led to a contradiction.

Let $C_{j,j+1}$ be the connected component of $B\cap X_{\rho_{i+1},\rho_i}$ or $\bar{B}^c\cap X_{\rho_{i+1},\rho_i}$ containing $c_{j,j+1}$. We denote by $C_{j,j+1}(\varepsilon)$ the set $C_{j,j+1}\cap\{v>\rho_i-\varepsilon\}$ where ε is a small positive number. By (c), it is connected, and consequently $\varDelta_{j,j+1}=\lim_{\varepsilon\downarrow 0}\overline{C_{j,j+1}(\varepsilon)}$ is connected and compact.

In virtue of (b), the connected component of $\overline{C_{j-1,j}}\cap\overline{C_{j,j+1}}\cap\{v\geqq v(q_j)\}$ containing q_j is sure to intersect with the set $\{v=\rho_i\}$. One of the intersecting points is denoted by q_j^*. We have easily $q_j^*\in\varDelta_{j-1,j}\cap\varDelta_{j,j+1}$, and q_j^* is of the same kind as q_j, that is, according as q_j is an $s.b.p.1.$, an $s.b.p.2.$ or an $o.b.p.$ q_j^* is so for $j=1,\cdots,2k$. Here we put $C_{0,1}=C_{2k,2k+1}=\varDelta_{0,1}=\varDelta_{2k,2k+1}=X$.

We shall take a point q'_{2j} $(j=1,\cdots,k-1)$ as follows:

(i) If $U(q_{2j}^*)\cap\partial B\cap\{v>\rho_i\}\neq\phi$ for any neighborhood $U(q_{2j}^*)$ of q_{2j}^*, we put $q'_{2j}=q_{2j}^*$.

(ii) Let us assume that q_{2j}^* has a neighborhood $U(q_{2j}^*)$ satisfying $U(q_{2j}^*)\cap\{v>\rho_i\}\subset B$. When any point q of $\varDelta_{2j,2j+1}$ has a neighborhood such that $U(q)\cap\{v>\rho_i\}\subset B$, we shall say that we can not take q'_{2j} or the number $2j$ is a missing number. In the other case, we give a neighborhood $V(q)$ to any $q\in\varDelta_{2j,2j+1}$ satisfying the followings: $V(q)\cap\{v>\rho_i\}$ is connected and if possible it is contained in B. If $g_{\rho_{i+1}}$ or f is given at q, then it is holomorphic in $V(q)$. If q is an $s.b.p.$ ($o.b.p.$), each point of $V(q)\cap\partial B$ is so. If $q\in\bar{B}^c$, then $V(q)\subset\bar{B}^c$. Since $\varDelta_{2j,2j+1}$ is compact, we can find finitely many such neighborhoods $V_1,\cdots,V_t$ satisfying $\bigcup_{\nu=1}^{t}V_\nu\supset\varDelta_{2j,2j+1}$, where $V_1=V(q_{2j}^*)$. We may assume that a chain of neighborhoods $V_1,\cdots,V_{t'}$ satisfies $V_\nu\cap V_{\nu+1}\cap\varDelta_{2j,2j+1}\neq\phi$, $V_\nu\cap\{v>\rho_i\}\subset B$ for $1\leqq\nu<t'$ and $V_{t'}\cap\{v>\rho_i\}\not\subset B$. We denote by q'_{2j} the point q such that $V_{t'}=V(q)$. Since $\varDelta_{2j,2j+1}\subset B^c$, we have $V_\nu\cap\varDelta_{2j,2j+1}\subset\partial B$ for $1\leqq\nu<t'$. From the construction, q_{2j}^* and q'_{2j} are of the same kind. And we have $U(q'_{2j})\cap\partial B\cap\{v>\rho_i\}\neq\phi$ for any neighborhood $U(q'_{2j})$

of q'_{2j}.

(iii) Let us assume that q^*_{2j} has a neighborhood $U(q^*_{2j})$ satisfying $U(q^*_{2j}) \cap \{v > \rho_i\} \subset \bar{B}^c$. Repeating the same method for $\Delta_{2j-1,2j}$, we take q'_{2j} if $2j$ is not a missing number. It plays an important role that q'_{2j} and q^*_{2j} are of the same kind and $U(q'_{2j}) \cap \partial B \cap \{v > \rho_i\} \neq \phi$ for any neighborhood $U(q'_{2j})$ of q'_{2j}.

