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On the existence of a Hall normal subgroup

Dedicated to Professor Yasuo AKIZUKI

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1. Introduction. A subgroup H of a finite group G is called a Hall subgroup if the order $|H|$ of H is relatively prime to the index $[G:H]$. A normal subgroup N of G is a normal complement to H if the conditions $NH=G$ and $N\cap H=\{1\}$ are satisfied. The purpose of this note is to prove a result giving a necessary and sufficient condition for H to have a normal complement. Let π denote a set of prime numbers. A π-number is an integer all of whose prime divisors belong to π. The complementary set to π is denoted by π'. A subgroup H is called a π-Hall subgroup if $|H|$ is a π-number but $[G:H]$ is a π'-number. The main result of this note is the following theorem.

THEOREM 1. *Let H be a π-Hall subgroup of G. Then H has a normal complement if and only if the following two conditions are satisfied:*

(1) *two elements of H which are conjugate in G are already conjugate in H; and*

(2) *if $x\in H$ satisfies the condition $C_G(x)\neq G$, then $C_H(x)$ is a π-Hall subgroup of $C_G(x)$ and has a normal complement in $C_G(x)$.*

We use the standard notation. $C_G(S)$ is the centralizer of a subset S and the normalizer is denoted by $N_G(S)$.

We will mention a few consequences. The classical theorem of Frobenius asserts that if a subgroup H of a finite group G satisfies

$$H\cap x^{-1}Hx=\{1\} \quad \text{for all} \quad x\notin H,$$

then H has a normal complement consisting of elements which are contained in none of the conjugate subgroups of H together with the identity element. In this case H is a Hall subgroup of G. If $1\neq x\in H$ and $y^{-1}xy\in H$, then y must belong to H. It is now easy to verify the conditions (1) and (2) for H. Therefore our theorem yields the theorem of Frobenius. Consider next the case when a Sylow subgroup S of G satisfies the condition $N_G(S)=C_G(S)$. Then S is necessary abelian and the classical theorem of Burnside asserts the existence of a normal complement to S. This is also proved by using Theorem 1. We will use induction on the order of G. The first condition

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(1) is precisely the lemma of Burnside. The second one is trivially satisfied by inductive hypothesis. More applications may be found in the last section.

2. Proof of Theorem 1. Suppose that H has a normal complement N. Let x and y be two elements of H such that $y=z^{-1}xz$ with $z\in G$. By assumption $G=NH$ so that $z=nh$ where $n\in N$ and $h\in H$. Then we have $y=h^{-1}n^{-1}xnh$. Hence

$$x^{-1}hyh^{-1}=x^{-1}n^{-1}xn\,.$$

The left side of this equation belongs to H, while the other belongs to N. Hence we have

$$1=x^{-1}hyh^{-1}=x^{-1}n^{-1}xn\,.$$

The first equality yields the condition (1). If in particular $x=y$, the above two equalities yield the condition (2) in Theorem 1.

Assume conversely that (1) and (2) are satisfied by H. Let H_0 denote the intersection of H and the center of G. Since H is a π-Hall subgroup, H_0 is the set of central π-elements of G. The following lemmas are proved under the assumptions (1) and (2).

LEMMA 1. *A π-element of G is conjugate to an element of H.*

PROOF. Let x be a π-element. If x is in the center of G, x is contained in $H_0\subseteqq H$. The element x is a product of mutually commuting elements $x_1, x_2, \cdots$ of prime power orders. If x is not in the center of G, at least one of the factors, say x_1, is not contained in the center of G. Then $C_G(x_1)\neq G$. By a Sylow's theorem x_1 is conjugate to an element y of H. Then x is conjugate to a π-element of $C_G(y)$. By (2), $C_G(y)$ is a semi-direct product of $C_H(y)$ and its complement. Hence by the theorem of Schur-Zassenhaus ([**4**], p. 132) x is conjugate to an element of H.

Let x be an element of G. Then x is a product of two commuting elements x_1 and x_2 where x_1 is a π-element and x_2 is a π'-element. This decomposition is unique. We call x_1 the π-factor of x.

Let θ be an irreducible character of H/H_0 with degree d. Define a function φ on G by the formula

$$\varphi(x)=\theta(y)-d$$

where y is an element of H conjugate to the π-factor of x. By Lemma 1 we can find such an element y and by (1), φ is well-defined. By definition φ is a class function. We want to prove that φ is a generalized character of G. According to a theorem of Brauer [**1**] it suffices to show that the restriction of φ on an elementary subgroup E is a generalized character. It suffices to consider the case when E is a subgroup of $C_G(x)$ with $x\in H-H_0$. The assertion follows then from the condition (2).

