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Projective modules over weakly noetherian rings

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Introduction. Let R be a commutative ring with a unit element. Then the space of prime ideals of R with the Zariski topology is called the *prime spectrum* of R which is denoted by spec(R), and the subspace of spec(R) of all maximal ideals of R is called the *maximal spectrum* of R which is denoted by m-spec(R). The dimension of such a space is the supremum of the lengths of chains of irreducible closed subsets. (See [**1**] and [**10**].) For brevity, we shall call a ring *weakly noetherian* if m-spec(R) satisfies the descending chain condition on closed subsets. Our main objective in this paper is to prove

THEOREM. *If R is a weakly noetherian ring and* dim(m-spec(R)) *is finite, then any projective R-module is a direct sum of finitely generated projective R-modules.*

From this we can easily deduce that, over a commutative indecomposable semilocal ring[1], any projective module is free[2].

Now let R be a commutative ring, M an R-module. Then M is called *faithfully flat* if M satisfies any one of the following equivalent conditions (see § 6.4 [5] p. 57):

(a) A sequence of R-modules $N' \to N \to N''$ is exact if and only if $M \underset{R}{\otimes} N' \to M \underset{R}{\otimes} N \to M \underset{R}{\otimes} N''$ is exact.

(b) M is flat and, for any R-module N, the relation $M \underset{R}{\otimes} N = (0)$ implies $N = (0)$.

(c) M is flat and, for any homomorphism $v: N \to N'$ of R-modules, the relation $1_M \otimes v = 0$ implies $v = 0$ where 1_M is the identity automorphism of M.

(d) M is flat and, for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}M \neq M$.

To prove the main theorem, we shall prove that, if R is an indecomposable weakly noetherian ring, any projective module ($\neq (0)$) is faithfully flat.

We shall always be dealing with rings with unit element and unitary modules. Further, unless the contrary is stated, "module" means "left module". Λ denotes a ring (not always commutative) and R denotes a commutative ring.

1) A ring is called indecomposable if it has no non-trivial idempotents. A commutative ring is called semilocal if the number of the maximal ideals is finite.

2) See [7].

1. Some basic lemmas and known results on projective modules.

We begin with a well-known

LEMMA 1.1 (Nakayama). *Let Λ be a ring, J the Jacobson radical of Λ, M a finitely generated module and M' a submodule of M. If $M'+JM=M, M'=M$.*

Now the following lemma is trivial and well-known.

LEMMA 1.2. *Let L, M, N be modules over a ring such that $L \supset M \supset N$. If N is a direct summand of L, then N is a direct summand of M.*

The following lemma is a generalization of Lemma 5 of [**10**].

LEMMA 1.3 ([**7**]). *Let P be a projective module over a ring Λ and p an element of P. If $p \notin \mathfrak{m}P$ for any maximal right ideal $\mathfrak{m}$ of Λ, then Λp is a direct summand of P and p is a free basis of Λp, where $\mathfrak{m}P$ is the image of $\mathfrak{m} \otimes_R P \rightarrow P$ by the natural map.*

LMMA 1.4.[3] *For a projective module $P(\neq (0))$ over a ring Λ, we have $JP \neq P$, where J is the Jacobson radical of Λ.*

PROPOSITION 1.5 (Eilenberg [**9**]). *Let P be a projective module over Λ. Then there exists a free module F such that $F \oplus P$ is free.*

REMARK. In this proposition, if R is a polynomial ring over a field and P is finitely generated, we may take a finitely generated free module as an F (Proposition 10 of [**10**]).

As a corollary we have

LEMMA 1.6. *Let Λ be a ring, P a projective module over Λ and p any element of P. Then there exists an integer $m(\geqq 0)$ such that $(\sum_{i=1}^{m} \oplus \Lambda_i) \oplus P, (\Lambda_i \cong \Lambda)$, contains a finitely generated free direct summand containg p.*

PROOF. By virtue of Lemma 1.5, there exist free modules U, F such that $U=F\oplus P$. Let

$\{u_i\}$ be a free basis of U,

$\{f_i\}$ a free basis of F,

π the projection from U to F,

(i.e., if $u \in U$, $u=f+p'$, $f\in F$, $p' \in P$, then $\pi(u)=f$),

$$p=\sum_{i=1}^{n} r_i u_i,\ r_i \in \Lambda,\ i=1, \cdots, n,$$

$$\pi(u_i)=\sum_{j=1}^{m_i} s_{ij} f_j,\ s_{ij} \in \Lambda, i=1,2,\cdots, j=1,\cdots, m_i,$$

$$m=\max(m_1, \cdots, m_n).$$

Put $F'=\sum_{i=1}^{m} \oplus Rf_i$, $P'=F'\oplus P$, $U'=\sum_{i=1}^{n} \oplus Ru_i$. Then $p \in U' \subset P' \subset U$ and U'

3) This is proposition 2.7 of [H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc., **95** (1960), 466–488]. simple proof is found in [**7**].

is a direct summand of U, hence of P' by Lemma 1.2. This completes the proof.

THEOREM 1.7 (Kaplansky [8]). *Let Λ be a ring, M a Λ-module which is a direct sum of (any number of) countably generated Λ-modules. Then any direct summand of M is likewise a direct sum of countably generated Λ-modules.*

From this we deduce easily

COROLLARY 1.8 (Kaplansky [8]). *Any projective module over a ring is a direct sum of countably generated projective modules.*

LEMMA 1.9 (Kaplansky [8]). *Let Λ be any ring, M a countably generated Λ-module. Assume that any direct summand N of M has the following property: any element of N can be embedded in a free (resp. finitely generated) direct summand of N. Then M is free (resp. a direct sum of finitely generated modules).*

2. Support of a module.

Let R be a commutative ring, S a multiplicatively closed set not containing 0 of R. As usual we denote by R_S the ring of quotient with respect to S, and if $S=R-\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R, we write $R_{\mathfrak{p}}$ for $R_{R-\mathfrak{p}}$. Similarly, for an R-module M, we denote by M_S the module of quotient with respect to S and we write $M_{\mathfrak{p}}$ for $M_{R-\mathfrak{p}}$, if $\mathfrak{p}$ is a prime ideal. We know that $M_S=M\underset{R}{\otimes}R_S$ and that there exists a canonical map $\varphi: M\to M_S$ and the kernel of this map is the S-component of (0) in M: $\operatorname{Ker}\varphi=\{m\in M \mid \text{there exists } s\in S \text{ such that } sm=0\}$.

Let M, M' be R-modules, such that $M\supset M'$. Then we use the following notation:

$(M':M)=$ the set of elements $x\in R$ such that $xM\subset M'$.

Let M be an R-module. Then the set of all maximal ideals $\mathfrak{m}$ of R such that $M_{\mathfrak{m}}\neq(0)$ is called the support of M and denoted by $\operatorname{Supp}(M)$.

LEMMA 2.1 (§ 1.7 of [5]). *Let M be an R-module. Then $M=(0)$ if and only if* $\operatorname{Supp}(M)=\phi$.

For: if $M_{\mathfrak{m}}=(0)$ for every maximal ideal $\mathfrak{m}$ of R, $(0:m)$ is contained in no maximal ideals of R, for any $m\in M$, hence $(0:m)=R$, i. e., $m=0$.

