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On pseudo-conformal transformations of hypersurfaces

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Given a real differentiable hypersurface $S_i, i=1,2$, of a complex manifold M_i, we say that a mapping f of S_1 into S_2 is *pseudo-conformal* if f extends to a holomorphic mapping of a neighborhood of S_1 in M_1 into that of S_2 in M_2. S_1 is called *pseudo-conformally equivalent* to S_2 by f if moreover f is bijective and f^{-1} is also pseudo-conformal. In this paper we shall consider pseudo-conformal transformations of a compact hypersurface S, which is by definition pseudo-conformally equivalent to itself by these transformations. The set of all the pseudo-conformal transformations of S forms a group, which becomes, with the natural topology, a Lie transformation group under some hypothesis (cf. Theorem 5 and Corollary in [**12**]), for instance, in the situation of Theorem 1 below (of course, without the assumption for G to be a Lie transformation group). Our aim is to classify all compact hypersurfaces admitting transitive pseudo-conformal transformation groups. The obtained results are shown in Theorems 1 and 2 (at the beginning of Section 2).

THEOREM 1. *Let S be a compact connected simply connected real analytic hypersurface of $\boldsymbol{C}^n$, the n-dimensional complex cartesian space, $n \neq 3, 7$. If S admits a connected Lie transformation group G of pseudo-conformal transformations which is transitive, then S is pseudo-conformally equivalent to the unit sphere in $\boldsymbol{C}^n$.*

This theorem was proved by E. Cartan [**4**] in the case $n=2$. In case $n=3$ or 7, we can only show that S is equivalent to the unit sphere *or else* to the hypersurface H of the complex manifold $V_n=\{(z_0, z_1, \cdots, z_n) \in \boldsymbol{C}^{n+1} \mid \sum_k (z_k)^2=1\}$, H consisting of the points with $\sum_k$ (the imaginary part of $z_k)^2$ $=$ constant >0. (V_3 is holomorphically equivalent to the group manifold $SL(2, \boldsymbol{C})$.) If a neighborhood of any compact set of V_n, $n=3$ or 7, can be imbedded into $\boldsymbol{C}^n$ (as a domain), then Theorem 1 will be false for this n, the converse being also true.

In Section 1, we shall give two examples (Propositions 1 and 2). The first shows that the converse of Theorem 1 is true. For the second, we shall give a "natural" complex structure to the tangent bundle space M of an arbitrary compact simply connected Riemannian symmetric space B of rank 1, namely the sphere, the complex, quaternionic or Cayley projective space (or plane). By means of Matsushima-Morimoto's theorem [**9**], we shall be able to prove that M is then a Stein manifold with a compact holomorphic (Lie) transformation group having a compact hypersurface S as an orbit. S is differentiably equivalent to the tangent sphere bundle of B. S will thus admit a compact, transitive, pseudo-conformal transformation group. This is the second example. (If B is the n-dimensional sphere, then the manifold given in this example is holomorphically equivalent to V_n mentioned above.) Section 2 will be devoted to the proof of Theorem 2, which, roughly speaking, states that a compact simply connected real hypersurface S in a Stein manifold M admitting a transitive pseudo-conformal transformation group G is necessarily one of the spaces mentioned above provided that the transformations in G extend to those of the whole space M. The demonstration of Theorem 2 is based on a theorem concerning compact Lie transformation groups [**11**] and a result about the homology groups of Stein manifolds. In Section 3, we shall prove Theorem 1. The first step is to show that Theorem 2 can be applied; S is contained in a bounded domain D, which will turn out to be a Stein manifold owing to the solution of Levi's problem, and the pseudo-conformal transformations of S extend to holomorphic transformations of D. The second step is to find the condition for D, which is differentiably equivalent to the tangent bundle of the space B mentioned above to be differentiably imbedded in the euclidean space $\boldsymbol{R}^{2n}$ of the same dimension, with the use of algebraic topology, especially concerning the Pontrjagin classes, and of the differential topology recently developped. We shall find that D is imbedded differentiably into $\boldsymbol{R}^{2n}$ if and only if B is the sphere of dimension 3 or 7.

