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On the finiteness of the derived normal ring of an affine ring

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Introduction. In this paper, a *ring* will mean a commutative ring with unity 1, and a *local* (or *semi-local*) *ring* a Noetherian local (or semi-local) ring. An element x will be called *algebraic* or *separably algebraic* over an integral domain $\mathfrak{o}$, if it is algebraic or separably algebraic, respectively, over the quotient field of $\mathfrak{o}$. We say an integral domain $\mathfrak{o}'$ is *separably generated* over a subring $\mathfrak{o}$, if the quotient field of $\mathfrak{o}'$ is separably generated over that of $\mathfrak{o}$. An integral domain $\mathfrak{o}$ will be called an *affine ring* over a subring I if it is a finitely generated extension ring of I. A ring P is called a *spot* over a subring I if there exists an affine ring $\mathfrak{o}$ over I which has a prime ideal $\mathfrak{p}$ such that $P=\mathfrak{o}_{\mathfrak{p}}$.

It has been shown by M. Nagata that if $\mathfrak{o}$ is an affine ring over a Dedekind domain (or a field) I and if $\mathfrak{o}$ is separably generated over I, then the derived normal ring of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module (see [**2**]).

Recently L. J. Ratliff, Jr. has proved that this theorem holds in case I is a regular local ring of rank 2 such that its residue field contains infinitely many elements (cf. [**5**]).

In the following lines, we shall show that the same is true if we replace the ground domain I by a Noetherian normal ring I' such that if $\mathfrak{p}$ is any prime ideal of I', then the local ring $I'_{\mathfrak{p}}$ is analytically unramified. (We say that a semi-local ring $\mathfrak{o}$ is *analytically unramified* if the completion of $\mathfrak{o}$ has no nilpotent element different from zero.)

This special type of ring I' (more general than a Dedekind domain or a regular local ring of rank 2) is termed, in this paper, "*a ground ring*".

The proof of the above result is performed making use of the following property: If P is a spot over a ground ring I' and if P is separably generated over I', then the derived normal ring of P is a finite P-module.

This property follows from the next theorem.

THEOREM 1. *Let $\mathfrak{o}$ be an analytically unramified local integral domain and let $\mathfrak{o}'=\mathfrak{o}[x_1, \cdots, x_n]$ be a separably generated integral domain over $\mathfrak{o}$. Then for any prime ideal $\mathfrak{p}'$ of $\mathfrak{o}'$, the local ring $\mathfrak{o}'_{\mathfrak{p}'}$ is analytically unramified.*

Therefore we shall first prove this theorem, and § 1 will be devoted to its proof. This theorem is a generalization of a result of D. Rees [**6**] (see Lemma 1 below) which plays an essential role in the course of the proof of the theorem.

In § 2 we shall prove the main theorem stated above.

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§ 1

First we recall the following result, due to D. Rees.

LEMMA 1. *Let $\mathfrak{o}$ be an analytically unramified local ring and let K be the total quotient ring of $\mathfrak{o}$. Then if $x_1, \cdots, x_n$ are elements of K and if $\mathfrak{p}'$ is any prime ideal of $\mathfrak{o}' = \mathfrak{o}[x_1, \cdots, x_n]$, the local ring $\mathfrak{o}'_{\mathfrak{p}'}$ is also analytically unramified.*

For the proof, see D. Rees [**6**] Theorem 1.6.

COROLLARY[1]. *Let $\mathfrak{o}$ be an analytically unramified semi-local integral domain and let K be its quotient field. Then, for any element x in K which is integral over $\mathfrak{o}$, the semi-local integral domain $\mathfrak{o}[x]$ is also analytically unramified.*

