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On the Gauss-Hecke sums

By Katsumi SHIRATANI

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§1. Introduction.

Let k be an algebraic number field of finite degree and $\mathfrak{d}$ be its different. Then for any non-zero number ω in k, we decompose the ideal $(\omega)\mathfrak{d}$ into the quotient of integral ideals $\mathfrak{a}, \mathfrak{b}$ which are mutually prime:

$$(\omega)\mathfrak{d}=\frac{\mathfrak{b}}{\mathfrak{a}}, \quad (\mathfrak{a}, \mathfrak{b})=1 .$$

The Gauss-Hecke sum is defined by an exponential sum

$$C(\omega)=\sum_{\kappa \bmod \mathfrak{a}} e^{2\pi i S(\omega\kappa^2)}, \tag{1}$$

where S means the absolute trace with respect to k and κ runs over a complete residue system of integers $\mathfrak{o}$ modulo $\mathfrak{a}$.

Hecke obtained the following formula from a transformation formula of the theta-function [**2**],

$$\frac{C(\omega)}{\sqrt{N\mathfrak{a}}}=\frac{\sqrt{N2\mathfrak{b}}}{N\mathfrak{b}_1} e^{\frac{\pi i}{4} S^*(\operatorname{sgn}\omega)} C\left(\frac{-\delta^2}{4\omega}\right) \tag{2}$$

where $\mathfrak{b}_1$ means an integral ideal satisfying $\frac{\mathfrak{a}}{4\mathfrak{b}}=\frac{\mathfrak{a}_1}{\mathfrak{b}_1}$, $(\mathfrak{a}_1, \mathfrak{b}_1)=1$ and δ a number in k determined by $\mathfrak{b}(\delta)=\mathfrak{g}$, $(\mathfrak{g}, \mathfrak{b}_1)=1$ and N the absolute norm, finally $S^*(\operatorname{sgn}\omega)$ the sum of signatures of real conjugates $\omega^{(p)}$ $(p=1, 2, \cdots, r_1)$ of ω, that is, $\sum_{p=1}^{r_1} \operatorname{sgn}\omega^{(p)}$.

Especially for two integers α, β in k satisfying $(\alpha, 2)=(\beta, 2)=(\alpha, \beta)=1$, Hecke derived from (2) for the quadratic power residue symbols $\left(\frac{\beta}{\alpha}\right), \left(\frac{\alpha}{\beta}\right)$, using some number ω,

$$\left(\frac{\beta}{\alpha}\right)\left(\frac{\alpha}{\beta}\right)=(-1)^g \frac{C\left(\frac{-\omega\alpha}{4}\right)C\left(\frac{-\omega\beta}{4}\right)}{C\left(\frac{-\omega}{4}\right)C\left(\frac{-\omega\alpha\beta}{4}\right)}, \tag{3}$$

$$g=g(\alpha, \beta)=\sum_{p=1}^{r_1} \frac{\operatorname{sgn}\alpha^{(p)}-1}{2}\,\frac{\operatorname{sgn}\beta^{(p)}-1}{2}.$$

Siegel [**3**] treated the following sum regarded as a generalization of the Gauss-Hecke sum,

(4) $$G(\omega, \tau)=\sum_{\kappa \bmod \mathfrak{a}} e^{\pi i S(\omega\kappa^2+\omega\kappa\tau)}\ [1],$$

where τ denotes a number which satisfies the condition

(4)′ $$S(\omega\alpha^2+\omega\alpha\tau)\equiv 0 \quad (\mathrm{mod}\ 2) \quad \text{for any } \alpha \text{ in } \mathfrak{a}.$$

If we have $(\mathfrak{a}\mathfrak{b}, 2)=1$, the reciprocity formula of Siegel reduces to the form:

(5) $$\frac{1}{\sqrt{N\mathfrak{a}}}G(\omega, \tau)=e^{\frac{\pi i}{4}\{S^*(\mathrm{sgn}\,\omega)-S(\omega\tau^2)\}}\frac{1}{\sqrt{N\mathfrak{b}}}G\Big(-\frac{\delta^2}{\omega}, \frac{\omega}{\delta}\tau\Big).$$

From this formula, he gave a simple and beautiful proof of Hasse's formula on the law of quadratic reciprocity in k.