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1. Examples

PROPOSITION 1. *Let S denote the hypersurface in $\boldsymbol{C}^n$ which is the boundary of the domain $D=\{(z_1,\cdots,z_n)\in\boldsymbol{C}^n \mid \sum_k |z_k|^2<1\}$. Then there exists an isomorphism from the pseudo-conformal transformation group G of S onto the holomorphic transformation group $A(D)$ of D: in particular, G is transitive.*

PROOF. Given an element f in G, f extends to a holomorphic mapping f' of a neighborhood of $\bar{D}=S\cup D$ into $\boldsymbol{C}^n$ by the classical theorem of Hartogs and Osgood. Restricted to $\bar{D}$, the extension f' is unique. The inverse f^{-1} extends also uniquely to a holomorphic mapping $(f^{-1})'$. We can see that $f'(D) \subset D$. In fact, since the Jacobian g of the transformation f is a holomorphic function defined on a neighborhood U of S and g does not vanish on a neighborbood V of S, the holomorphic functions g and $1/g$ extend to holomorphic functions $\tilde{g}$ and h on a neighborhood of $\bar{D}$. Clearly one has $h \cdot \tilde{g} = 1$ on V and so on $\bar{D}$. Hence $\tilde{g}$ does not vanish on $\bar{D}$. Since the Jacobian of f' coincides with $\tilde{g}$ on $\bar{D}$, f' is a local homeomorphism. If $f'(D) \not\subset D$, there would exist a point $p \in D$ such that $f(p)$ is on the boundary of $f'(D)$, which contradicts to the local homeomorphism of f'. Hence $f' \circ (f^{-1})'$ and $(f^{-1})' \circ f'$ are defined on D. It follows that $f' \circ (f^{-1})'$ and $(f^{-1})' \circ f'$, which are extensions of the identity $= f \circ f^{-1} = f^{-1} \circ f$, coincide with the identity mapping on a neighborhood of D, or in other words $(f^{-1})'$ is the inverse of f'. Therefore f' is a homeomorphism leaving D invariant. We thus obtain an isomorphism α of G into $A(D)$ by assigning to f the restriction of f' to D. It remains to show that α is surjective. The domain D with the group $A(D)$ is a bounded symmetric domain. Hence D is imbedded into the compact form, the complex projective space, so that any element F of $A(D)$ extends to a holomorphic (projective) transformation of the complex projective space. This implies, in particular, that F extends to a holomorphic homeomorphism of a neighborhood of $\bar{D}$ into another one leaving S invariant. So F belongs to the image $\alpha(G)$, and the proposition is proved.

To give another example, we consider a compact, simply connected, symmetric space, B, of rank 1. B is a simply connected homogeneous manifold K/L of a compact connected Lie group K, characterized by the property that the isotropy subgroup L operates on the tangent space $T_o(B)$ to B at the point $o, L(o) = o$, *s-irreducibly*, where an orthogonal representation $\lambda : L \to O(m)$ of a group L is called s-irreducible when $\lambda(L)$ is transitive on the unit sphere in $\boldsymbol{R}^m$. We identify an arbitrary transformation of B with its differential, and we take K as a transformation group of the tangent bundle $T(B)$ of B in this way. Since K is compact, B admits a K-invariant Riemannian metric, which is unique up to a constant multiplier. Let $S(B, c), c \geqq 0$, denote the set of the tangent vectors to B of constant length c with respect to that metric. K operating on $T(B)$, each K-orbit is one of $S(B, c)$ for some c. As differentiable manifolds, $S(B, 0)$ is B and $S(B, c), c > 0$, is the tangent sphere bundle of B.

PROPOSITION 2. *With the conventions given above, the tangent bundle $M = T(B)$ of a compact simply connected symmetric space $B = K/L$ of rank 1 admits a complex structure J which is invariant under K, with respect to which M is a Stein manifold, and K is a transitive pseudo-conformal transformation* *group of the tangent sphere bundle* $S(B, c), c>0$.