LEMMA 2. *The function φ is equal to $\chi-d$ where χ is an irreducible character of G.*

PROOF. Since φ is a generalized character, the assertion follows from the equations

$$(1/|G|)\sum_{x\in G}\varphi(x)=-d \quad \text{and} \quad (1/|G|)\sum_{x\in G}|\varphi(x)|^2=1+d^2\,.$$

The computation is easy. Let $x_1,\cdots,x_m$ be the set of representatives of conjugate classes of H in $H-H_0$. Then $\varphi(x)=0$ unless the π-factor of x is conjugate to one of the elements x_i $(i=1,2,\cdots,m)$. Hence

$$(1/|G|)\sum_{x\in G}\varphi(x)=(1/|G|)\sum_{i}[G:C_G(x_i)]\sum{}'\varphi(x_i y)$$

where in the second summation y ranges over π'-elements in $C_G(x_i)$. Hence the above summation is equal to $(1/|H|)\sum_{x\in H}(\theta(x)-d)=-d$. The second equation is proved similarly.

Let $N(\theta)$ be the kernel of the representation with character χ in Lemma 2. Then $N(\theta)$ contains all the π'-elements of G. Let N_0 be the intersection of $N(\theta)$ where θ ranges all the irreducible characters of H/H_0. It follows easily that $N_0H=G$ and $N_0\cap H=H_0$. H_0 is by definition a central Hall subgroup of N_0. Hence by a theorem of Schur [**4**], N_0 is a direct product of H_0 and a subgroup N. Since N is a characteristic subgroup of N_0, N is the normal complement of H.

3. The second formulation. We say that a π-Hall subgroup H of G satisfies the condition F_π if every nilpotent π-subgroup of G is contained in a conjugate subgroup of H. If we omit the word nilpotent, then we obtain the condition D_π of P. Hall. F_π is weaker than D_π. In fact F_π does not imply the conjugacy of two distinct π-Hall subgroups. By a theorem of Schur-Zassenhaus ([**4**], p. 132), the existence of a normal complement to H implies the condition F_π. As a matter of fact the existence of a normal complement to H implies D_π but the derivation of D_π requires a deep result of Feit and Thompson [**2**].

LEMMA 3. *Let H be a π-Hall subgroup satisfying* (1) *of Theorem* 1 *and F_π. Then for $x\in H$, $C_H(x)$ is a π-Hall subgroup of $C_G(x)$.*

PROOF. Let P be a Sylow group of $C_H(x)$. It suffices to show that P is a Sylow group of $C_G(x)$. By a theorem of Sylow P is a part of a Sylow group S of $C_G(x)$. The group E generated by S and x is elementary. Hence by F_π a conjugate subgroup E^t is contained in H. Since both x and x^t are in H, they are conjugate in H by (1). Hence there exists an element u of H so that $x=x^{tu}$. Then $E^{tu}\subseteqq H$ and $S^{tu}\subseteqq H$. Since P is a Sylow group of H we have $|P|\geqq|S^{tu}|=|S|$. Hence $P=S$.

By the same method we can prove that $C_H(x)$ satisfies F_π. This suggests the following formulation.

THEOREM 2. *Let H be a π-Hall subgroup of G. H has a normal complement if and only if the following two conditions are satisfied:*

(1) *for any subset S of H, two elements of $C_H(S)$ are conjugate in $C_G(S)$ if and only if they are conjugate in $C_H(S)$; and*

(2) *H satisfies the condition F_π.*

PROOF. Use induction on the order of G. By Lemma 3 and the remark just made $C_H(S)$ satisfies the same assumptions as H. Hence if $C_G(S) \neq G$ the inductive hypothesis says that $C_H(S)$ has a normal complement in $C_G(S)$. Theorem 1 is applicable and yields the existence of a normal complement to H under (1) and (2).

4. Applications. In the introduction we derived the transfer theorem of Burnside. In the same way we can prove a theorem of Frobenius asserting the existence of a normal p-complement of a group G under the condition that $N_G(U)/C_G(U)$ is a p-group whenever U is a p-subgroup of G. Recently Kochendörffer and Zappa have remarked that a normal complement to a Hall subgroup H gives a "distinguished" set of representatives from cosets of H. A set of elements T is a distinguished set of coset representatives if T contains one and only one element of each coset of H and if $T^h = T$ for $h \in H$. They have verified that the existence of a distinguished set of coset representatives and the condition D_π are necessary and sufficient conditions for the existence of a normal complement under the various restrictions on the structure of H. The weakest restriction on H given in [3] is the solvability of H. If T is a distinguished set of coset representatives and if an element $t \in T$ transforms an element $x \in H$ into H, then t commutes with x. Hence the existence of T implies the condition (1) of Theorem 2. Hence without assuming the solvability of H we have the same conclusion as a theorem of Zappa.

THEOREM 3. *Let H be a π-Hall subgroup of G. There exists a normal complement to H if and only if there is a distinguished set of coset representatives of H and H satisfies the condition F_π.*

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