LEMMA 2.2 (§ 1.7 of [5]). *If an R-module M is a sum of a family of submodules $\{M_\lambda\}$, we have* $\operatorname{Supp}(M)=\bigcup_\lambda \operatorname{Supp}(M_\lambda)$.

LEMMA 2.3 (§ 1.7 of [5]). *If M is a finitely generated R-module,* $\operatorname{Supp}(M)$ *is the set of maximal ideals containing* $((0):M)$.

LEMMA 2.4. *Let R be a commutative ring, $\mathfrak{m}$ a maximal ideal of R, P a projective module and M a submodule of P which is contained in a finitely generated submodule M' of P. Then we have $M+\mathfrak{m}P=P$ if and only if $(P/M)_{\mathfrak{m}}=(0)$, i. e., $\mathfrak{m}\notin\operatorname{Supp}(P/M)$.*

PROOF. Let N be any module. Then we have

$$N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}}=(N\otimes_R R_{\mathfrak{m}})\otimes_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})=N\otimes_R(R/\mathfrak{m})=N/\mathfrak{m}N\,.$$

Therefore, we have

$$\begin{aligned}M+\mathfrak{m}P=P &\Leftrightarrow P/\mathfrak{m}P=(M+\mathfrak{m}P)/\mathfrak{m}P\\ &\Leftrightarrow P_{\mathfrak{m}}/\mathfrak{m}P_{\mathfrak{m}}=(M_{\mathfrak{m}}+\mathfrak{m}P_{\mathfrak{m}})/\mathfrak{m}P_{\mathfrak{m}}\\ &\Leftrightarrow M_{\mathfrak{m}}+\mathfrak{m}P_{\mathfrak{m}}=P_{\mathfrak{m}}\Rightarrow M'_{\mathfrak{m}}+\mathfrak{m}P_{\mathfrak{m}}=P_{\mathfrak{m}}\,.\end{aligned}$$

We shall prove that

$$M_{\mathfrak{m}}+\mathfrak{m}P_{\mathfrak{m}}=P_{\mathfrak{m}}\Leftrightarrow (P/M)_{\mathfrak{m}}=(0)\,.$$

Now $P_{\mathfrak{m}}$ is a projective module over a local ring $R_{\mathfrak{m}}$, hence $P_{\mathfrak{m}}$ is free. Let $\{u_i\}$ be a free basis for $P_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$. Then there exists an integer n such that $M'_{\mathfrak{m}}\subset\sum_{i=1}^{n}\oplus R_{\mathfrak{m}}u_i$. Put $\sum_{i=1}^{n}\oplus R_{\mathfrak{m}}u_i=P'$. Then P' is a direct summand of $P_{\mathfrak{m}}$, hence there exists a submodule P'' such that $P_{\mathfrak{m}}=P'\oplus P''$. The above relation $M'_{\mathfrak{m}}+\mathfrak{m}P_{\mathfrak{m}}=P_{\mathfrak{m}}$ implies that $P'\oplus P''=P_{\mathfrak{m}}=P'+\mathfrak{m}P_{\mathfrak{m}}=P'\oplus\mathfrak{m}P''$. Therefore, we have $\mathfrak{m}P''=P''$. By Lemma 1.4 we have $P''=(0)$. Therefore $P_{\mathfrak{m}}=P'$ is finitely generated over $R_{\mathfrak{m}}$. By Lemma 1.1 we have $(P/M)_{\mathfrak{m}}=P_{\mathfrak{m}}/M_{\mathfrak{m}}=(0)$. The converse implication is obvious. Thus we have completed the proof.

3. Maximal spectrum with the Zariski topology.

To a commutative ring R, we associate a topological space m-spec(R) (maximal spectrum of R): m-spec(R) is the set of maximal ideals in R with the Zariski topology.

We shall constantly use the following notations:

$X=m\text{-spec}(R)$,

$V(\mathfrak{a})=$ the set of elements $\mathfrak{x}\in X$ such that $\mathfrak{x}\supset\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R,

$D(\mathfrak{a})=X-V(\mathfrak{a})$,

$\mathfrak{c}(\mathfrak{x})=$ the set of elements $c\in R$ such that there exists an element $s\in R$ satisfying $s\notin\mathfrak{x}$, $sc=0$ where $\mathfrak{x}$ is an element of X,

$\mathfrak{c}(\mathfrak{X})=\bigcap_{\mathfrak{x}\in\mathfrak{X}}\mathfrak{c}(\mathfrak{x})$ where $\mathfrak{X}$ is a subset of X.

LEMMA 3.1 (cf. § 1.1, p. 80 [5]). *We have the following properties:*

i) $V((0))=X$, $V(R)=\phi$.

ii) *The relation* $\mathfrak{a}\subset\mathfrak{b}$ *implies* $V(\mathfrak{a})\supset V(\mathfrak{b})$.

iii) *For any family of ideals* $\{\mathfrak{a}_\lambda\}$ *of* R, $V(\bigcup_\lambda\mathfrak{a}_\lambda)=\bigcap_\lambda V(\mathfrak{a}_\lambda)$.

iv) $\mathfrak{c}(\mathfrak{x})$ *is an ideal of* R *and* $\mathfrak{x}\supset\mathfrak{c}(\mathfrak{x})$.

v) $\mathfrak{X} \subset V(\bigcap_{\mathfrak{x} \in \mathfrak{X}} \mathfrak{x}) \subset V(\mathfrak{c}(\mathfrak{X}))$.

vi) *The sets of the form* $V(\mathfrak{a})$, $\mathfrak{a}$ *ranging over the set of ideals of* R, *are the closed sets with respect to the Zariski topology of* X.

A closed set $\mathfrak{F}$ of X is called irreducible if it is not empty and is not a finite union of proper closed subsets. We define $ht(\mathfrak{F})$ for a closed set $\mathfrak{F}$ of X as follows: if $\mathfrak{F}$ is irreducible, $ht(\mathfrak{F})$ is the (possibly infinite) supremum of all integers n for which there is a strictly increasing chain

$$\mathfrak{F} = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_n$$

of irreducible closed sets $\mathfrak{F}_i$ of X; in general $ht(\mathfrak{F})$ is the infimum of the heights of the irreducible closed subsets of $\mathfrak{F}$ if $\mathfrak{F} \neq \phi$; $ht(\phi) = \infty$. We also write $\dim X = \sup ht(\mathfrak{F})$, where $\mathfrak{F}$ ranges over the non-empty closed sets in X. If every closed set $\mathfrak{F}$ in X is a finite union of irreducible closed sets, $\mathfrak{F} = \bigcup_i \mathfrak{F}_i$, we call X a decomposition space. We see easily that there is then such a decomposition, unique up to order, for which no $\mathfrak{F}_i$ is in the union of the remaining $\mathfrak{F}_j$; the $\mathfrak{F}_i$ in this decomposition are called the components of $\mathfrak{F}$. It is well known, and elementary, that a noetherian space is a decomposition space.

We know that a closed set $\mathfrak{F}$ is irreducible if and only if $\bigcap_{\mathfrak{f} \in \mathfrak{F}} \mathfrak{f}$ is a prime ideal and that, if an irreducible set $\mathfrak{F}$ is contained in a union of closed sets $\mathfrak{G} \cup \mathfrak{H}$, $\mathfrak{F}$ is contained in $\mathfrak{G}$ or in $\mathfrak{H}$. (Cf. [**1**] and [**5**]).