Similarly we take a point q'_{2j+1} having the same properties as q'_{2j} ($j=1, \cdots, k-1$). When any point q in $\Delta_{2j,2j+1}$ has a neighborhood $U(q)$ satisfying $U(q) \cap \{v > \rho_i\} \subset B$, we can take neither q'_{2j} nor q'_{2j+1}. Then q^*_{2j} and q^*_{2j+1} are of the same kind. Besides assume that we can not take q'_{2j-1}. Since $\Delta_{2j-1,2j} \cap \Delta_{2j,2j+1} \neq \phi$, any point of $\Delta_{2j-1,2j}$ has a neighborhood U such that $U \cap \{v > \rho_i\} \subset B$ and hence we can not take q'_{2j-2}, and q^*_{2j-1} is of the same kind as q^*_{2j-2}. Consequently, we have $\prod_{j=2}^{2k-1} q'_j = \prod_{j=2}^{2k-1} q^*_j$ in the group $\mathfrak{G}$, where $q'_j = e$ if j is a missing number.

If $k > 1$, we may assume that q_2 and q_{2k-1} are $s.b.p.1.$. In fact, if q_{2k-1} is an $s.b.p.2.$, we may consider $c_{2k-1,2k}$ as c from the beginning. If q_{2k-1} is an $o.b.p.$, then q_{2k-2} is so. In this case we consider a part of c between q_1 and q_{2k-2} as c and repeat the same process. Thus we may assume that q_{2k-1} is an $s.b.p.1.$. Let q_{2j_1} be an $s.b.p.1.$ such that $q_{2j_1+1} \cdots q_{2k-2}$ is the unit of $\mathfrak{G}$. Since c is cancelable, we can find such q_{2j_1}. Since $c_{2j_1-1,2j_1} \subset B$, we may assume $q_{2j_1-1}, \cdots, q_{2j_1-2\lambda}$ are $o.b.p.$ and $q_{2j_1-2\lambda-1}$ is not an $o.b.p.$ ($\lambda = 0, 1, \cdots, j_1-1$). If $q_{2j_1-2\lambda-1}$ is an $s.b.p.2.$, then we consider $q_{2j_1-2\lambda-1}$ as q_1 from the beginning. If $q_{2j_1-2\lambda-1}$ is an $s.b.p.1.$, we may assume $\lambda \neq 0$ and we consider $q_{2j_1-2\lambda}$ as q_{2k} and repeat the same process. Thus we may assume q_2 is an $s.b.p.1.$.

We shall take q'_{2k}. If q^*_{2k} has a neighborhood $U(q^*_{2k})$ such that $U(q^*_{2k}) \cap \{v > \rho_i\} \subset \bar{B}^c$, applying the same method as above to $\Delta_{2k-1,2k}$ we take q'_{2k}. In this case, since q^*_{2k-1} and q^*_{2k} are not of the same kind we can always take q'_{2k}. In the other case we put $q'_{2k} = q^*_{2k}$. Similarly we take q'_1.

Let us assume that neither $2j$ nor λ ($> 2j$) is a missing number and any ν satisfying $2j < \nu < \lambda$ is a missing number. Then λ must be odd. We put $\lambda = 2j + 2\mu + 1$ ($1 \leqq j \leqq k-1$; $0 \leqq \mu \leqq k-j-1$). We have always $\Delta_{2\nu,2\nu+1} \subset B^c$ for any ν satisfying $1 \leqq \nu \leqq k-1$. If $\mu \neq 0$, since $2j$ is not a missing number and $2j+1$ is a missing number, we have $\Delta_{2j+1,2j+2} \subset B^c$. Similarly we obtain $\Delta_{2\nu+1,2\nu+2} \subset B^c$ for any ν satisfying $j \leqq \nu \leqq j+\mu-1$. If q'_{2j} is not in $\Delta_{2j,2j+1}$ but in $\Delta_{2j-1,2j}$, we remember the existence of a chain of neighborhoods $\{V_\nu\}$ used in taking q'_{2j}. It is the same for $q'_{2j+2\mu+1}$. Thus we can find finitely many points $q'_{2j,0}, \cdots, q'_{2j,t_{2j}}$ and their neighborhoods $V(q'_{2j,0}), \cdots, V(q'_{2j,t_{2j}})$ satisfying the followings: $q'_{2j,0} = q'_{2j}$, $q'_{2j,t_{2j}} = q'_{2j+2\mu+1}$, $q'_{2j,\nu} \in \bigcup_{\alpha=2j-1}^{2j+2\mu+1} \Delta_{\alpha,\alpha+1}$, $V(q'_{2j,\nu}) \cap \bar{B}^c \neq \phi$ and $V(q'_{2j,\nu}) \cap \{v > \rho_i\}$ is connected. If $g_{\rho_{i+1}}$ or f is holomorphic at $q'_{2j,\nu}$, it is holomorphic in $V(q'_{2j,\nu})$. If $q'_{2j,\nu} \in \bar{B}^c$, $V(q'_{2j,\nu})$ is contained in $\bar{B}^c$. If $q'_{2j,\nu} \in \partial B$, any point of $\partial B \cap V(q'_{2j,\nu}) \cap \{v > \rho_i\}$ can be joined to q'_{2j}, by a curve in $\partial B \cap \{v > \rho_i\}$ $(\nu = 0, 1, \cdots, t_{2j})$. The set $V(q'_{2j,\nu}) \cap V(q'_{2j,\nu+1}) \cap \left(\bigcup_{\alpha=2j-1}^{2j+2\mu+1} \varDelta_{\alpha,\alpha+1}\right)$ is not empty $(\nu = 0, 1, \cdots, t_{2j}-1)$.