PROOF. Let $\mathfrak{m}$ be an arbitrary maximal ideal of $\mathfrak{o}' = \mathfrak{o}[x]$. Then $\mathfrak{p} = \mathfrak{m} \cap \mathfrak{o}$ is a maximal ideal of $\mathfrak{o}$, and $\mathfrak{o}'_{\mathfrak{m}} = \mathfrak{o}_{\mathfrak{p}}[x]_{\mathfrak{q}}$ where $\mathfrak{q} = \mathfrak{m}\mathfrak{o}'_{\mathfrak{m}} \cap \mathfrak{o}_{\mathfrak{p}}[x]$. Since the completion $\mathfrak{o}^*_{\mathfrak{p}}$ of $\mathfrak{o}_{\mathfrak{p}}$ is a direct summand of the completion $\mathfrak{o}^*$[2], $\mathfrak{o}_{\mathfrak{p}}$ is analytically unramified. Therefore $\mathfrak{o}'_{\mathfrak{m}}$ is also analytically unramified by Lemma 1. On the other hand, $\mathfrak{o}'^* = \mathfrak{o}'_{\mathfrak{m}_1}{}^* \oplus \cdots \oplus \mathfrak{o}'_{\mathfrak{m}_r}{}^*$ (direct sum) where $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ are all the maximal ideals of $\mathfrak{o}'$. Hence $\mathfrak{o}'$ is analytically unramified.

Next we shall prove the following:

LEMMA 2. *Let $\mathfrak{o}$ be an analytically unramified semi-local integral domain, and let x be a separably integral element over $\mathfrak{o}$ (in some field containing $\mathfrak{o}$). Then $\mathfrak{o}[x]$ is an analytically unramified semi-local integral domain.*

PROOF. Let $\bar{\mathfrak{o}}$ be the derived normal ring of $\mathfrak{o}$, then $\bar{\mathfrak{o}}$ is a finite $\mathfrak{o}$-module. (Cf. M. Nagata [**2**], Appendix I, Proposition 1.) Hence $\bar{\mathfrak{o}}$ is an analytically unramified semi-local integral domain by the Corollary to Lemma 1. Furthermore $\mathfrak{o}[x]$ is a subspace of $\bar{\mathfrak{o}}[x]$. Therefore we have only to prove this lemma under the assumption that $\mathfrak{o}$ is normal.

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be all the prime ideals of height zero of $\mathfrak{o}^*$, and set $\mathfrak{o}_i = \mathfrak{o}^*/\mathfrak{p}_i$ for $i = 1, \cdots, r$. Then, by assumption, $\mathfrak{o}^*$ is a subring of $\mathfrak{o}_1 \oplus \cdots \oplus \mathfrak{o}_r$. Since x is integral over $\mathfrak{o}$, we have

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1) This corollary and Lemma 2 below can be also proved easily without using Lemma 1.

2) In this paper, the completion of a (semi-) local ring $\mathfrak{o}$ is denoted by $\mathfrak{o}^*$.

$$\mathfrak{o}[x]^* \cong \mathfrak{o}[x] \otimes_{\mathfrak{o}} \mathfrak{o}^* \subseteqq \mathfrak{o}[x] \otimes_{\mathfrak{o}} (\mathfrak{o}_1 \oplus \cdots \oplus \mathfrak{o}_r)$$
$$= (\mathfrak{o}[x] \otimes_{\mathfrak{o}} \mathfrak{o}_1) \oplus \cdots \oplus (\mathfrak{o}[x] \otimes_{\mathfrak{o}} \mathfrak{o}_r).$$

On the other hand, since any non-zero-divisor of $\mathfrak{o}$ is also that of $\mathfrak{o}^*$, we have $\mathfrak{p}_i \cap \mathfrak{o} = (0)$ for each i. Hence $\mathfrak{o}$ is considered as a subring of the integral domain $\mathfrak{o}_i$ for $i = 1, \cdots, r$. Then, as is well known, each $\mathfrak{o}[x] \otimes_{\mathfrak{o}} \mathfrak{o}_i$ has no nilpotent element other than 0, because $\mathfrak{o}$ is normal. Therefore $\mathfrak{o}[x]^*$ also has no nilpotent element other than 0, which proves our lemma.