LEMMA 3.2. *Let* R *be a commutative ring. Then* $X = m\text{-spec}(R)$ *is a noetherian space if and only if, for any closed set* $\mathfrak{F} = V(\mathfrak{b})$, *there is a finitely generated ideal* $\mathfrak{a} \subset \mathfrak{b}$ *of* R *such that* $\mathfrak{F} = V(\mathfrak{a})$.

PROOF. If a_1 is any element of $\mathfrak{b}$, we have $(a_1) \subset \mathfrak{b}$, hence $V((a_1)) \supset V(\mathfrak{b})$. If $V((a_1)) \neq V(\mathfrak{b})$, there is an element $\mathfrak{x} \in X$ such that $\mathfrak{x} \in V((a_1))$, $\notin V(\mathfrak{b})$, i.e., $\mathfrak{x} \supset (a_1)$, $\mathfrak{x} \not\supset \mathfrak{b}$. Let a_2 be any element of $\mathfrak{b}$ such that $\mathfrak{x} \not\ni a_2$. Then we have $(a_1) \subset (a_1, a_2) \subset \mathfrak{b}$, $V((a_1)) \supsetneqq V((a_1, a_2)) \supset V(\mathfrak{b})$. In this way, we have a descending sequence of closed sets $V((a_1)) \supsetneqq V((a_1, a_2)) \supsetneqq \cdots \supset V(\mathfrak{b})$. If X is noetherian, there exists an integer n such that $V((a_1, \cdots, a_n)) = V(\mathfrak{b})$.

Conversely, let $\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \cdots \supset \mathfrak{F}_n \supset \cdots$ be a descending chain of closed sets of X. Let $\mathfrak{a}_i$ be finitely generated ideals such that $V(\mathfrak{a}_i) = \mathfrak{F}_i$. Put $\mathfrak{b}_i = \mathfrak{a}_1 + \cdots + \mathfrak{a}_i$. We have that $V(\mathfrak{b}_i) = \mathfrak{F}_i$ and that $\mathfrak{F} = \cap \mathfrak{F}_i = V(\cup \mathfrak{b}_i)$. By assumption there is a finitely generated ideal $\mathfrak{b}$ such that $\mathfrak{F} = V(\mathfrak{b})$, $\mathfrak{b} \subset \cup \mathfrak{b}_i$. Thus $\mathfrak{b} \subset \bigcup_{i=1}^{n} \mathfrak{b}_i$ for a suitable n, hence $\mathfrak{F} = V(\mathfrak{b}) = V(\bigcup_{i=1}^{n} \mathfrak{b}_i) = \bigcap_{i=1}^{n} \mathfrak{F}_i = \mathfrak{F}_n$. This completes the proof.

The following Lemma 3.3 is Lemma 4 of [**10**] and Lemma 3.1 of [**1**].

LEMMA 3.3 (Chinese Remainder Theorem). *If* M *is an* R*-module,* $\mathfrak{x}_i$ *distinct elements of* X, *and* $m_i \in M$, $i = 1, \cdots, n$, *then there is an element* $m \in M$

for which $m \equiv m_i$ (mod. $\mathfrak{x}_i M$).

Let P be an R-module and $p_1, \cdots, p_n$ elements of P. If $p_1, \cdots, p_n$ are linearly independent mod. $\mathfrak{x}P$ over $R/\mathfrak{x}$, (i. e., $p_1, \cdots, p_n$ are a free basis mod. $\mathfrak{x}P$ over $R/\mathfrak{x}$ of a direct summand of $P/\mathfrak{x}P$), we say $p_1, \cdots, p_n$ are *free* (or a *free basis*) at $\mathfrak{x}$ in P. The set $\mathfrak{X}$ of all $\mathfrak{x} \in X$ at which $p_1, \cdots, p_n$ are not a free basis is called the *singular set* of $p_1, \cdots, p_n$ in P.

Now the following lemma is Lemma 3 in [**10**], and the proof is the same as in [**10**].

LEMMA 3.4. *Let $p_1, \cdots, p_n$ be elements of a projective R-module P. Then the singular set of $p_1, \cdots, p_n$ is closed in X.*

PROOF. Let F be a free R-module such that $F = P \oplus Q$, $\{u_i\}$ a free basis of F. Assume that

$$p_i = \sum_{j=1}^{m} s_{ij} u_j, \quad i = 1, \cdots, n.$$

Then $p_1, \cdots, p_n$ are free at $\mathfrak{x}$ if and only if the rank of the $n \times m$ matrix $(s_{ij}) = S$ is n mod. $\mathfrak{x}$. Let $S_1, \cdots, S_t$ be the set of $n \times n$ minors of S. Then the singular set is equal to $\bigcap_{i=1}^{t} V((\det(S_i))) = V((\det(S_1), \cdots, \det(S_t)))$. Therefore, the singular set is closed.

4. Faithfully flat modules.

Let R be an indecomposable commutative ring. Then we know that any finitely generated projective module is faithfully flat (cf. Lemma 4.2 of [**4.**]) and that, if R is an integral domain, any projective module is faithfully flat.

Now we recall the

DEFINITION. A commutative ring R is called *weakly noetherian* if the maximal spectrum $X = m\text{-spec}(R)$ is a noetherian space.

The following theorem is a direct consequence of Theorem 5.1 of the next section, but we give another proof here.

THEOREM 4.1. *Let R be a weakly noetherian ring. Then every projective module $P (\neq (0))$ is faithfully flat if and only if the ring R is indecomposable.*

PROOF. Let $F = P \oplus Q$, $\{u_i\}$ a free basis of F, $u_i = p_i + q_i$, $p_i = \sum_{j=1}^{n_i} s_{ij} u_j$, $\mathfrak{a} = (\{s_{ij}\})$ the ideal generated by the set $\{s_{ij}; i = 1, 2, \cdots, j = 1, 2, \cdots, n_i\}$. It is evident that $\mathfrak{a}$ is the smallest ideal such that $\mathfrak{a}P = P$. Put $\mathfrak{M} = \{\mathfrak{m} \in X \mid \mathfrak{m}P = P\}$, $\mathfrak{N} = \{\mathfrak{n} \in X \mid \mathfrak{n}P \neq P\}$. Then we have that $\mathfrak{M} \cap \mathfrak{N} = \phi$, $\mathfrak{M} \cup \mathfrak{N} = X$ and that $\mathfrak{c}(\mathfrak{M}) \cap \mathfrak{c}(\mathfrak{N}) = (0)$. For: $c \in \mathfrak{c}(\mathfrak{M}) \cap \mathfrak{c}(\mathfrak{N})$ implies $(0 : c) \not\subset \mathfrak{x}$ for each maximal ideal $\mathfrak{x}$ of R, hence $(0 : c) = R$, i. e., $c = 0$. Now we have $\mathfrak{c}(\mathfrak{M}) \supset \mathfrak{a}$. For: let s_{ij} be an element of the set of generators $\{s_{ij}\}$ of $\mathfrak{a}$. Then

$p_i=\sum_{k=1}^{n_i} s_{ik}u_k$. If $\mathfrak{m}\in\mathfrak{M}, \mathfrak{m}P=P$. Therefore, $P_{\mathfrak{m}}=(0)$. Hence there exists an element $s_i'\notin\mathfrak{m}$ such that $s_i'p_i=0$. Therefore, $s_i's_{ij}=0$. This implies that $\mathfrak{c}(\mathfrak{M})\supset\mathfrak{a}$. Now we have, if $\mathfrak{n}\in\mathfrak{N}, \mathfrak{n}\not\supset\mathfrak{a}$. For: $\mathfrak{n}\supset\mathfrak{a}$ implies $\mathfrak{n}P\supset\mathfrak{a}P=P$, a contradiction. Therefore, we have $\mathfrak{n}\not\supset\mathfrak{c}(\mathfrak{M})$, for all $\mathfrak{n}\in\mathfrak{N}$.