It will be convenient to investigate the special $B=SO(n+1)/SO(n)=$ the sphere, $n>1$, before the proof. We identify B with $\{X\in \boldsymbol{R}^{n+1} \mid$ the inner product $\langle X, X\rangle=1\}$ and $M=T(B)$ with $\{(X, Y)\in \boldsymbol{R}^{n+1}\times \boldsymbol{R}^{n+1} \mid \langle X, X\rangle=1, \langle X, Y\rangle=0\}$. Let V_n be the complex submanifold $\{(z_1, \cdots, z_{n+1})\in \boldsymbol{C}^{n+1} \mid \sum_k (z_k)^2 =1\}=\{W+\sqrt{-1}\,Y \mid W, Y\in \boldsymbol{R}^{n+1}, \langle W, W\rangle-\langle Y, Y\rangle=1, \langle W, Y\rangle=0\}$ of $\boldsymbol{C}^{n+1}$. The complex orthogonal group $O(n+1, \boldsymbol{C})$ operating on $\boldsymbol{C}^{n+1}$ has V_n as an orbit, and V_n is a complex homogeneous manifold $O(n+1, \boldsymbol{C})/O(n, \boldsymbol{C})$. V_n is a Stein manifold, since V_n is a closed complex submanifold of $\boldsymbol{C}^{n+1}$. If δ denotes the mapping of M onto V_n defined by $\delta(X, Y)=(1+\langle Y, Y\rangle)^{1/2}X+\sqrt{-1}\,Y$, then δ is not only a diffeomorphism but also an equivariant mapping[1] as regards the transformation group $K=SO(n+1)$, the maximal compact subgroup of $O(n+1, \boldsymbol{C})$; i.e. by δ the operation of K on M is carried onto that of K on V_n and K becomes a subgroup of the transformation group of V_n. Identifying M with V_n by δ, we are lead to the conclusion: $K=SO(n+1)$ is a holomorphic transformation group of a Stein manifold M, having the hypersurface $S(B, c), c>0$, as an orbit.

PROOF of PROPOSITION 2. First we will verify.

LEMMA 1. *K^C [resp. L^C] denoting the complex form of K [resp. L], the complex homogeneous manifold $M=K^C/L^C$ is diffeomorphic with $T(B), B=K/L$ defined in Proposition 2, and the diffeomorphism is equivariant with respect to the transformation group K.*

K is the maximal compact subgroup of the complex Lie group K^C. Since L is the identity component of $L^C\cap K, B$ is the universal covering manifold of the K-orbit $B'=K/(L^C\cap K)$ in M. Except in the already investigated case $B=$ the sphere, B' is, however, known to be simply connected, on account of the fact that the isotropy subgroup $L^C\cap K\supset L$ is s-irreducible on the tangent space $\mathfrak{b}$ to B' at the point left fixed by it. Therefore B' is diffeomorphic with B, and we have $L^C\cap K=L$; we, identifying B' with B, consider B as a submanifold of M. L is s-irreducible on $\mathfrak{b}$ and so on $J\mathfrak{b}$, where J is the given (integrable) almost complex structure of M. Some neighborhood U of B in the normal bundle $N(B)=K\times_L J\mathfrak{b}$ is naturally imbedded in M with a K-invariant Riemannian metric in such a way that U is K-invariant. The K-orbits $\neq B$ in U are hypersurfaces. (See [**11**] for the details.) Thus we can apply the following Lemma to U (hence M).

LEMMA 2. *Assume that a compact connected Lie group K is a Lie transformation group of a connected paracompact non-compact manifold M. If there exists a K-orbit which is a hypersurface, then there exists an equivariant diffeo-*

1) That is to say, δ naturally gives rise to an *isomorphism* of the transformation groups.

morphism of M onto the normal bundle $N(B)$ of some K-orbit $B=K/L$, the operation of K on $N(B)$ being naturally defined on the homogeneous vector bundle $N(B)$ ([**11**]).

In our case the K-invariant almost complex structure $J: T(M)\rightarrow T(M)$ gives rise to a K-equivariant bundle-isomorphism of $T(B)$ onto $N(B)$. Lemma 1 is thereby proved.

K^C/L^C is a Stein manifold by Matsushima's theorem [**9**], on which K operates as a holomorphic transformation group having compact simply-connected hypersurfaces $S(B, c), c>0$, as orbits, in view of the above arguments. The Proof of Proposition 2 is completed.

REMARK. To find the imbedding of K/L into K^C/L^C, we used the assumption on S to be simply connected. But this is not necessary by an unpublished result of Iwahori and Sugiura, stating that any connected homogeneous space $K/L, K$ compact, is naturally imbedded in K^C/L^C.