Now we obtain the following:

PROPOSITION 1. *Let $\mathfrak{o}$ be an analytically unramified local integral domain and let x be a separably algebraic element over $\mathfrak{o}$ (in some field containing $\mathfrak{o}$). Then, for any prime ideal $\mathfrak{p}'$ of $\mathfrak{o}' = \mathfrak{o}[x]$, the local ring $\mathfrak{o}'_{\mathfrak{p}'}$ is analytically unramified.*

PROOF. We can find an element $a (\neq 0)$ of $\mathfrak{o}$ such that ax is integral over $\mathfrak{o}$. If we set $\mathfrak{o}'' = \mathfrak{o}[ax]_{\mathfrak{p}' \cap \mathfrak{o}[ax]}$ and $\mathfrak{p}'' = \mathfrak{p}' \mathfrak{o}'_{\mathfrak{p}'} \cap \mathfrak{o}''[x]$, we have $\mathfrak{o}'_{\mathfrak{p}'} = \mathfrak{o}''[x]_{\mathfrak{p}''}$ and x belongs to the quotient field of $\mathfrak{o}''$. Hence, for the proof of the proposition, it is sufficient to show that $\mathfrak{o}''$ is analytically unramified (by virtue of Lemma 1). In other words, we may assume originally that x is separably integral over $\mathfrak{o}$. Now the proposition follows easily from Lemma 2 and Lemma 1.

Secondly we shall prove the following proposition.

PROPOSITION 2. *Let $\mathfrak{o}$ be an analytically unramified local integral domain and let x be a transcendental element over $\mathfrak{o}$. Then, for any prime ideal $\mathfrak{p}'$ of $\mathfrak{o}' = \mathfrak{o}[x]$, the local ring $\mathfrak{o}'_{\mathfrak{p}'}$ is analytically unramified.*

PROOF. First we remark that we may assume that $\mathfrak{p}'$ is a maximal ideal of $\mathfrak{o}'$ and $\mathfrak{p}' \cap \mathfrak{o}$ coincides with the unique maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Indeed, if we replace $\mathfrak{o}$ by $\mathfrak{o}_{\mathfrak{p}' \cap \mathfrak{o}}$, we may suppose that $\mathfrak{p}' \cap \mathfrak{o} = \mathfrak{m}$ (by virtue of Lemma 1). On the other hand, let $\mathfrak{P}$ be a maximal ideal of $\mathfrak{o}'$ containing $\mathfrak{p}'$, then $\mathfrak{o}'_{\mathfrak{p}'} = (\mathfrak{o}'_{\mathfrak{P}})_{\mathfrak{p}' \mathfrak{o}'_{\mathfrak{P}}}$. Therefore if we show that $\mathfrak{o}'_{\mathfrak{P}}$ is analytically unramified, it follows from Lemma 1 that $\mathfrak{o}'_{\mathfrak{p}'}$ is also analytically unramified. Therefore we suppose that $\mathfrak{p}'$ is maximal and $\mathfrak{p}' \cap \mathfrak{o} = \mathfrak{m}$.

Then there exists a monic polynomial $f(x)$ in $\mathfrak{o}'$ which is irreducible modulo $\mathfrak{m}$, such that $\mathfrak{p}' = \mathfrak{m}\mathfrak{o}' + f(x)\mathfrak{o}'$. Hence we have $\mathfrak{o}'^*_{\mathfrak{p}'} = \mathfrak{o}^*\{f\}[x]$ where $f = f(x)$ and $\mathfrak{o}^*\{f\}$ is the ring of formal power series in f with coefficients in $\mathfrak{o}^*$. More precisely,

$$\mathfrak{o}'^*_{\mathfrak{p}'} \cong \mathfrak{o}^*\{f\}[X]/(f(X) - f)\mathfrak{o}^*\{f\}[X],$$

where X is an indeterminate. Set $F(X) = f(X) - f$, then $F(X)$ is irreducible modulo the maximal ideal of $\mathfrak{o}^*\{f\}$. What we shall show is that $\mathfrak{o}^*\{f\}[X] / F(X)\mathfrak{o}^*\{f\}[X]$ has no nilpotent element different from 0.