Thus $\mathfrak{M}$ is a closed set, $\mathfrak{M}=V(\mathfrak{c}(\mathfrak{M}))=V(\mathfrak{a})$. Therefore by Lemma 3.2 there is a finite set of elements $\{a_1,\cdots,a_n\}$ of $\mathfrak{a}$ such that $\mathfrak{M}=V((a_1,\cdots,a_n))$. Then there is an integer m such that $(a_1,\cdots,a_n)\subset(\{s_{ij}\,;i=1,\cdots,m,j=1,\cdots,n_i\})=\mathfrak{b}$. Let $P'=\sum_{i=1}^{m}Rp_i$ and $\mathfrak{m}$ any element of $\mathfrak{M}$. Then there is an element s of R such that $s\notin\mathfrak{m}$ and $sP'=(0)$ since P' is finitely generated. Let $\mathfrak{n}$ be an element of $\mathfrak{N}$. Then we have $\mathfrak{n}\not\supset\mathfrak{b}$ since $\mathfrak{M}=V(\mathfrak{b})$. Hence $P'\not\subset\mathfrak{n}F$, i.e., $P'\not\subset\mathfrak{n}P$. Let $p_{\mathfrak{n}}$ be an element of P' such that $p_{\mathfrak{n}}\notin\mathfrak{n}P$. Then the image $\varphi(p_{\mathfrak{n}})$ of $p_{\mathfrak{n}}$ in $P_{\mathfrak{n}}$ by the canonical map $\varphi:P\to P_{\mathfrak{n}}$ is a free basis of $\varphi(Rp_{\mathfrak{n}})$ which is a direct summand of $P_{\mathfrak{n}}$ by Lemma 1.3. Now $sP'=(0)$ implies $sp_{\mathfrak{n}}=0$, hence $\psi(s)\varphi(p_{\mathfrak{n}})=0$ in $P_{\mathfrak{n}}$ where ψ is the canonical map: $R\to R_{\mathfrak{n}}$. Since $\varphi(p_{\mathfrak{n}})$ is a free basis of $\varphi(Rp_{\mathfrak{n}})$, we have $\psi(s)=0$ in $R_{\mathfrak{n}}$. Therefore, there is an element $s_{\mathfrak{n}}$ of R such that $s_{\mathfrak{n}}\notin\mathfrak{n}$ and $s_{\mathfrak{n}}s=0$. This implies that $s\in\mathfrak{c}(\mathfrak{n})$ for any element $\mathfrak{n}$ of $\mathfrak{N}$. Hence we have $s\in\bigcap_{\mathfrak{n}}\mathfrak{c}(\mathfrak{n})=\mathfrak{c}(\mathfrak{N})$. Since $s\notin\mathfrak{m}$, we have $\mathfrak{m}\not\supset\mathfrak{c}(\mathfrak{N})$. This holds for any element $\mathfrak{m}$ of $\mathfrak{M}$. Therefore, we have proved that $\mathfrak{c}(\mathfrak{M})+\mathfrak{c}(\mathfrak{N})=R$ and that $\mathfrak{c}(\mathfrak{M})\cap\mathfrak{c}(\mathfrak{N})=(0)$. Thus we have $\mathfrak{c}(\mathfrak{M})\oplus\mathfrak{c}(\mathfrak{N})=R$. Now generally we have $\mathfrak{N}\neq\phi$. If not, we have $\mathfrak{M}=X$ and $\mathfrak{x}P=P$ for any $\mathfrak{x}\in X$. This implies $P_{\mathfrak{x}}=(0)$ for any $\mathfrak{x}\in X$, i.e., $P=(0)$ by Lemma 2.1. If $\mathfrak{M}\neq\phi$, $\mathfrak{c}(\mathfrak{M})$ and $\mathfrak{c}(\mathfrak{N})$ are proper ideals, i.e., $\neq R$ and $\neq(0)$, since $\mathfrak{c}(\mathfrak{M})\not\subset\mathfrak{n}$ for any $\mathfrak{n}\in\mathfrak{N}$ and $\mathfrak{c}(\mathfrak{N})\not\subset\mathfrak{m}$ for any $\mathfrak{m}\in\mathfrak{M}$. Therefore, if R is indecomposable, $\mathfrak{M}$ must be void, i.e., P must be faithfully flat. Since the converse is obvious, we have completed the proof.

5. Weakly noetherian rings.

DEFINITION. Let R be a commutative ring, P a projective module which is not finitely generated over R and M a finitely generated submodule of P. An element $\mathfrak{x}$ of X $(=m\text{-spec}(R))$ is said to be *redundant* with respect to M for P if $M+\mathfrak{x}P=P$. If there exists no such a submodule, $\mathfrak{x}$ is said to be *irredundant* for P.

NOTATION. Let P be a projective module and M a submodule of P. Then we write

$$\mathfrak{S}(M,P)=\{\mathfrak{x}\in X\mid M+\mathfrak{x}P\neq P\}\,,$$

$$\mathfrak{T}(M,P)=\{\mathfrak{x}\in X\mid M+\mathfrak{x}P=P\}\,,$$

$\mathfrak{S}(P)=$ the set of all irredundant elements of X,

$\mathfrak{T}(P)=$ the set of all redundant elements of X.

THEOREM 5.1. *Let R be a weakly noetherian ring and P a projective module which is not finitely generated. Then there exists a finitely generated submodule M of P such that $\mathfrak{S}(M,P)=\mathfrak{S}(P)$ and $\mathfrak{T}(M,P)=\mathfrak{T}(P)$ and the set $\mathfrak{S}(P)$ (resp. $\mathfrak{T}(P)$) of all irredundant (resp. redundant) elements for P of X is open and closed. Furthermore, we have $\mathfrak{S}(P)=V(\mathfrak{c}(\mathfrak{S}(P))), \mathfrak{T}(P)=V(\mathfrak{c}(\mathfrak{T}(P)))$.*

PROOF. For brevity, we assume that P is countably generated. Let F be a free module such that $F=P\oplus Q$; $\{u_i\}, i=1,2,\cdots$, a free basis for F; π the projection from F to P (i.e., if $f\in F$, $f=p+q$, $p\in P$, $q\in Q$, then $\pi f=p$); $\pi u_i=p_i, p_i=\sum_{j=1}^{n_i} s_{ij}u_j, s_{ij}\in R$. In this proof, we fix the free basis $\{u_j\}$.