2. Hypersurfaces in Stein manifolds

THEOREM 2. *Let G be a connected Lie transformation group of holomorphic transformations of a Stein manifold M. If G leaves invariant a compact connected simply connected hypersurface S and G is transitive on S, then S is pseudo-conformally equivalent either to the unit sphere in $\boldsymbol{C}^n$ (see Proposition* 1) *or to a tangent sphere bundle $S(B, c), c>0$, of a compact simply connected symmetric space B of rank* 1 (*see Proposition* 2). (*In the latter case M is differentiably the tangent bundle of B.*)

This section is devoted to the proof of this theorem. Since S is simply connected, the maximal compact subgroup K of G is transitive on S by the well known theorem of Montgomery. We can assume that M is connected. Since M is a Stein manifold, M is paracompact but not compact. Hence Lemma 2 applies and gives that M is differentiable and equivariantly hemeomorphic with the normal bundle $N(B)$ of some orbit $B=K/L$, K naturally operating on $N(B)$.

LEMMA 3. *If the set B reduces to a point, then S is pseudo-conformally equivalent to the unit sphere in $\boldsymbol{C}^n, n=\dim_C M$.*

PROOF. $N(B)$ is the tangent space to M at *the point* B, and K is an orthogonal group on $N(B)$, leaving the complex structure $J(B)$ of the vector space $N(B)$, where $J(B)$ is the value taken at B of the almost complex structure J of M. Thus $N(B)$ can be identified with $\boldsymbol{C}^n$ on which K operates as a unitary group $\subset U(n)$. And there exists a diffeomorphism α of $\boldsymbol{C}^n$ onto M which is equivariant with respect to K. Each element k of K is the differential of $\alpha k\alpha^{-1}$ restricted to the tangent sapce $N(B)=\boldsymbol{C}^n$ at B to M. We consider the Lie algebra $\mathfrak{k}$ of K as a set of vector fields on $\boldsymbol{C}^n$ in the usual way. α induces an isomorphism α' of $\mathfrak{k}$ onto the Lie algebra $\alpha'(\mathfrak{k})$ consisting of the infinitesimal transformations corresponding to K as a transformation group of M. We extend α' to the linear mapping α'' of $\mathfrak{k}^c=\{u+iv \mid u,v\in\mathfrak{k}\}$, $i=\sqrt{-1}=J(B)$, onto the vector space $\{\alpha'(u)+J\alpha'(v) \mid u,v\in\mathfrak{k}\}$ by setting $\alpha''(iv)=Jv$. $\alpha''(\mathfrak{k}^c)$ consists of holomorphic vector fields on M. $\alpha''(\mathfrak{k}^c)$ is moreover a Lie algebra. α'' is a homomorphism. α'' is shown to be injective. In fact, if $\alpha'(u)+J\alpha'(v)$ vanishes identically on M, then the infinitesimal transformations $-\alpha'(u)$ and $J\alpha'(v)$ (both of which vanish at B) induce the same infinitesimal linear transformations $-u$ and iv on the tangent space $N(B)=\boldsymbol{C}^n$ to M at B. But, since K is compact, u,v are skew-hermitian matrices and this would imply $-u=iv=0$ and that α'' is injective. $\mathfrak{k}^c$ generates a subgroup of special linear group $SL(n,\boldsymbol{C})$. The subgroup is transitive on $\boldsymbol{C}^n-\alpha^{-1}(B)$, and $\alpha''(\mathfrak{k}^c)$ is locally transitive on $M-B$. For the proof of Lemma 3, we have to show that, for any point $p\neq B$, the isotropy subalgebra $\alpha''(\mathfrak{k}^c)_p$ (i.e. the subalgebra formed by all the vector fields vanishing at p) of $\alpha''(\mathfrak{k}^c)$ "essentially" coincides with the isotropy subalgebra $\mathfrak{k}^c_q$ at a point of the unit sphere on $\boldsymbol{C}^n$. But we shall prove a stronger statement:

(2.0) *Under the hypothesis of Lemma* 3, *there exists a holomorphic homeomorphism of M into $\boldsymbol{C}^n$ which carries S onto the unit sphere.*