(6) $$\Big(\frac{\beta}{\alpha}\Big)\Big(\frac{\alpha}{\beta}\Big)=(-1)^{g+S\left(\frac{1-\alpha}{2}\frac{1-\beta}{2}\right)}, \quad (\alpha, \beta)=1,\ \alpha\equiv\beta\equiv 1\ (\mathrm{mod}\ 2).$$

In this paper we prove in §2 that Siegel's sum (4), under the assumption $(\mathfrak{a}\mathfrak{b}, 2)=1$, just coincides with the Gauss-Hecke sum $C(2\omega)$, and in §3 therefrom we show that Hasse's formula (6) can be derived from (2). In §4 we prove also the formula of complementary law from (2) in a similar way. One remarkable fact is that $C\left(\frac{\omega}{8}\right)$ with $(\omega)\mathfrak{d}=\mathfrak{b}$, $(\mathfrak{b}, 2)=1$ consists essentially of only one term.

§2. A special sum $C\left(\frac{\omega}{8}\right)$.

First we quote some important properties of the parameter τ in (4).

In the case where $(\mathfrak{a}\mathfrak{b}, 2)=1$, a number τ which satisfies (4)′ for given ω can be chosen in $\mathfrak{a}$ and is determined uniquely modulo $2\mathfrak{a}$.[2] Hence it is legitimate to write such as $G(\omega, \tau)=G(\omega)$. We shall use this notation in the sequel.

Furthermore for the τ chosen in this way, we have

$$\omega\tau^2\equiv 1 \quad \Big(\mathrm{mod}\ \frac{2}{\mathfrak{d}}\Big)^{3)}.$$

Now we will simplify the sum $C\left(\frac{\omega}{8}\right)$ under the conditions $(\omega)\mathfrak{d}=\frac{\mathfrak{b}}{\mathfrak{a}}$, $(\mathfrak{a}\mathfrak{b}, 2)=(\mathfrak{a}, \mathfrak{b})=1$. First

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1) In the case were $2|\mathfrak{b}$ holds, we may take 0 as τ, and (4) becomes (1). Furthermore note that (2) can be derived from the general formula of Siegel in [**3**].

2) See [**3**, p. 11].

3) See Hilfssatz 2 in [**3**].

$$C\left(\frac{\omega}{8}\right)=\sum_{\kappa \bmod 8\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}\kappa^2\right)}=2^n \sum_{\kappa \bmod 4\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}\kappa^2\right)}.$$

By making use of an integer α_0 such that $\mathfrak{a}\mathfrak{c}=(\alpha_0)$, $(\mathfrak{c}, 2\mathfrak{a}\mathfrak{b})=1$, we have

$$\begin{aligned}\sum_{\kappa \bmod 4\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}\kappa^2\right)} &= \sum_{\kappa_1 \bmod \mathfrak{a},\, \kappa_2 \bmod 4} e^{2\pi iS\left(\frac{\omega}{8}(4\kappa_1+\alpha_0\kappa_2)^2\right)},\\ &= \sum_{\kappa_3 \bmod 2}\ \sum_{\kappa_1 \bmod \mathfrak{a},\, \kappa_2 \bmod 2} e^{2\pi iS\left(\frac{\omega}{8}(4\kappa_1+2\kappa_2\alpha_0+\kappa_3\alpha_0)^2\right)},\\ &= \sum_{\mathfrak{a} \mid \tau' \bmod 2\mathfrak{a}} H(\omega, \tau'),\end{aligned}$$

where we set $H(\omega, \tau')=\sum_{\mathfrak{a} \mid \kappa \bmod 2\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\kappa+\tau')^2\right)}$. The numbers τ' belong to $\mathfrak{a}$ and are determined modulo $2\mathfrak{a}$.

We assert that $H(\omega, \tau')=0$ if τ' does not coincide with Siegel's number τ indicated in the first part of this section.

When τ' is not equal to τ, there exists a number α' in $\mathfrak{a}$ such that $S(\omega\alpha'^2+\omega\tau'\alpha')\not\equiv 0 \pmod 2$. Hence

$$\begin{aligned}C=\sum_{\mathfrak{a}\mid\alpha \bmod 2\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\alpha+\tau')^2\right)} &= \sum_{\mathfrak{a}\mid\alpha \bmod 2\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\alpha+2\alpha'+\tau')^2\right)},\\ &= e^{2\pi iS\left(\frac{\omega}{2}(\alpha'^2+\alpha'\tau')\right)}\cdot \sum_{\mathfrak{a}\mid\alpha \bmod 2\mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\alpha+\tau')^2\right)},\end{aligned}$$

from which we conclude $C=0$.

Now a short calculation shows that

$$H(\omega, \tau')=e^{-2\pi iS\left(\frac{\omega}{8}\tau'^2\right)}\cdot C\cdot \sum_{2\mid\kappa \bmod \mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\kappa+\tau')^2\right)}.$$

Hence our assertion is valid.