Now let M be a finitely generated submodule of P, $\{m_1,\cdots,m_n\}$ a system of generators for $M, m_i=\sum_{j=1}^{m} s'_{ij}u_j, i=1,2,\cdots,n, s'_{ij}\in R$ and at least one of $\{s'_{1m},\cdots,s'_{nm}\}$ not zero. Then we write $F(M)=\sum_{j=1}^{m}\oplus Ru_j$, $\bar{M}=F(M)\cap P$ and $\bar{\bar{M}}=\sum_{j=1}^{m}Rp_j$. Now we have that

$$M\subset\bar{M}\subset\bar{\bar{M}},$$
$$\mathfrak{S}(M,P)\supset\mathfrak{S}(\bar{M},P)\supset\mathfrak{S}(\bar{\bar{M}},P)\supset\mathfrak{S}(P),$$
$$\mathfrak{T}(M,P)\subset\mathfrak{T}(\bar{M},P)\subset\mathfrak{T}(\bar{\bar{M}},P)\subset\mathfrak{T}(P).$$

Put $\mathfrak{a}(M)=(\{s_{ij}:i=1,2,\cdots,j=m+1,m+2,\cdots,n_i\})$. Let s_{ij} be any element of the system of generators $\{s_{ij}:j>m\}$ of the ideal $\mathfrak{a}(M)$. Then we have $m+1\leqq j\leqq n_i, p_i=\sum_{k=1}^{n_i}s_{ik}u_k$. Now let $\mathfrak{m}$ be any element of $\mathfrak{T}(\bar{M},P)$, then we have $\bar{M}+\mathfrak{m}P=P$, hence $(P/\bar{M})_{\mathfrak{m}}=(0)$ by Lemma 2.4 since $\bar{\bar{M}}$ is finitely generated and contains $\bar{M}$. Therefore there exists an element s_i of R such that $s_i\notin\mathfrak{m}$, $s_ip_i\in\bar{M}$. Thus we have $s_ip_i=\sum_{k=1}^{n_i}s_is_{ik}u_k\in\bar{M}\subset\sum_{j=1}^{m}\oplus Ru_j$. This implies that $s_is_{ik}=0$ for $k=m+1,\cdots,n_i$. Thus, if $j>m$ and $\mathfrak{m}\in\mathfrak{T}(\bar{M},P)$, $s_{ij}\in\mathfrak{c}(\mathfrak{m})$. Thus we have proved that $\mathfrak{a}(M)\subset\bigcap_{\mathfrak{m}\in\mathfrak{T}(\bar{M},P)}\mathfrak{c}(\mathfrak{m})=\mathfrak{c}(\mathfrak{T}(\bar{M},P))\subset\bigcap_{\mathfrak{m}\in\mathfrak{T}(\bar{M},P)}\mathfrak{m}$. Now of course we have $F(M)+\mathfrak{a}(M)F\supset P$ and this implies that $\bar{\bar{M}}+\mathfrak{a}(M)P=P$. Therefore, if $\mathfrak{m}\in V(\mathfrak{a}(M))$, we have $\bar{\bar{M}}+\mathfrak{m}P=P$, i.e., $\mathfrak{m}\in\mathfrak{T}(\bar{\bar{M}},P)$, i.e., $\mathfrak{m}\notin\mathfrak{S}(\bar{\bar{M}},P)$. Thus we have

$$\mathfrak{T}(\bar{\bar{M}},P)\supset V(\mathfrak{a}(M))\supset V(\mathfrak{c}(\mathfrak{T}(\bar{M},P)))\supset\mathfrak{T}(\bar{M},P),$$
$$\mathfrak{S}(\bar{\bar{M}},P)\subset D(\mathfrak{a}(M))\subset\mathfrak{S}(\bar{M},P)\subset\mathfrak{S}(M,P).$$

By Lemma 3.2, there exists a finitely generated ideal $\mathfrak{a}'=(a_1,\cdots,a_{t'})$ contained in $\mathfrak{a}(M)$ such that $V(\mathfrak{a}')=V(\mathfrak{a}(M))$. Assume that $\mathfrak{a}'\subset(\{s_{ij}:i=1,\cdots,t,j=m+1,\cdots,n_i\})$, $\max(n_1,\cdots,n_t)=m'$. Put $N=\sum_{i=1}^{t}Rp_i$. Let $\mathfrak{m}$ be any element of $\mathfrak{T}(\bar{M},P)$.

Then we have $(P/\bar{M})_{\mathfrak{m}}=(0)$, hence $((\bar{M}+N)/\bar{M})_{\mathfrak{m}}=(0)$. Therefore, there exists an element $s \in R$ such that $s \notin \mathfrak{m}, sN \subset \bar{M}$, i.e., $sp_i \in \bar{M}$ for $i=1, 2, \cdots, t$. If $\mathfrak{x} \notin V(\mathfrak{a}')$, there exists an s_{ij} such that $1 \leqq i \leqq t$, $m+1 \leqq j \leqq m'$, $s_{ij} \notin \mathfrak{x}$. Now we have $p_i=\sum_{k=1}^{n_i} s_{ik}u_k$, $sp_i \in \bar{M} \subset \sum_{k=1}^{m} \oplus Ru_k$. Therefore, we have that $ss_{ij}=0$ for $j=1, \cdots, t$, $j=m+1, \cdots, m'$, hence $s \in \mathfrak{c}(\mathfrak{x})$ since $s_{ij} \notin \mathfrak{x}$. Thus we have $s \in \mathfrak{c}(D(\mathfrak{a}'))$, hence $\mathfrak{m} \not\supset \mathfrak{c}(D(\mathfrak{a}'))$ if $\mathfrak{m} \in \mathfrak{T}(\bar{M}, P)$. Therefore, $\mathfrak{T}(\bar{M}, P) \cap V(\mathfrak{c}(D(\mathfrak{a}')))=\phi$, hence $V(\mathfrak{c}(D(\mathfrak{a}'))) \subset \mathfrak{S}(\bar{M}, P)$. Thus we have proved that $\mathfrak{S}(\bar{\bar{M}}, P) \subset D(\mathfrak{a}(M)) \subset V(\mathfrak{c}(D(\mathfrak{a}(M)))) \subset \mathfrak{S}(\bar{M}, P)$.

Let M_1 be a finitely generated submodule of P. Then we have

$$\mathfrak{S}(M_1, P) \supset \mathfrak{S}(\bar{M}_1, P) \supset \mathfrak{S}(\bar{\bar{M}}_1, P) \supset \mathfrak{S}(P).$$

If $\mathfrak{S}(\bar{\bar{M}}_1, P) \neq \mathfrak{S}(P)$, $\mathfrak{S}(\bar{\bar{M}}_1, P)$ contains an element $\mathfrak{x} \in X$ such that there exists a finitely generated submodule M_1' satisfying $M_1'+\mathfrak{x}P=P$. Then we have that, if we put $M_2=\bar{\bar{M}}_1+M_1'$, $M_2+\mathfrak{x}P=P$ and that $\mathfrak{S}(\bar{\bar{M}}_1, P) \supsetneqq \mathfrak{S}(M_2, P) \supset \mathfrak{S}(P)$. In this way, we may make an ascending sequence of finitely generated submodules $M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$ and a descending sequence of closed subsets of X

$$V(\mathfrak{c}(D(\mathfrak{a}(M_1)))) \supsetneqq V(\mathfrak{c}(D(\mathfrak{a}(M_2)))) \supsetneqq \cdots \supset \mathfrak{S}(P).$$