First we note that K contains either the special unitary group $SU(n)$ or the symplectic group $Sp(m)$ (in case $n=2m$) among the known compact Lie transformation groups transitive on the $(2n-1)$-dimensional sphere (see A. Borel, C.R. Paris, **230** (1950), 1378-1380), simply due to the condition that the elements of K commute with $i=J(B)$). For the proof of (2.0) (hence Lemma 3), we can assume that K coincides with $SU(n)$ or $Sp(m)$. $\alpha''(\mathfrak{k}^c)_p, p\neq B$, is a complex subalgebra of $\alpha''(\mathfrak{k}^c)$, $\alpha''(\mathfrak{k}^c)_p$ contains the isotropy subalgebra $\alpha''(\mathfrak{k})_p=\alpha'(\mathfrak{k})_p$, and its complex dimension equals $\dim_c(\mathfrak{k}^c)-n$. In case $K=SU(n)$ or $Sp(m)$, it is an elementary matter to see that the normalizer of $\alpha''(\mathfrak{k}^c)_p, p\neq B$, in $\alpha''(\mathfrak{k}^c)$ has complex dimension greater than $\alpha''(\mathfrak{k}^c)_p$ by just one. Since M is simply connected, it follows[2] that there exists a vector field w $(\neq 0)$ on M which commutes with any element of $\alpha''(\mathfrak{k}^c)$. By this property, w is a holomorphic vector field, since $\alpha''(\mathfrak{k}^c)$ is locally transitive on $M-B$. $\alpha'(\mathfrak{k})$ and w span a Lie algebra which is locally transitive on $M-B$. To fix the notion, we assume that the sense of w is "inward" at some point of $M-B$. Since

2) In general, let $\mathfrak{g}$ be a locally transitive Lie algebra of vector fields on a simply connected manifold M. The centralizer of $\mathfrak{g}$ in the lie algebra of all vector fields on M is isomorphic with $\mathfrak{n}(\mathfrak{g}_0)/\mathfrak{g}_0$ where $\mathfrak{n}(\mathfrak{g}_0)$ is the normalizer of $\mathfrak{g}_0$ in $\mathfrak{g}$ and $\mathfrak{g}_0$ denotes the totality of the vector fields in $\mathfrak{g}$ which vanish at a point o of M. (compare Proposition 7.1 in T. Nagano, Sci. Papers Coll. Gen. Ed. Univ. Tokyo, **10** (1960), 17-27.)

all the K-orbits $\neq B$ are compact connected two-sided hypersurfaces and w carries K-orbits to K-orbits, this assumption implies that, given any point p of M, $(\exp tw)(p)$ is defined for any non-negative t and converges to B when t tends to the infinity. Now it is evident that, given a point p of S and a point q of the unit sphere in $\boldsymbol{C}^n$, there exists a holomorphic homeomorphism β of $M-B$ into $\boldsymbol{C}^n-\alpha^{-1}(B)$ which is equivariant with respect to K and to the semi-group $\{\exp tw \mid t \geqq 0\}$ and satisfies $\beta(p)=q$. β extends to a holomorphic homeomorphism of M into $\boldsymbol{C}^n$ by the Hartogs-Osgood theorem. (2.0) is thus proved.

To continue the demonstration of Theorem 2, we assume that B does not reduce to a point. B is then a compact connected submanifold of dimension $\geqq 1$. Since M is a Stein manifold, it follows that B is not a complex submanifold. Hence the tangent spaces are not invariant under J. Let $\mathfrak{b}$ be the tangent space to $B=K/L$ at $o=L(o)$. Then $\mathfrak{b}+J\mathfrak{b} \neq \{0\}$ is invariant under L and $J=J(o)$. The normal space $\mathfrak{n}$ to B at o (with respect to some K-invariant Riemannian metric on M) therefore intersects $\mathfrak{b}+J\mathfrak{b}$ non-trivially;

(2.1) *The space* $(\mathfrak{b}+J\mathfrak{b})\cap\mathfrak{n} \neq \{0\}$ *is invariant under* L.

By Lemma 2, B is a deformation retract of M. On the other hand the integral homology group $H_p(M)$ is trivial for $p>n=\dim_C M$ and $H_n(M)$ has no torsion, because M is a Stein manifold (see Andreotti-Frankel [**1**], for instance). Therefore we have $H_p(B)=0$ for $p>n$, and $H_n(B)$ has no torsion. In particular we find

(2.2) $$\dim B \leqq n\,.$$

Since B is simply connected, n is greater than 1. Hence M is not homeomorphic with $\boldsymbol{R}\times B$, $\boldsymbol{R}=$ the line, by (2.2). Hence the following lemma can be used:

LEMMA 4. *Under the hypotheses of Lemma* 2, *assume moreover that* K *does not operate on* M *trivially in the sense that the isotropy subgroups are not all conjugate to each other. Then the structure group* L *of the vector bundle* $N(B)$ *is Lemma* 2 *is s-irreducible on the fiber, therefore real irreducible* ([**11**]).