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be all the prime ideals of height zero in $\mathfrak{o}^*$, then each

$\mathfrak{o}_i = \mathfrak{o}^*/\mathfrak{p}_i$ $(i=1, \cdots, r)$ is a complete local integral domain containing $\mathfrak{o}$ as a subring, and $\mathfrak{o}^*\{f\}[X]/F(X)\mathfrak{o}^*\{f\}[X]$ is a subring of the direct sum of all $\mathfrak{o}_i\{f\}[X]/F(X)\mathfrak{o}_i\{f\}[X], i=1, \cdots, r$. Furthermore each $\mathfrak{o}_i\{f\}[X]/F(X)\mathfrak{o}_i\{f\}[X]$ is isomorphic to the completion of $\mathfrak{o}_i[x]_{\mathfrak{p}'_i}$, where $\mathfrak{p}'_i = \mathfrak{m}\mathfrak{o}_i[x] + f(x)\mathfrak{o}_i[x]$ and $f(x)$ is irreducible modulo the maximal ideal $\mathfrak{m}\mathfrak{o}_i$ of $\mathfrak{o}_i$. Hence, from the beginning, we may assume that $\mathfrak{o}$ is a complete local integral domain.

Now let $\bar{\mathfrak{o}}$ be the derived normal ring of $\mathfrak{o}$. Then, as is well known, $\bar{\mathfrak{o}}$ is a complete normal local integral domain and is a finite $\mathfrak{o}$-module. Set $\mathfrak{o}'' = \bar{\mathfrak{o}}[x]$ and $S = \mathfrak{o}' - \mathfrak{p}'$ (complementary set of $\mathfrak{p}'$ in $\mathfrak{o}' = \mathfrak{o}[x]$), then $\mathfrak{o}''_S$ is a finite $\mathfrak{o}'_{\mathfrak{p}'}$-module, and hence $\mathfrak{o}'_{\mathfrak{p}'}{}^*$ is a subring of $\mathfrak{o}''_S{}^*$. On the other hand, if we denote the maximal ideals of $\mathfrak{o}''$ lying over $\mathfrak{p}'$ by $\mathfrak{p}''_1, \cdots, \mathfrak{p}''_t$, then $\mathfrak{o}''_S{}^* = \mathfrak{o}''_{\mathfrak{p}''_1}{}^* \oplus \cdots \oplus \mathfrak{o}''_{\mathfrak{p}''_t}{}^*$ (direct sum). *Therefore we may assume that* $\mathfrak{o}$ *is a complete normal local integral domain.* We have only to show that $\mathfrak{o}\{f\}[X]/F(X)\mathfrak{o}\{f\}[X]$ has no nilpotent element other than 0, on this assumption.

Let K be the quotient field of $\mathfrak{o}$. Then, since $\mathfrak{o}$ is normal, $f(x)$ is irreducible in $K[x]$, and hence $F(X) = f(X) - f(x)$ is also irreducible in $K\{f\}[X]$. But $K\{f\}$ is an unique factorization domain, and therefore the ideal $F(X)K\{f\}[X]$ is prime. Hence $F(X)\mathfrak{o}\{f\}[X] = F(X)K\{f\}[X] \cap \mathfrak{o}\{f\}[X]$ is also a prime ideal, and consequently $\mathfrak{o}\{f\}[X]/F(X)\mathfrak{o}\{f\}[X]$ is an integral domain. This completes the proof.

From Propositions 1 and 2 we obtain the first main theorem.