On the other hand if τ' is equal to τ, then

$$H(\omega, \tau)=2^n \sum_{\mathfrak{o}\mid\kappa \bmod \mathfrak{a}} e^{2\pi iS\left(\frac{\omega}{8}(2\kappa+\tau)^2\right)}.$$

Therefore we obtain

$$C\left(\frac{\omega}{8}\right)=4^n\cdot e^{\pi iS\left(\frac{\omega}{4}\tau^2\right)}\cdot G(\omega). \tag{7}$$

In particular, we see for $\mathfrak{a}=\mathfrak{o}$,

$$C\left(\frac{\omega}{8}\right)=4^n\cdot e^{\pi iS\left(\frac{\omega}{4}\tau^2\right)}, \tag{8}$$

where the integer τ satisfies the condition $S(\omega\nu^2+\omega\nu\tau)\equiv 0 \pmod 2$ for any ν in $\mathfrak{o}$.

By the way, for $(\tilde{\omega})\mathfrak{d}=\frac{\mathfrak{b}}{\mathfrak{a}_1\mathfrak{a}_2}$, $(\mathfrak{a}_1, \mathfrak{a}_2)=1$, choose two numbers α_1, α_2 such that $\mathfrak{a}_1\mathfrak{c}_1=(\alpha_1)$, $\mathfrak{a}_2\mathfrak{c}_2=(\alpha_2)$, $(\mathfrak{c}_1, \mathfrak{a}_2)=(\mathfrak{c}_2, \mathfrak{a}_1)=1$ hold, then we have by Hecke [**2**],

$$C(\tilde{\omega})=C(\tilde{\omega}\alpha_1^2)\cdot C(\tilde{\omega}\alpha_2^2). \tag{9}$$

Therefore for our number ω, we have

$$C\Big(\frac{\omega}{8}\Big)=C\Big(\frac{\omega\alpha_1^2}{8}\Big)\cdot C(8\omega), \tag{10}$$

where α_1 means an integer determined by $\mathfrak{a}\mathfrak{c}_1=(\alpha_1)$, $(\mathfrak{c}_1, 2)=1$.

Hence we obtain

$$C\Big(\frac{\omega}{8}\Big)=4^n\cdot e^{\frac{\pi i}{4}S(\omega\alpha_1^2\xi^2)}\cdot C(8\omega). \tag{11}$$

Herein ξ means a number which satisfies the condition $S(\omega\alpha_1^2\nu^2)\equiv S(\omega\alpha_1^2\nu\xi)$ (mod 2) for any integer ν.

Set $\alpha_1\xi=\tau_0$, then τ_0 satisfies $S(\omega\alpha^2)\equiv S(\omega\alpha\tau_0)$ (mod 2) for any α in (α_1). Hence the number τ in (7) satisfies especially $S(\omega\alpha\tau)\equiv S(\omega\alpha\tau_0)$ (mod 2) for any α in (α_1), whence $(\omega)(\alpha_1)(\tau-\tau_0)\subset 2\mathfrak{d}^{-1}$. Therefore $\tau\equiv\tau_0$ $\Big(\text{mod } \frac{2}{\mathfrak{b}\mathfrak{c}_1}\Big)$ which gives us $\tau\equiv\tau_0$ (mod $2\mathfrak{a}$). This shows that we can choose τ_0 as τ in (7).

By (7), (11), we finally obtain $G(\omega)=C(8\omega)$.

Now we have in general

$$C(\beta\omega)=\Big(\frac{\beta}{\mathfrak{a}}\Big)C(\omega), \tag{12}$$

when $(\mathfrak{a}, 2)=(\mathfrak{a}, \beta)=1$, $\beta\in\mathfrak{o}$ hold [**2**].

Hence we obtain

$$G(\omega)=\Big(\frac{8}{\mathfrak{a}}\Big)C(\omega)=\Big(\frac{2}{\mathfrak{a}}\Big)C(\omega)=C(2\omega). \tag{13}$$

This is an interesting equality which explains us a relationship between Siegel's sum and the Gauss-Hecke sum under our assumption $(\mathfrak{a}\mathfrak{b}, 2)=1$.

§ 3. Hasse's formula.

We shall show that the formula (6) of Hasse can be obtained from (2) by a slight modification of Siegel's proof in [3]. We assume that $(\alpha, \beta)=1$, $\alpha\equiv\beta\equiv1$ (mod 2).