Together with (2.1), this lemma gives

(2.3) $$(\mathfrak{b}+J\mathfrak{b})\cap\mathfrak{n}=\mathfrak{n}\,.$$

The tangent space $T_o(M)$ to M at $o\in B$ being the direct sum of $\mathfrak{b}$ and $\mathfrak{n}$, it follows from (2.3) that we have $2\dim B=2\dim\mathfrak{b}=\dim\mathfrak{b}+\dim J\mathfrak{b}\geqq\dim(\mathfrak{b}+J\mathfrak{b}) =\dim M=2n$, and hence $\dim B\geqq n$. By (2.2), we thus find that $\dim B=n$ and $\mathfrak{b}\cap J\mathfrak{b}=0$. From Lemma 4, we therefore conclude that

(2.4) *The normal bundle* $N(B)$ *is equivalent to the tangent bundle* $T(B)$, *and the isotropy subgroup of* K *operating on* B *is s-irreducible on the fiber of* $T(B)$.

This implies that B is a compact symmetric space of rank 1. By Lemma

4, the simply connected K-orbit S is an $(n-1)$-sphere bundle over B. Hence B is also simply connected. We consider the Lie algebra $\mathfrak{k}$ of K as a space of vector fields on M. $\{u+Jv \mid u, v \in \mathfrak{k}\}$ is the complexification $\mathfrak{k}^c$ of $\mathfrak{k}$, as is easily seen from the facts that $T_o(M)$ is the direct sum of $T_o(B)$ and $J(T_o(B))$ and that K is effective and transitive on B. Also one finds that the complexification $\mathfrak{l}^c$ of the Lie algebra $\mathfrak{l}$ of L is the isotropy subalgebra of $\mathfrak{k}^c$ at o. It follows that a K-equivariant holomorphic imbedding β_r of $\bigcup_{t<r} S(B, t), r \geqq 0$, into M extends to a K-equivariant holomorphic imbedding of $\bigcup_{t<r+\varepsilon} S(B, t)$ into M for some positive number ε, provided that β_r is not surjective.

Let R be the lowest upper bound of such r's as β_r's are defined. β_R is surjective. We have only to prove this when R is the infinity. Suppose that β_∞ is defined but not surjective. Then K^c/L^c would admit sufficiently many non-constant bounded holomorphic functions, because $\beta_\infty(K^c/L^c)$ is relatively compact in a Stein manifold M. Thus K^c/L^c would admit a Kählerian metric which is invariant under all holomorphic transformations, as is proved by considering the kernel functions for the bounded domains. Thus the isotropy subgroup H of the group A of all the holomorphic transformations of M (at the point left fixed by L^c) would be compact. On the other hand $H (\subset L^c)$ is irreducible on the tangent space at that point. Hence H would be a maximal compact subgroup of A. Therefore K^c/L^c must be homeomorphic with a euclidean space, contrary to the fact that the compact manifold B is a deformation retract of K^c/L^c. We have proved that β_R is surjective. So S is $\beta_R(S(B, c))$ for some c.

REMARK. It may be possible to verify the conclusion of Theorem 2 under the hypothesis on G that G is merely a transitive pseudo-conformal transformation group of S, instead of the one supposed in the theorem that the elements of G are holomorphic transformations of M. For that it will be necessary to show that the theorem of Hartogs and Osgood used for the proof of Theorem 1 (and Proposition 1) is valid for arbitrary Stein manifolds, not only for $\boldsymbol{C}^n$.