Since X is a noetherian space, there exists an integer w such that $V(\mathfrak{c}(D(\mathfrak{a}(M_w)))) = V(\mathfrak{c}(D(\mathfrak{a}(M_{w'}))))$ if $w' \geqq w$. If we put $M^{\#}=\bar{\bar{M}}_w$, then we have $\mathfrak{S}(M^{\#}, P) = V(\mathfrak{c}(D(\mathfrak{a}(M^{\#}))))=\mathfrak{S}(P)$ and $\mathfrak{T}(M^{\#}, P)=V(\mathfrak{a}(M^{\#}))=\mathfrak{T}(P)$. Thus $\mathfrak{S}(P)$ and $\mathfrak{T}(P)$ are open and closed since $\mathfrak{S}(P) \cap \mathfrak{T}(P)=\phi$ and $\mathfrak{S}(P) \cup \mathfrak{T}(P)=X$. Now we have that

$$\mathfrak{T}(P)=V(\mathfrak{c}(\mathfrak{T}(M^{\#}, P)))=\mathfrak{T}(M^{\#}, P).$$

Thus we have $\mathfrak{T}(P)=V(\mathfrak{c}(\mathfrak{T}(P)))$. Similarly we have

$$\mathfrak{S}(P)=D(\mathfrak{a}(M^{\#}))=V(\mathfrak{c}(D(\mathfrak{a}(M^{\#})))).$$

Thus we have $\mathfrak{S}(P)=V(\mathfrak{c}(\mathfrak{S}(P)))$. Therefore, we have completed the proof.

Now we can restate Theorem 4.1 in a stronger form.

COROLLARY 5.2. *Let R be a weakly noetherian ring and P a projective module which is not finitely generated. Then the set $\mathfrak{T}(P)$ of all redundant elements of X for P is void if R is indecomposable.*

PROOF. By Theorem 5.1, we have $\mathfrak{S}(P)=V(\mathfrak{c}(\mathfrak{S}(P)))$, $\mathfrak{T}(P)=V(\mathfrak{c}(\mathfrak{T}(P)))$. While we have $\mathfrak{c}(\mathfrak{S}(P))+\mathfrak{c}(\mathfrak{T}(P))=R$. For: let $\mathfrak{m}$ be any maximal ideal of R. Then $\mathfrak{m} \in \mathfrak{S}(P)$ or $\in \mathfrak{T}(P)$. If $\mathfrak{m} \in \mathfrak{S}(P)$, we have $\mathfrak{m} \not\supset \mathfrak{c}(\mathfrak{T}(P))$. If $\mathfrak{m} \in \mathfrak{T}(P)$, we have $\mathfrak{m} \not\supset \mathfrak{c}(\mathfrak{S}(P))$. Further, $\mathfrak{c}(\mathfrak{S}(P)) \cap \mathfrak{c}(\mathfrak{T}(P))=(0)$ since $\mathfrak{S}(P) \cup \mathfrak{T}(P)=X$. Thus we have

$$R=\mathfrak{c}(\mathfrak{S}(P)) \oplus \mathfrak{c}(\mathfrak{T}(P)).$$

Since R is indecomposable we have $\mathfrak{c}(\mathfrak{S}(P))=(0)$ or $\mathfrak{c}(\mathfrak{T}(P))=(0)$. But $\mathfrak{c}(\mathfrak{T}(P))$

can not be the zero ideal. For: if $\mathfrak{c}(\mathfrak{T}(P))=(0), \mathfrak{T}(P)=V(\mathfrak{c}(\mathfrak{T}(P)))=V((0))=X$. By Theorem 5.1, there exists a finitely generated submodule M of P such that $\mathfrak{T}(M, P)=\mathfrak{T}(P)=X$, i.e., $(P/M)_{\mathfrak{m}}=(0)$ for every element $\mathfrak{m}$ of X, and this implies that $(P/M)=(0)$ by Lemma 2.1, i.e., $P=M$. But this contradicts our assumption that P is not finitely generated. Thus $\mathfrak{c}((\mathfrak{S}(P))=(0)$ and $\mathfrak{S}(P)=V(\mathfrak{c}(\mathfrak{S}(P)))=X$, hecce $\mathfrak{T}(P)=\phi$. This completes the proof.

6. Preliminaries for the main theorem.

The following lemma is obvious but, for the completeness, we give the proof.

LEMMA 6.1. *A weakly noetherian ring is a direct sum of a finite number of indecomposable weakly noetherian rings.*

PROOF. First we note that $R/\mathfrak{a}$ is weakly noetherian for any ideal $\mathfrak{a}$ of R, hence any direct summand of R is weakly noetherian. Let $\mathfrak{S}$ be the set of ideals consisting of all ideals $\mathfrak{a}(\neq R)$ such that $R/\mathfrak{a}$ is not a direct sum of a finite number of indecomposable subrings. Let $\mathfrak{a}$ be an element of $\mathfrak{S}$ such that $V(\mathfrak{a})$ is minimal in the set $\{V(\mathfrak{a}), \mathfrak{a}\in\mathfrak{S}\}$. By assumption $R/\mathfrak{a}$ is decomposable, i.e., there exist proper ideals $\mathfrak{b}, \mathfrak{c}$ in R such that $\mathfrak{b}=(e_1, \mathfrak{a}), \mathfrak{c}=(e_2, \mathfrak{a})$, $\mathfrak{b}+\mathfrak{c}=R, \mathfrak{b}\cap\mathfrak{c}=\mathfrak{a}$ where e_1, e_2 are orthogonal idempotents mod. $\mathfrak{a}$. Now there exist maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$, such that $\mathfrak{m}_1\supset\mathfrak{c}, \mathfrak{m}_2\supset\mathfrak{b}$, then we have $\mathfrak{m}_1\not\ni e_1, \mathfrak{m}_2\not\ni e_2$. Thus we have that $V(\mathfrak{b})\subsetneqq V(\mathfrak{a})$, $V(\mathfrak{c})\subsetneqq V(\mathfrak{a})$, hence $\mathfrak{b}, \mathfrak{c}\notin\mathfrak{S}$. Thus $\bar{R}_1=\mathfrak{b}/\mathfrak{a}$ and $\bar{R}_2=\mathfrak{c}/\mathfrak{a}$ are direct sums of a finite number of indecomposable subrings. This is a contradiction since $\bar{R}=R/\mathfrak{a}=\bar{R}_1\oplus\bar{R}_2$. This completes the proof.

The following theorem is essentially due to Serre [**10**] and Bass [**1**]. Deleting the finiteness assumption in Theorem 4 of [**1**] and in Theorem 2 of [**10**], we have

THEOREM 6.2 (Serre). *Let R be a weakly noetherian ring for which* $\dim(m\text{-}\mathrm{spec}(R))$ *is finite. Let P be a projective module and M a submodule of P such that* $\dim((M+\mathfrak{x}P)/\mathfrak{x}P : R/\mathfrak{x})=\infty$ *at all $\mathfrak{x}\in X$. We are given the data:*

i) *$\mathfrak{F}$ a closed set in X.*
ii) *$\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ distinct elements of $\mathfrak{F}$.*
iii) *$v_1, \cdots, v_n$ with $v_i\in M, i=1, \cdots, n$.*
iv) *$p_1, \cdots, p_h\in P$ which are free outside $\mathfrak{F}$.*
v) *An integer $k\geqq 0$.*

Then there exist $p\in M$ and a closed set $\mathfrak{F}'$ in X such that

(a) *$p\equiv v_i$ (mod. $\mathfrak{x}_iP$), $i=1, \cdots, n$.*
(b) *$p_1, \cdots, p_h, p$ are free outside $\mathfrak{F}\cup\mathfrak{F}'$.*
(c) *$ht(\mathfrak{F}')\geqq k$.*

We prove this theorem for the completeness.