THEOREM 1. *Let* $\mathfrak{o}$ *be an analytically unramified local integral domain and let* $\mathfrak{o}'$ *be an affine ring over* $\mathfrak{o}$. *If* $\mathfrak{o}'$ *is separably generated over* $\mathfrak{o}$, *then, for any prime ideal* $\mathfrak{p}'$ *of* $\mathfrak{o}'$, *the quotient ring* $\mathfrak{o}'_{\mathfrak{p}'}$ *is also analytically unramified.*

PROOF. Let $x_1, \cdots, x_n$ be the elements of $\mathfrak{o}'$ such that $\mathfrak{o}' = \mathfrak{o}[x_1, \cdots, x_n]$. We may assume that the subset $\{x_1, \cdots, x_r\}$ $(r = dim_{\mathfrak{o}}\mathfrak{o}')$ is a separating transcendence base of $\mathfrak{o}'$ over $\mathfrak{o}$[3]. Then (if $r < n$) the remaining elements $x_{r+1}, \cdots, x_n$ are separably algebraic over $\mathfrak{o}[x_1, \cdots, x_r]$.

Now we will construct a finite sequence of local rings $\mathfrak{o}_1, \cdots, \mathfrak{o}_n$ as follows: first we set $\mathfrak{o}_1 = \mathfrak{o}[x_1]_{\mathfrak{p}_1}$ where $\mathfrak{p}_1 = \mathfrak{o}[x_1] \cap \mathfrak{p}'\mathfrak{o}'_{\mathfrak{p}'}$, and if $\mathfrak{o}_{i-1}$ $(2 \leqq i \leqq n)$ has been already defined, then we set $\mathfrak{o}_i = \mathfrak{o}_{i-1}[x_i]_{\mathfrak{p}_i}$ where $\mathfrak{p}_i = \mathfrak{o}_{i-1}[x_i] \cap \mathfrak{p}'\mathfrak{o}'_{\mathfrak{p}'}$. Then, by induction on i, we see easily that each $\mathfrak{o}_i$ is analytically unramified, by Proposition 2 (if $i \leqq r$) and by Proposition 1 (if $i > r$). Hence especially $\mathfrak{o}_n = \mathfrak{o}'_{\mathfrak{p}'}$ is analytically unramified. This completes the proof.

§2

In this section, by a *ground ring* I we shall mean a Noetherian normal ring such that, whenever $\mathfrak{p}$ is a prime ideal of I, the quotient ring $I_{\mathfrak{p}}$ is

3) If $r = 0$, this set is empty.

always analytically unramified.

From Theorem 1 and definitions above, we see easily the following:

THEOREM 2. *Let P be a spot over a ground ring I. If P is separably generated over I, then P is analytically unramified, and the derived normal ring of P is a finite P-module.*

Now we come to the second main theorem, which is a generalization of M. Nagata's result (cf. [**2**] Appendix 2, Proposition 4) and also a result of Ratliff (cf. [**5**]).

THEOREM 3. *Let $\mathfrak{o}$ be an affine ring over a ground ring I. If $\mathfrak{o}$ is separably generated over I, then the derived normal ring of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module.*

The proof is similar to that of Theorem 1.3. in [1].

Let $\mathfrak{o}'$ be the derived normal ring of $\mathfrak{o}$. By the normalization theorem (cf. M. Nagata [**2**] Chapter 3, Theorem 4), we can find a separating transcendence base $y_1, \cdots, y_n$ of $\mathfrak{o}$ over I, and an element a ($\neq 0$) of I such that $\mathfrak{o}[a^{-1}]$ is integral over $I[a^{-1}, y_1, \cdots, y_n]$. Then $\mathfrak{o}'[a^{-1}]$ is the integral closure of $I[a^{-1}, y_1, \cdots, y_n]$ in the quotient field L of $\mathfrak{o}$.

Since L is separable over $I[a^{-1}, y_1, \cdots, y_n]$ which is normal, $\mathfrak{o}'[a^{-1}]$ is a finite $I[a^{-1}, y_1, \cdots, y_n]$-module, and consequently it is a finite $\mathfrak{o}[a^{-1}]$-module. (Cf. M. Nagata [**4**] Corollary (10. 16).)