3. The proof of Theorem 1

Let S be a compact connected simply connected hypersurface of $\boldsymbol{C}^n$, on which transitively operates a connected Lie transformation group G of pseudo-conformal transformations. By the Jordan-Brouwer theorem, $\boldsymbol{C}^n - S$ has two connected components, one of which, D, is relatively compact. We shall first prove that

(3.1) *D is a Stein manifold.*

Since S is a compact hypersurface of a euclidean space, there exists a point p on S in a neighborhood of which S is convex. We define a Gauss-mapping ν of S into the unit sphere with center $o=(0,\cdots,0)$ by assigning to $x\in S$ the point $\nu(x)$ in such a way that the vector $\overrightarrow{o\nu(x)}$ is parallel to the unit normal vector at x and ν is differentiable. By Sard's theorem the Jacobian is different from zero at a point s sufficiently near to p. The second fundamental form S is definite at s. Hence, at s, S satisfies the Levi-Krzoska condition. Since a pseudo-conformal transformation group K is transitive on S, D is a Krzoska pseudo-convex domain. Hence D is a domain of holomorphy, as was proved by Oka and others (see Grauert [**7**]), and finally a Stein manifold.

As in the proof of Proposition 1, G is isomorphic with a subgroup of the holomorphic transformation group $A(D)$ of D, which is a Lie transformation group (H. Cartan [**5**]). The isomorphism is continuous with respect to the modified compact-open topology (see Gleason-Palais [**6**], for instance), on account of the maximal principle concerning holomorphic functions; in particular the image is a Lie subgroup of $A(D)$ by the Kuranishi-Yamabe theorem [**13**]. Naturally G is considered to be a topological transformation group of the bounded manifold $\bar{D}=D\cup S$. Since S is simply connected, the maximal compact subgroup K of G is transitive on S, and a K-orbit sufficiently near to S is homeomorphic with S by a well known theorem on compact transformation groups (see Borel [**2**], for instance). For the moment we observe this orbit, and denote it by the same S. Thus, by (3.1), Theorem 2 applies; we see that S is pseudo-conformally equivalent to the unit sphere in $\boldsymbol{C}^n$, or else to a tangent sphere bundle of a compact symmetric space of rank 1. Since S is a compact hypersurface of a euclidean space, the Pontrjagin class $p(S)$ must be trivial. In this connection, we shall show:

PROPOSITION 3. *The Pontrjagin class $p(S)$ is not trivial, if S is the tangent sphere bundle of a complex m-dimensional complex projective space, $P^m(\boldsymbol{C}), m>2$, of a quaternionic projective space other than the projective line, or of the Cayley projective plane.* (*The projective lines are homeomorphic with spheres.*)

PROOF. Let $B=K/L$ be one of these symmetric spaces, and D be the tangent bundle of B, with the projection $\pi:D\to B$. If $\iota:B\to D$ denotes the inclusion mapping, then the bundle $\iota^{-1}(T(D))$ induced from $T(D)$ is equivalent to the Whitney sum $T(B)+N(B)$, where $T(X)$ is the tangent bundle of a manifold X. On the other hand, $\kappa:S\to D$ denoting the inclusion mapping, $\kappa^{-1}(T(D))$ is the Whitney sum of $T(S)$ and the trivial line bundle. Therefore we obtain the relation between Pontrjagin classes:

$$p(S)=(\pi\circ\kappa)^*(p(B)^2)\,. \tag{3.2}$$

$\pi\circ\kappa$ is the projection of the sphere bundle S onto the base space B. $p(B)$

has been calculated by Borel and Hirzebruch [3]. Assume $B=P^m(\boldsymbol{C})$, for instance. Then $p(B)=(1+\beta^2)^{m+1}$, where β is the generator of the integral cohomology group $H^2(P^m(\boldsymbol{C}))=\boldsymbol{Z}$. The Gysin sequence applied to the $(2m-1)$-sphere bundle S over $B=P^m(\boldsymbol{C})$:

$$\cdots \to H^{i-1-k}(B)\to H^i(B)\to H^i(S)\to H^{i-k}(B)\to \cdots,\ k=2m-1\,,$$

gives that the projection $\pi\circ\kappa: S\to B$ induces an isomorphism of $H^4(B)$ onto $H^4(S)$, if m is greater than 2. It follows from (3.2) that the first Pontrjagin class $(\pi\circ\kappa)^*(2(m+1)\beta^2)$ of S does not vanish. This argument is valid for the quaternionic projective space and the Cayley projective plane both different from the line, and Proposition 3 is proved in the same way.