Following Siegel [3], we define numbers γ, δ, ω_i and an ideal $\mathfrak{c}$ by

$$\mathfrak{d}(\gamma)=\mathfrak{c}, \qquad (\mathfrak{c}, 2\alpha)=1,$$

$$\omega=\omega_1=\frac{\beta}{\alpha}\gamma, \qquad \omega_2=\frac{1}{\alpha}\gamma, \qquad \omega_3=\gamma, \qquad \omega_4=\beta\gamma.$$

$$(\omega)\mathfrak{d}=\frac{(\beta)}{(\alpha)}\mathfrak{c}, \qquad ((\alpha\beta)\mathfrak{c}, 2)=1.$$

$$\mathfrak{d}(\delta)=\mathfrak{g}\,, \qquad (\mathfrak{g}, 2(\beta)\mathfrak{c})=1\,, \qquad \mathfrak{g} \text{ integral}.$$

Furthermore we put $\omega_i^*=-\frac{\delta^2}{\omega_i}$ $(i=1,2,3,4)$. Then we have

$$C(2\omega_1)=C(2\beta\omega_2)=\left(\frac{\beta}{\alpha}\right)C(2\omega_2)\,,$$

$$C(2\omega_1)=\frac{\sqrt{N\alpha}}{4^n\sqrt{N\beta\mathfrak{c}}}\cdot e^{\frac{\pi i}{4}S^*(\operatorname{sgn}\omega_1)}\cdot C\left(\frac{-\delta^2}{8\omega_1}\right),$$

$$C(2\omega_2)=\frac{\sqrt{N\alpha}}{4^n\sqrt{N\mathfrak{c}}}\cdot e^{\frac{\pi i}{4}S^*(\operatorname{sgn}\omega_2)}\cdot C\left(\frac{-\delta^2}{8\omega_2}\right).$$

Therefore

$$\left(\frac{\beta}{\alpha}\right)=\frac{1}{\sqrt{N\beta}}\cdot e^{\frac{\pi i}{4}\{S^*(\operatorname{sgn}\omega_1)-S^*(\operatorname{sgn}\omega_2)\}}\cdot\frac{C\left(\frac{-\delta^2}{8\omega_1}\right)}{C\left(\frac{-\delta^2}{8\omega_2}\right)}.$$

Take an integer γ' such that $\mathfrak{c}\mathfrak{c}'=(\gamma'), (\mathfrak{c}', 2)=1$, then from (9) follow

$$C\left(\frac{-\delta^2}{8\omega_1}\right)=C\left(\frac{\omega_1^*}{8}\right)=C\left(-\frac{\alpha\delta^2}{8\beta\gamma}\beta^2\gamma'^2\right)\cdot C\left(-\frac{8^2\alpha\delta^2}{8\beta\gamma}\right),$$

$$C\left(\frac{-\delta^2}{8\omega_2}\right)=C\left(\frac{\omega_2^*}{8}\right)=C\left(-\frac{\alpha\delta^2}{8\gamma}\gamma'^2\right)\cdot C\left(-\frac{8^2\alpha\delta^2}{8\gamma}\right).$$

Since $(\gamma'\gamma^{-1}\delta)=\mathfrak{c}\mathfrak{c}'\mathfrak{c}^{-1}\mathfrak{g}=\mathfrak{c}'\mathfrak{g}$, we see that $\gamma'\gamma^{-1}\delta$ is an integer. Hence we have

$$C\left(\frac{\omega_1^*}{8}\right)=C\left(-\frac{\alpha\beta\gamma}{8}\right)\cdot C\left(-8\frac{\alpha\delta^2}{\beta\gamma}\right)$$

$$=4^n\cdot e^{\frac{\pi i}{4}S(-\alpha\beta\gamma\tau^2)}\cdot\left(\frac{\alpha}{(\beta)\mathfrak{c}}\right)\cdot C\left(-\frac{2\delta^2}{\beta\gamma}\right),$$

$$C\left(\frac{\omega_2^*}{8}\right)=C\left(-\frac{\alpha\gamma}{8}\right)\cdot C\left(-8\frac{\alpha\delta^2}{\gamma}\right)$$

$$=4^n\cdot e^{\frac{\pi i}{4}S(-\alpha\gamma\tau^2)}\cdot\left(\frac{\alpha}{\mathfrak{c}}\right)\cdot C\left(-\frac{2\delta^2}{\gamma}\right).$$

Here the assumption $\alpha\equiv\beta\equiv1 \pmod 2$ shows that τ satisfies $S(\gamma\kappa^2)\equiv S(\gamma\kappa\tau)$ (mod 2) for any integer κ. Therefore

$$\left(\frac{\beta}{\alpha}\right)=\frac{1}{\sqrt{N\beta}}e^{\frac{\pi i}{4}\{S^*(\operatorname{sgn}\omega_1)-S^*(\operatorname{sgn}\omega_2)-S(\alpha\beta\gamma\tau^2)+S(\alpha\gamma\tau^2)\}}\cdot\left(\frac{\alpha}{\beta}\right)\cdot\frac{C(2\omega_4^*)}{C(2\omega_3^*)}.$$