PROOF. We proceed by induction on k.

$k=0$. Take $\mathfrak{F}'=X$. Then (b) is vacuous and (a) can be accomplished by Lemma 3.3.

$k\geqq 1$. By inductive assumption, there exist $u\in M$ and a closed set $\mathfrak{G}$ in X such that

(a′) $u\equiv v_i$ (mod. $\mathfrak{x}_iP$), $i=1,\cdots,n$;

(b′) $p_1,\cdots,p_n$, u are free outside $\mathfrak{F}\cup\mathfrak{G}$.

(c′) $ht(\mathfrak{G})\geqq k-1$.

There is no loss in assuming that $\mathfrak{G}=\mathfrak{G}_1\cup\cdots\cup\mathfrak{G}_m$ where the $\mathfrak{G}_\alpha$ are the components of the singular set of $p_1,\cdots,p_h$, u which are not contained in $\mathfrak{F}$. (Note, if $\mathfrak{G}=\phi$, $m=0$.) With this done, we may choose $\mathfrak{y}_\alpha\in\mathfrak{G}_\alpha-(\bigcup_{\beta\neq\alpha}\mathfrak{G}_\beta)\cup\mathfrak{F}$, $\alpha=1,\cdots,m$. Since $\dim((M+\mathfrak{x}P)/\mathfrak{x}P: R/\mathfrak{x})=\infty$ at all $\mathfrak{x}\in X$ by assumption, we may choose $w_\alpha\in M$ so that

(1) $p_1,\cdots,p_h,u+w_\alpha$ are free at $\mathfrak{y}_\alpha$, $\alpha=1,\cdots,m$.

We now apply induction again, this time to

i) $\mathfrak{F}\cup\mathfrak{G}$,

ii) $\mathfrak{x}_1,\cdots,\mathfrak{x}_n,\mathfrak{y}_1,\cdots,\mathfrak{y}_m$ which are distinct elements of $\mathfrak{F}\cup\mathfrak{G}$,

iii) $0,\cdots,0$ (n zeros of P) and $w_1,\cdots,w_m$,

iv) $p_1,\cdots,p_h,u$ which are free outside $\mathfrak{F}\cup\mathfrak{G}$,

v) $k-1$.

We obtain $t\in M$ and $\mathfrak{H}$ a closed set in X such that

(a″) $t\equiv 0$ (mod. $\mathfrak{x}_iP$), $t\equiv w_\alpha$ (mod. $\mathfrak{y}_\alpha P$), $i=1,\cdots,n$, $\alpha=1,\cdots,m$,

(b″) $p_1,\cdots,p_h,u,t$ are free outside $\mathfrak{F}\cup\mathfrak{G}\cup\mathfrak{H}$,

(c″) $ht(\mathfrak{H})\geqq k-1$.

As before, we may assume $\mathfrak{H}=\mathfrak{H}_1\cup\cdots\cup\mathfrak{H}_d$ with the $\mathfrak{H}_\beta$'s the components of the singular set of $p_1,\cdots,p_h,u,t$ not contained in $\mathfrak{F}\cup\mathfrak{G}$. (If $\mathfrak{H}=\phi$, $d=0$.) Then we may choose $\mathfrak{z}_\beta\in\mathfrak{H}_\beta-(\bigcup_{\gamma\neq\beta}\mathfrak{H}_\gamma)\cup\mathfrak{F}\cup\mathfrak{G}$, $\beta=1,\cdots,d$, whereon

(2) $p_1,\cdots,p_h,u$ are free at $\mathfrak{z}_\beta$, $\beta=1,\cdots,d$.

Now since $\mathfrak{x}_1,\cdots,\mathfrak{x}_n,\mathfrak{y}_1,\cdots,\mathfrak{y}_m,\mathfrak{z}_1,\cdots,\mathfrak{z}_d$ are distinct we may choose $f\in R$, by Lemma 3.3, so that

$$f\equiv 0\ (\text{mod. } \mathfrak{x}_i),\quad i=1,\cdots,n,$$

$$f\equiv 1\ (\text{mod. } \mathfrak{y}_\alpha),\quad \alpha=1,\cdots,m,$$

$$f\equiv 0\ (\text{mod. } \mathfrak{z}_\beta),\quad \beta=1,\cdots,d.$$

Finally we set $p=u+ft$ and take for $\mathfrak{F}'$ the union of the components of the singular set of $p_1,\cdots,p_h,p$ not contained in $\mathfrak{F}$. Then $p\in M$ and (b) is automatic and (a) is verified by the computation:

$$p=u+ft\equiv v_i\ (\text{mod. } \mathfrak{x}_iP),\ i=1,\cdots,n.$$

To establish (c), we first note the obvious fact that if $p_1, \cdots, p_h$, $p=u+ft$ are not free at $\mathfrak{x}$, then neither are $p_1, \cdots, p_h, u, t$. Hence, the singular set of $p_1, \cdots, p_h, p$ is contained in that of $p_1, \cdots, p_h, u, t$ the latter being contained in $\mathfrak{F} \cup \mathfrak{G} \cup \mathfrak{H}$, by (b″). Therefore $\mathfrak{F}' \subset \mathfrak{F} \cup \mathfrak{G} \cup \mathfrak{H}$; but, by our choice of $\mathfrak{F}'$, it follows that $\mathfrak{F}' \subset \mathfrak{G} \cup \mathfrak{H}$, and from this it follows that $ht(\mathfrak{F}') \geqq k-1$. If $\mathfrak{F}' = \phi$ we are done, so we assume $\mathfrak{F}' \neq \phi$ and we must show $ht(\mathfrak{F}') \neq k-1$. If not, let $\mathfrak{K}$ be a component of $\mathfrak{F}'$ of height $k-1$. Then clearly $\mathfrak{K}$ must be a component of either $\mathfrak{G}$ or $\mathfrak{H}$, i. e., $\mathfrak{K} =$ some $\mathfrak{G}_\alpha$ or some $\mathfrak{H}_\beta$. Therefore, some $\mathfrak{y}_\alpha \in \mathfrak{K}$ or some $\mathfrak{z}_\beta \in \mathfrak{K}$. But $\mathfrak{y}_\alpha \in \mathfrak{F}'$ contradicts (1) and $\mathfrak{z}_\beta \in \mathfrak{F}'$ contradicts (2). Thus we have completed the proof.

