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On the functional inequality $\left|f\left(\frac{x+y}{2}\right)\right| \leqq \frac{|f(x)|+|f(y)|}{2}$

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§1. Considering the Cauchy's functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}, \tag{1}$$

where $f(z)$ is an entire function of z, we have the following functional inequality:

$$\left|f\left(\frac{x+y}{2}\right)\right| \leqq \frac{|f(x)|+|f(y)|}{2}. \tag{2}$$

In this paper we shall determine all the entire functions $f(z)$ which satisfy (2).

THEOREM. *If $f(z)$ is an entire function of z, then all the functions which satisfy* (2) *are $(\alpha z+\beta)^n$ and $\exp(\alpha z+\beta)$ where α, β are arbitrary complex constants and n is an arbitrary natural number, and only these.*

PROOF. We may assume that $f(z)\not\equiv 0$. Putting $z=s+it$ (s, t real), $\varphi(s,t)=|f(z)|$ and using a real parameter τ[1], the function

$$F(\tau)=\varphi(a+h\tau, b+k\tau)+\varphi(a-h\tau, b-k\tau)$$

has a minimum $2\varphi(a,b)$ at $\tau=0$ by (2). Here a, b, h, k are arbitrary real constants which satisfy $f(a+ib)\neq 0$. Hence we have $F''(0)\geqq 0$. Since

$$F''(0)=2\{\varphi_{ss}(a,b)h^2+2\varphi_{st}(a,b)hk+\varphi_{tt}(a,b)k^2\},$$

we have

$$\varphi_{ss}(a,b)h^2+2\varphi_{st}(a,b)hk+\varphi_{tt}(a,b)k^2\geqq 0.$$

Since h, k are arbitrary, we have

$$\varphi_{st}^2(a,b)-\varphi_{ss}(a,b)\varphi_{tt}(a,b)\leqq 0. \tag{3}$$

Since $f(a+ib)\neq 0$, there exists a regular branch $g(z)$ of $\sqrt{f(z)}$ in a properly chosen vicinity V of $z=\gamma=a+ib$.

Using the Cauchy-Riemann equations, we have

$$\{\varphi_{st}(a,b)\}^2-\varphi_{ss}(a,b)\varphi_{tt}(a,b)=4\{|g(\gamma)g''(\gamma)|^2-|g'(\gamma)|^4\}.$$

By (3) we have

(4) $$|g(\gamma)g''(\gamma)| \leqq |g'(\gamma)|^2.$$

Since $f(z)=g^2(z)$ in V, by (4) we have

$$|f(\gamma)f''(\gamma)| \leqq |f'(\gamma)|^2.$$

Hence we have the following inequality at every point z

(5) $$|f(z)f''(z)| \leqq |f'(z)|^2.$$

Let us put $G(z)=\dfrac{f(z)f''(z)}{\{f'(z)\}^2}$ and assume that z_0 is an arbitrary complex number. We may assume $f'(z)\not\equiv 0$, otherwise $f(z)\equiv$ const., which surely satisfies the original functional inequality. Then there exists a positive number δ such that $f'(z)\neq 0$ in $0<|z-z_0|<\delta$. By Riemann's theorem $G(z)$ is regular at $z=z_0$, and $|G(z_0)|\leqq 1$. Hence $G(z)$ is an entire function satisfying $|G(z)|\leqq 1$ in $|z|<+\infty$. By Liouville's theorem $G(z)$ is a constant.

Hence we have

(6) $$f(z)f''(z)=A\{f'(z)\}^2,$$

where A is a constant with $|A|\leqq 1$.

Solving this differential equation (6), the assertion is proved.

REMARK. If $f(z)$ is a meromorphic function and satisfies (2) in $|z|<+\infty$, then we can easily prove that $f(z)$ is an entire function of z.

§ 2. Now, using the above theorem we shall solve the following functional equations under the hypothesis that $f(z)$, $g(z)$ are both entire functions of z:

(1) $$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2},$$

(2) $$f(x+y)=f(x)+f(y),$$

(3) $$f(x+y)=f(x)f(y),$$

(4) $$f(xy)=f(x)f(y),$$

(5) $$|f(x+y)|+|f(x-y)|=2|f(x)|+2|g(y)|,$$

(6) $$|f(x+y)|+|f(x-y)|=2|f(x)|+2|f(y)|.$$

Solution of (1). $f(z)=\alpha z+\beta$. It is clear.

Solution of (2). $f(z)=\alpha z$. It is also clear.

Solution of (3). We have

$$\left|f\left(\frac{x+y}{2}\right)\right|=\left|f\left(\frac{x}{2}\right)\right|\left|f\left(\frac{y}{2}\right)\right|\leqq\frac{\left|f^2\left(\frac{x}{2}\right)\right|+\left|f^2\left(\frac{y}{2}\right)\right|}{2}$$

$$=\frac{|f(x)|+|f(y)|}{2}.$$

Hence $f(z)$ satisfies the condition of our theorem. Hence we have $f(z)\equiv 0$

or $f(z)=\exp(\alpha z)$ where α is an arbitrary complex constant.

Solution of (4). Putting $g(z)=f(e^z)$, by (4) we have

$$g(x+y)=g(x)g(y).$$

By the above result we have $f(z)\equiv 0$ or $f(z)\equiv 1$ or $f(z)=z^n$ where n is an arbitrary natural number.

Solution of (5). By (5) we have

$$|f(x+y)|+|f(x-y)|\geqq 2|f(x)|,$$

which implies

$$\left|f\left(\frac{x+y}{2}\right)\right|\leqq\frac{|f(x)|+|f(y)|}{2}.$$

Hence $f(z)$ satisfies the condition of our theorem, and we can conclude $f(z)=(\alpha z+\beta)^2$, $g(z)=\alpha^2 z^2$ where α, β are both arbitrary complex constants.

Solution of (6). By the above result we have $f(z)=\alpha z^2$ where α is an arbitrary complex constant.

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