For q'_{2j+1}, we can similarly find points $q'_{2j+1,\nu}$. Skipping the missing numbers and numbering afresh $q'_1, \cdots, q'_{2k}$ in the order, we denote them by $\underline{q_1}, \cdots, \underline{q_{2l}}$. We denote by $\underline{q_{j,\nu}}$ such a point $q'_{j,\nu}$ between $\underline{q_j}$ and $\underline{q_{j+1}}$ as above. We may assume $V(\underline{q_{j,t_j}}) = V(\underline{q_{j+1,0}})$, which is denoted by $V(\underline{q_{j+1}})$. We take note that $\underline{q_1} = q'_1$ and $\underline{q_{2l}} = q'_{2k}$. We take a point $\underline{q''_j}$ in $V(\underline{q_j}) \cap \partial B \cap \{v > \rho_i\}$.

The point $\underline{q_1}$ can be joined to $\underline{q_{2l}}$ by a curve c' in $X_{\rho_i,\rho_{i-1}}$ satisfying the following: c' runs through $V(\underline{q_{1,0}}), \cdots, V(\underline{q_{1,t_1-1}}), V(\underline{q_{2,0}}), \cdots, V(\underline{q_{2l-1,0}}), \cdots, V(\underline{q_{2l-1,t_{2l-1}-1}})$ and $V(\underline{q_{2l,0}})$ in this order. It satisfies the assumptions given in the beginning of this section n°. 3. Every $\underline{q''_j}$ belongs to c'. The curve c' runs from the interior of B to the exterior of B at $\underline{q''_{2j}}$ and contrary at $\underline{q''_{2j+1}}$, $(1 \leqq j \leqq l-1)$.

Under these assumptions on c', the number of points of $c' \cap \partial B$ between $\underline{q_{2j-1}}$ and $\underline{q_{2j}}$ is even. Denoting them by $R_1, \cdots, R_{2r}$, we show that $R_{2\lambda-1}$ and $R_{2\lambda}$ are of the same kind $(1 \leqq \lambda \leqq r)$. In fact, it never happens by the assumption (IV) that the one is an $s.b.p.2.$ and the other is an $o.b.p.$. We may assume that a part $c'_{2\lambda-1,2\lambda}$ of c' between $R_{2\lambda-1}$ and $R_{2\lambda}$ is covered by a chain of neighborhoods $V(\underline{q_{2j-1,\alpha}}), V(\underline{q_{2j-1,\alpha+1}}), \cdots, V(\underline{q_{2j-1,\alpha+\beta}})$. By the properties of $V(\underline{q_{2j-1,\nu}})$, we have $V(\underline{q_{2j-1,\nu}}) \cap B \neq \phi$. On the other hand $c'_{2\lambda-1,2\lambda}$ is contained in $\bar{B}^c$, and so we have $V(\underline{q_{2j-1,\nu}}) \cap \partial B \neq \phi$. This implied $\underline{q_{2j-1,\nu}} \in \partial B$ $(\nu = \alpha, \cdots, \alpha+\beta)$. Therefore both $g_{\rho_{i+1}}$ and f are single-valued and holomorphic in $V(\underline{q_{2j-1,\nu}})$, and consequently we have the same at $R_{2\lambda}$ according as $f = g_{\rho_i}$ or $f \neq g_{\rho_i}$ at $R_{2\lambda-1}$. This shows that $R_{2\lambda-1}$ and $R_{2\lambda}$ are of the same kind.