PROPOSITION 4. *Let B be an n-dimensional, compact, orientable, differentiable manifold with the properties: 1) $H^1(B)=0$, and 2) the Euler-Poincaré characteristic $\chi(B)$ does not vanish. Then the tangent bundle $T(B)$ of B cannot be differentiably imbedded into the $2n$-dimensional euclidean space.*

PROOF. Assume the contrary: $T(B)$ is differentiably imbedded into the $2n$-dimensional sphere S^{2n}. The imbedded tangent sphere bundle S divides S^{2n} into two connected components D and D'; $S^{2n}=\bar{D}\cup\bar{D}'$, $S=\bar{D}\cap\bar{D}'$. We may assume that D is homeomorphic with $T(B)$. S is a subcomplex of S^{2n} with some triangulation. Due to the Mayer-Vietoris formula, we have

$$H^i(S)=H^i(\bar{D})+H^i(\bar{D}'),\qquad 0<i<2n-1\,. \tag{3.3}$$

Since B is a deformation retract of $\bar{D}$, (3.3) shows

$$H^n(S)=H^n(B)+H^n(\bar{D}')=\boldsymbol{Z}+H^n(\bar{D}')\,. \tag{3.4}$$

Applying Gysin's formula to the sphere bundle S over B, we get the exact sequence:

$$H^0(B)\xrightarrow{\lambda} H^n(B)\to H^n(S)\to H^1(B)=0\,,$$

where, with the identification $H^0(B)=H^n(B)=\boldsymbol{Z}$, λ denotes the multiplication by $\chi(B)$. (Gysin's sequence can actually be applied to S, because the structure group is connected.) Consequently we find $H^n(S)=\boldsymbol{Z} \bmod \chi(B)$, contrary to (3.4).

As a corollary to Proposition 4 just proved, we have

(3.5) *The tangent bundle of $B=P^2(\boldsymbol{C})$ cannot be differentiably imbedded into $\boldsymbol{C}^4$.*

Finally consider the case where B is a sphere, a space of the remaining class of compact, simply connected, symmetric spaces of rank 1.

PROPOSITION 5. *The tangent bundle $T(B)$ of an n-dimensional sphere B can be differentiably imbedded into the $2n$-dimensional euclidean space, if and only if $n=1$, 3, or 7.*

PROOF. Assume that $T(B)$ is imbedded into the $2n$-dimensional euclidean space. First we consider the case where $n>2$. Haefliger's theorem [8] gives that the differentiable imbedding of B into the $2n$-dimensional euclidean space is unique up to a differentiable isotopy; in particular, the differentiable normal bundle D of the imbedded B is trivial. Hence the tangent bundle $T(B)$, equivalent to D, is trivial. It follows from Milnor's result [**10**] that n equals 3 or 7. As regards the case $n=2$ (or, more generally, $n=$even), Proposition 5 is a corollary to Proposition 4. The converse is patent.

Hitherto S has been an orbit near S mentioned in Theorem 1. Hence the demonstration of Theorem 1 is immediate from the above arguments, if the following proposition is proved.

PROPOSITION 6. *Let S be a compact, connected, real analytic hypersurface of $\boldsymbol{C}^n$ and let K be a compact connected Lie transformation group of pseudo-conformal transformations of S, then there exists a bounded domain D' of $\boldsymbol{C}^n$ containing S such that every element of K extends to a holomorphic transformation of D'.*

PROOF. Since S is a real analytic hypersurface, every vector field (on S) which belongs to the Lie algebra $\mathfrak{k}$ of K extends to a holomorphic vector field on a neighborhood of S, according to Tanaka [**12**] (Proposition 1, p. 404). $\mathfrak{k}$ being finite-dimensional, $\mathfrak{k}$ can thus be considered as the Lie algebra of vector fields on a neighborhood of S. Given a neighborhood U of S, there exist neighborhoods V of S and W of the identity of K such that any element f in W extends to a holomorphic homeomorphism of V into U, since S is compact. Using this fact repeatedly, one finds that, for any positive integer m, V and W can be so chosen that any element f in $W^m=\{f_1f_2\cdots f_m \mid f_1,\cdots,f_m\in W\}$ extends to a holomorphic homeomorphism of V into U. Since K is compact and connected, K coincides with W^m for sufficiently large m. Here one may assume V is a domain. Let D' be the set $\bigcup_{f\in K} f(V)$, for any element g in K, g extends uniquely to a holomorphic homeomorphism $gf\circ f^{-1}$ of $f(V)$ into D', and hence every element of K extends to a holomorphic transformation of D'. If the arbitrarily given neighborhood U of the compact space S is a bounded domain, then so is D' and Proposition 6 is proved.

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