LEMMA 6.3. *Let R be an indecomposable weakly noetherian ring for which* $\dim(m\text{-spec}(R))$ *is finite. Let P be a projective module which is not finitely generated, u any element of P and M a submodule of P such that $Ru+M=P$. Then there exists an element $m \in M$ such that $R(u+m)$ is a direct summand of P and $u+m$ is a free basis of $R(u+m)$.*

PROOF. Let $\mathfrak{F}$ be the singular set of $u, ht(\mathfrak{F}) = k$ and $\mathfrak{F} = \mathfrak{F}_1 \cup \cdots \cup \mathfrak{F}_n$ where the $\mathfrak{F}_\alpha$ are the components of $\mathfrak{F}$. Select $\mathfrak{x}_\alpha \in \mathfrak{F}_\alpha - (\bigcup_{\beta \neq x} \mathfrak{F}_\beta)$, $\alpha = 1, \cdots, n$. Since R is indecomposable, any maximal ideal of R is irredundant for P, by Corollary 5.2. Therefore, $\dim(P/\mathfrak{x}P : R/\mathfrak{x}) = \infty$, hence $\dim(M+\mathfrak{x}P/\mathfrak{x}P : R/\mathfrak{x}) = \infty$ for any element $\mathfrak{x}$ of X since $Ru+M=P$. Thus we may choose $w_\alpha \in M$ so that

(1) $u+w_\alpha$ is free at $\mathfrak{x}_\alpha, \alpha = 1, \cdots, n$.

Now we have the data:

i) $\mathfrak{F}$ a closed set in X,
ii) $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ which are distinct elements of $\mathfrak{F}$,
iii) $w_1, \cdots, w_n$ with $w_i \in M$, $i = 1, \cdots, n$,
iv) u which is free outside $\mathfrak{F}$,
v) $k' = \dim X + 1$.

By Theorem 6.2, we obtain $t \in M$ and $\mathfrak{H}$ a closed set in X such that

(a) $t \equiv w_\alpha \pmod{\mathfrak{x}_\alpha P}$, $\alpha = 1, \cdots, n$,
(b) u, t are free outside $\mathfrak{F} \cup \mathfrak{H}$,
(c) $ht(\mathfrak{H}) \geqq k'$, (hence $\mathfrak{H} = \phi$).

We set $u_1 = u+t$ and take for $\mathfrak{F}'$ the singular set of u_1. If u_1 is not free at $\mathfrak{x}$, then neither are u, t, the latter being contained in $\mathfrak{F}$, by (b) and (c). Therefore $\mathfrak{F}' \subset \mathfrak{F}$, and from this it follows that $ht(\mathfrak{F}') \geqq k$. If $\mathfrak{F}' = \phi$ we are done, so we assume $\mathfrak{F}' \neq \phi$ and we show $ht(\mathfrak{F}') \neq k$. If not, let $\mathfrak{K}$ be a component of $\mathfrak{F}'$ of height k. Then $\mathfrak{K}$ must be a component of $\mathfrak{F}$; i. e., $\mathfrak{K} =$ some $\mathfrak{F}_\alpha$. Therefore, some $\mathfrak{x}_\alpha \in \mathfrak{K}$. But $\mathfrak{x}_\alpha \in \mathfrak{K}$ contradicts (1). Thus $ht(\mathfrak{F}') \geqq k+1$ and $m_1 = t \in M$. Inductively we have elements $m_1, m_2, \cdots$ of M and the singular sets $\mathfrak{F}^{(i)}$ of $u_i = u + \sum_{j=1}^{i} m_j$ such that $ht(\mathfrak{F}^{(i)}) \geqq k+i$. If $i \geqq k'-k$, $ht(\mathfrak{F}^{(i)}) \geqq k+i \geqq k'$, therefore, $\mathfrak{F}^{(i)} = \phi$. Thus, if we set $m = \sum_{j=1}^{k'-k} m_j$, we have $m \in M$, and $p = u+m$ is free at all $\mathfrak{x} \in X$. Thus Rp is a direct summand of P and p is a free basis of Rp by Lemma 1.3. This completes the proof.

7. The main theorem.

We rewrite our main theorem.

THEOREM 7.1. *Let R be a weakly noetherian ring for which* $\dim(m\text{-spec}(R))$ *is finite and let P be a projective R-module. Then P is a direct sum of finitely generated projective modules.*

PROOF. By Lemma 6.1, R is a direct sum of a finite number of indecomposable rings: $R = R_1 \oplus \cdots \oplus R_n$. Then R_iP is R_i- and R-projective and $P = \sum_{i=1}^{n} \oplus R_iP$ and R_i is weakly noetherian. Therefore, we may assume that R is indecomposable without loss of generality.

Now, by virtue of Corollary 1.8 and Lemma 1.9, the following Lemma 7.2 suffices to complete the proof of our theorem[4].

LEMMA 7.2. *Let R be an indecomposable weakly noetherian ring for which* $\dim(m\text{-spec}(R))$ *is finite. Let P be a projective R-module and p any element of P. Then p can be embedded in a finitely generated direct summand of P.*

PROOF. We may assume that P is not finitely generated. By Lemma 1.6, there exists an integer m such that

$$P' = (\sum_{i=1}^{m} \oplus Rf_i) \oplus P = K_1 \oplus K_1', \quad K_1 \ni p$$

where $f_1, \cdots, f_m$ are independent variables and K_1 is a finitely generated projective module. Under these conditions, we shall prove that there exists a finitely generated projective module K_2 such that

$$P'' = (\sum_{i=2}^{m} \oplus Rf_i) \oplus P = K_2 \oplus K_2', \quad K_2 \ni p.$$

Now we have

$$P' = Rf_1 \oplus P'' = K_1 \oplus K_1', \quad P'' \cap K_1 \ni p.$$

Let π be the projection from P' to K_1', $\pi f_1 = u$ and $\pi P'' = M$. Then we have $Ru + M = K_1'$ and K_1' is a projective module which is not finitely generated. Thus by Lemma 6.3, there exists an element m of M such that $R(u+m)$ is a direct summand of K_1'. Let $\pi p'' = m$. Then we have

$$P' = R(f_1 + p'') \oplus P'' = K_1 \oplus R(u+m) \oplus U$$

4) This method of the proof is the same as in [8].

where U is a submodule of K_1'. Let π' be the projection from P' to U. Then we have $\pi' P'' = U$. For: let u' be any element of $U, u' = r(f_1 + p'') + q$, $r \in R$, $q \in P''$, then $u' = \pi u' = r(u+m) + \pi q = \pi' r(u+m) + \pi' \pi q = \pi' q$. Therefore, we have an exact sequence

$$0 \longrightarrow K_2 \longrightarrow P'' \xrightarrow{\pi'} U \longrightarrow 0 .$$

This sequence splits since U is projective. $K_2 = P'' \cap (K_1 \oplus R(u+m))$ since $K_2 = \{p''' \in P'' \mid \pi' p''' = 0\}$ and $\pi' p''' = 0$ if and only if $p''' \in K_1 \oplus R(u+m)$. Now K_2 is a direct summand of P'', hence of P', and contained in $K_1 \oplus R(u+m)$ which is finitely generated. Thus K_2 is a direct summand of $K_1 \oplus R(u+m)$ by Lemma 1.2, hence K_2 is a finitely generated projective module. Now p is contained in $P'' \cap K_1$ hence in $P'' \cap (K_1 \oplus R(u+m)) = K_2$. Thus we have proved that p is contained in K_2 which is a finitely generated projective direct summand of P''. Repeating this process, we get a finitely generated direct summand $\bar{K}$ of P which contains p. Thus we have completed the proof.

REMARK. Seshadri proved that, if R is a principal ideal ring, any finitely generated projective module over $R[X]$ is free (Proposition 9 of [**10**]). Combining this with our Theorem 7.1, we can delete the finiteness assumption in Seshadri's theorem. For example, let $R = k[X, Y]$ where k is a field. Then any projective module over R is free.

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