Now

$$C(2\omega_4^*)=\frac{\sqrt{N\beta\mathfrak{c}}}{4^n\sqrt{N\mathfrak{g}^2}}\cdot e^{\frac{\pi i}{4}S^*(\operatorname{sgn}\omega_i^*)}\cdot C\left(\frac{-\delta^2}{8\omega_4^*}\right),$$

$$C(2\omega_3^*)=\frac{\sqrt{N\mathfrak{c}}}{4^n\sqrt{N\mathfrak{g}^2}}\cdot e^{\frac{\pi i}{4}S^*(\operatorname{sgn}\omega_3^*)}\cdot C\left(\frac{-\delta^2}{8\omega_3^*}\right).$$

Hence we have

$$\frac{C(2\omega_1^*)}{C(2\omega_3^*)}=e^{\frac{\pi i}{4}\{S^*(\mathrm{sgn}\ \omega_1^*)-S^*(\mathrm{sgn}\ \omega_3^*)\}}\sqrt{N\beta}\ \frac{C\left(\frac{\beta\gamma}{8}\right)}{C\left(\frac{\gamma}{8}\right)}.$$

Therefore

$$\left(\frac{\beta}{\alpha}\right)\left(\frac{\alpha}{\beta}\right)=e^{\frac{\pi i}{4}\{S^*(\mathrm{sgn}\ \alpha\beta\gamma)-S^*(\mathrm{sgn}\ \alpha\gamma)-S^*(\mathrm{sgn}\ \beta\gamma)+S^*(\mathrm{sgn}\ \gamma)\}}$$

$$\times e^{\frac{\pi i}{4}\{-S(\alpha\beta\gamma\tau^2)+S(\alpha\gamma\tau^2)+S(\beta\gamma\tau^2)-S(\gamma\tau^2)\}}.$$

By making use of that $\gamma\tau^2\equiv 1\ \left(\mathrm{mod}\ \frac{2}{\mathfrak{d}}\right)$, we see

$$\frac{1}{4}\{S^*(\mathrm{sgn}\ \alpha\beta\gamma)-S^*(\mathrm{sgn}\ \alpha\gamma)-S^*(\mathrm{sgn}\ \beta\gamma)+S^*(\mathrm{sgn}\ \gamma)\}\equiv g(\alpha,\beta)\quad(\mathrm{mod}\ 2),$$

$$\frac{1}{4}\{-S(\alpha\beta\gamma\tau^2)+S(\alpha\gamma\tau^2)+S(\beta\gamma\tau^2)-S(\gamma\tau^2)\}\equiv S\left(\frac{1-\alpha}{2}\ \frac{1-\beta}{2}\right)\quad(\mathrm{mod}\ 2).$$

This completes the proof of (6).

§ 4. Complementary law.

The formula $\left(\frac{-1}{\alpha}\right)=(-1)^{S\left(\frac{1-\alpha}{2}\right)}$, where α is a totally positive number and $\alpha\equiv 1$ (mod 2), follows from (6). Our concern is to obtain the formula for $\left(\frac{2}{\alpha}\right)$.

Suppose that $\alpha\equiv 1$ (mod 2), then we have

$$\left(\frac{2}{\alpha}\right)=\frac{C\left(\frac{-\alpha\omega'}{8}\right)C\left(\frac{-\omega'}{4}\right)}{C\left(\frac{-\omega'}{8}\right)C\left(\frac{-\alpha\omega'}{4}\right)},$$

where ω' means a number determined by $(\omega')\mathfrak{d}=\mathfrak{b}$, $(\mathfrak{b},2)=1$. This formula follows easily from (2) by Hecke's method as in the previous section.

Hence in the case where $\alpha\equiv 1$ (mod 4) from (8) follows

$$\left(\frac{2}{\alpha}\right)=e^{\frac{\pi i}{4}S((1-\alpha)\omega'\tau^2)}.$$

By making use of that $\omega'\tau^2\equiv 1\ \left(\mathrm{mod}\ \frac{2}{\mathfrak{b}}\right)$, we obtain

$$\left(\frac{2}{\alpha}\right)=(-1)^{S\left(\frac{1-\alpha}{4}\right)}.$$

This is also a well-known formula of Hasse [**1**].

General Education Department,
Kyushu University

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