In virtue of these considerations, we get the fact that c' is a cancelable curve joining $\underline{q_1}$ and $\underline{q_{2l}}$, where $\underline{q_1} = q'_1$ is an $s.b.p.2.$ and $\underline{q_{2l}} = q'_{2k}$ is an $o.b.p.$. We distinguish four cases by the relation among q'_1, q'_{2k} and $\overline{(X-B) \cap \{v > \rho_i\}}$. In any cases, we are led to a contradiction by the definition of B^*_{i+1} and the assumption (IV).

Thus we have proved the existences of B^*_{i+1} and $g_{\rho_{i+1}}$ satisfying the assumptions (I)$\sim$(IV). Hence, by induction we have an $\mathcal{F}$-valued holomorphic function $g_{-\infty}$, which is defined on D and coincides with f on $(D-B) \cap \{v > \rho_0\}$. Since $D-K$ is connected, it coincides with f on $D-K$. This completes the proof.

REMARK: If we assume only that $\{\inf v(K) - \varepsilon < v < \sup v(K) + \varepsilon\} \Subset X$, we have an $\mathcal{F}$-valued holomorphic function $g_{\rho_{s+1}}$ by induction, which is defined on $(D \cup B^*_{s+1}) \cap \{v > \rho_{s+1}\}$ and coincides with f on $(D - B^*_{s+1}) \cap \{v > \rho_{s+1}\}$, where

ρ_{s+1} means $\inf v(B)-\frac{1}{2}\varepsilon$ (cf. n°. 1). Since $B_{s+1}^{*}\supset B\supset K$, $g_{\rho_{s+1}}$ is defined and $\mathfrak{F}$-valued holomorphic on a set containing K and $g_{\rho_{s+1}}=f$ holds on a non-empty open subset of $D-K$.

§ 8. Some generalizations.

1. Let X be a complex manifold. For an arbitrary locally convex complete Hausdorff topological vector space $\mathfrak{F}$, the function space $A(X, \mathfrak{F})$ of all $\mathfrak{F}$-valued holomorphic functions on X is also a topological vector space of the same type with the topology of compact uniform convergence.

LEMMA 8. *For two complex manifolds X and Y, $A(X, A(Y, \mathfrak{F}))$ is isomorphic to $A(X\times Y, \mathfrak{F})$ as a topological vector space.*

PROOF. To an element $f(p, q)$ in $A(X\times Y, \mathfrak{F})$ we correspond the mapping $f^{*}(p)(q):=f(p, q)$ of X into the space of $\mathfrak{F}$-valued functions on Y. Obviously $f^{*}(p)$ is a continuous function with values in $A(Y, \mathfrak{F})$. To see the holomorphy of $f^{*}(p)$, we take an arbitrary point p in X and a system of coordinates $z_{1}, \cdots, z_{n}$ in a neighborhood of p, where $n=\dim_{p} X$ and p corresponds to the origin. The Cauchy's formula (1) implies

$$f(z_{1}, \cdots, z_{n}, q)=\frac{1}{(2\pi i)^{n}}\int_{|\zeta_i|=\rho_i}\cdots\int\frac{f(\zeta_{1}, \cdots, \zeta_{n}, q)}{(\zeta_{1}-z_{1})\cdots(\zeta_{n}-z_{n})}d\zeta_{1}\cdots d\zeta_{n}$$

for any q in Y. Therefore, we have

$$f^{*}(z_{1}, \cdots, z_{n})=\frac{1}{(2\pi i)^{n}}\int_{|\zeta_i|=\rho_i}\cdots\int\frac{f^{*}(\zeta_{1}, \cdots, \zeta_{n})}{(\zeta_{1}-z_{1})\cdots(\zeta_{n}-z_{n})}d\zeta_{1}\cdots d\zeta_{n}$$

and for any continuous linear functional u

$$uf^{*}(z_{1}, \cdots, z_{n})=\frac{1}{(2\pi i)^{n}}\int_{|\zeta_i|=\rho_i}\cdots\int\frac{uf^{*}(\zeta_{1}, \cdots, \zeta_{n})}{(\zeta_{1}-z_{1})\cdots(\zeta_{n}-z_{n})}d\zeta_{1}\cdots d\zeta_{n}.$$

This shows the complex-valued function $uf^{*}(z)$ is holomorphic in a neighborhood of p and hence $f^{*}(p)$ is an $A(Y, \mathfrak{F})$-valued holomorphic function.