Therefore if $a^{-1} \in \mathfrak{o}, \mathfrak{o}'$ is a finite $\mathfrak{o}$-module. Hence we may assume that $a^{-1} \notin \mathfrak{o}$. Then there exist a finite number of elements $c_1, \cdots, c_r$ in $\mathfrak{o}'$ such that $\mathfrak{o}'[a^{-1}] = \mathfrak{o}[a^{-1}, c_1, \cdots, c_r]$. Set $\mathfrak{o}_1 = \mathfrak{o}[c_1, \cdots, c_r]$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be all the prime divisors (*not necessarily minimal*) of $a\mathfrak{o}_1$. Then, by Theorem 2, the derived normal ring of $(\mathfrak{o}_1)_{\mathfrak{p}_i}$ is a finite $(\mathfrak{o}_1)_{\mathfrak{p}_i}$-module and coincides with the quotient ring $\mathfrak{o}'_{S_i}$ of $\mathfrak{o}'$ with respect to $S_i = \mathfrak{o}_1 - \mathfrak{p}_i$, for each i ($1 \leqq i \leqq s$). Hence we can find the elements $c'_1, \cdots, c'_t$ of $\mathfrak{o}'$ such that $(\mathfrak{o}_1)_{\mathfrak{p}_i}[c'_1, \cdots, c'_t]$ is a normal ring for *every* i. Set $\mathfrak{o}_2 = \mathfrak{o}_1[c'_1, \cdots, c'_t]$. Then we can prove the following assertion as in [**1**].

For any ring $\mathfrak{S}$ such tnat $\mathfrak{o}_2 \subseteqq \mathfrak{S} \subseteqq \mathfrak{o}'$ and for any prime ideal $\mathfrak{p}$ of height 1 *in $\mathfrak{S}$, the ring $\mathfrak{S}_{\mathfrak{p}}$ is a normal ring.*

For, if $a \notin \mathfrak{p}$, the assertion is obvious. If $a \in \mathfrak{p}$, the ideal $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{o}_1$ is a prime divisor of $a\mathfrak{o}_1$. (Cf. M. Nagata [**3**], Lemma 3.) Hence, $(\mathfrak{o}_1)_{\mathfrak{p}'}[c'_1, \cdots, c'_t]$ is normal and consequently $\mathfrak{S} \subseteqq \mathfrak{o}' \subseteqq \mathfrak{S}_{\mathfrak{p}}$. Therefore $\mathfrak{S}_{\mathfrak{p}}$ is a normal ring.

Now our theorem follows from the above facts in exactly the same way as the proof of Theorem 1.3. in [**1**].

If we define the notions of *function fields* over a ground ring I, *models* of a function field and *normal models* in a similar way as in [**1**], we obtain the following assertion directly from Theorem 3.

COROLLARY. *Let M be a model of a function field L over a ground ring I. If L is separably generated over I, then there exists the derived normal* *model of M.*

REMARK. A ground ring I need not be normal if it is a semi-local ring. More precisely, *if I is an analytically unramified semi-local integral domain and if we name it a "ground ring", then all the results in this section hold.* For, the validity of Theorem 2 is obvious and that of Theorem 3 is easily seen from the facts that the derived normal ring I' of I is a finite I-module and I' is a ground ring in the sense of our definition at the beginning of this section.

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Bibliography

[1] M. Nagata, A general theory of algebraic geometry over Dedekind domains I, Amer. J. Math., **78** (1956), 78-116.
[2] M. Nagata, A general theory of algebraic geometry over Dedekind domains II, Amer. J. Math., **80** (1958), 382-420.
[3] M. Nagata, On the derived normal rings of Noetherian integral domains, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **29** (1955), 293-303.
[4] M. Nagata, Local Rings, Interscience Publishers, New York, 1962.
[5] L. J. Ratliff, Jr., Affine rings over rank two regular local rings, Amer. J. Math., **83** (1961), 709-722.
[6] D. Rees, A note on analytically unramified local rings, J. London Math. Soc., **36** (1961), 24-28.