Conversely, we take an element $f^{*}(p)$ in $A(X, A(Y, \mathfrak{F}))$. Evidently the $\mathfrak{F}$-valued function $f(p, q):=f^{*}(p)(q)$ is continuous on $X\times Y$. Now, we consider the complex-valued function $g(p, q):=u(f(p, q))$ for any continuous linear functional u on $\mathfrak{F}$, which is holomorphic on Y for each fixed point p in X because $f^{*}(p)\in A(Y, \mathfrak{F})$. For each point q in Y the functional u induces the continuous functional u_{q}' on $A(Y, \mathfrak{F})$ where u_{q}' is defined to be $u(f(q))$ for each f in $A(Y, \mathfrak{F})$. By definition, $g(p, q)=u_{q}'(f^{*}(p))$ is holomorphic on X for each fixed point q in Y. This shows the holomorphy of $g(p, q)$ on $X\times Y$, and hence $f(p, q)\in A(X\times Y, \mathfrak{F})$. The other parts to be proved are well known facts. We omit the details.

THEOREM 6. *Let X be a complex manifold and (D, K) be an H-problem on X which is solvable. Then for an arbitrary complex manifold Y any $\mathfrak{F}$-valued holomorphic function in $(D-K)\times Y$ has the unique holomorphic continuation in $D\times Y$.*

PROOF. We have

$$\begin{aligned} A((D-K)\times Y, \mathfrak{F}) &= A(D-K, A(Y, \mathfrak{F})) && \text{(from Lemma 8)} \\ &= A(D, A(Y, \mathfrak{F})) && \text{(from the solvability of the } H\text{-problem)} \\ &= A(D\times Y, \mathfrak{F}) && \text{(from Lemma 8).} \end{aligned}$$

This proves Theorem 6.

2. Let X be a differentiable manifold of class $C^k (0 \leqq k \leqq \infty)$. We can define naturally $\mathfrak{F}$-valued differentiable functions of class C^k on X. The function space $C^k(X, \mathfrak{F})$ of all differentiable functions of class C^k is considered as a locally convex complete Hausdorff topological vector space with the topology of compact uniform convergence of functions and their derivatives.

LEMMA 9. *Let X be a complex manifold and Y be a differentiable manifold of class C^k. An $\mathfrak{F}$-valued function $f(p, q)$ on $X\times Y$ induces a $C^k(Y, \mathfrak{F})$-valued holomorphic function $f^*(p)(q) := f(p, q)$ if and only if $f(p, q)$ is a k-times continuously differentiable family of $\mathfrak{F}$-valued holomorphic functions, that is, $f(p, q)$ is holomorphic on X for each fixed point q in Y and has k-th derivatives referred to each local coordinates in Y which are continuous with respect to the topology of $X\times Y$.*

PROOF. We take a k-times continuously differentiable family of holomorphic functions $f(p, q)$. Obviously, $f^*(p)(q) := f(p, q) \in C^k(Y, \mathfrak{F})$ for a fixed p in X, and $f^*(p)$ is $C^k(Y, \mathfrak{F})$-valued continuous function because of the continuity of k-th derivatives. And the holomorphy of $f^*(p)$ is also obvious from the Cauchy's integral formula as in the proof of Lemma 8.

The converse follows immediately from the definition of the topology of $C^k(Y, \mathfrak{F})$.

THEOREM 7. *Take a k-times continuously differentiable family of $\mathfrak{F}$-valued holomorphic functions $f(p, t)$ on an open subset G of a complex manifold with a parameter t in a differentiable manifold of class C^k. Under appropriate conditions of G, the continued functions $g(p, t)$ into some larger open set D as in Theorems 1～6 etc. constitute also a k-times continuously differentiable family of $\mathfrak{F}$-valued holomorphic functions.*

PROOF. From Lemma 9 a k-times continuously differentiable family of $\mathfrak{F}$-valued holomorphic functions is regarded as a $C^k(Y, \mathfrak{F})$-valued holomorphic function. Theorem 7 is a special case of Theorems 1～6 etc. q. e. d.

Nagoya University and Chuo University

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