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Classification of $SO(n)$-bundles over the quaternion projective plane

By Seiya SASAO and Itiro TAMURA

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Let Ω be the quaternion projective plane and let $SO(n)$ be the rotation group of $(n-1)$-sphere S^{n-1}. It is well known that the equivalence classes of $SO(n)$-bundles over Ω are 1-1 correspondence with the homotopy classes of maps of Ω into the classifying space B_n of $SO(n)$, therefore the classification of $SO(n)$-bundles over Ω reduces to the computation of the homotopy classes of maps $f:\Omega\to B_n$. Since the cases $n=1,2$ are trivial we are interested in cases $n\geqq 3$. We denote by $\Omega(n)$ the set of the equivalence classes of $SO(n)$-bundles over Ω.

We shall prove

THEOREM. *$\Omega(3)$ is in 1-1 correspondence with the pairs $(m, \mu(m)Z_2)$ such that $\frac{m(m-1)}{2}\equiv 0$ (mod 12), where $\mu(m)=0$ if m is even, $\mu(m)=1$ if m is odd.*

$\Omega(4)$ is in 1-1 correspondence with the triples $(m, l, \bar{\mu}_1Z_2+\bar{\mu}_2Z_2)$ such that $\frac{m(m-1)}{2}\equiv 0$, $\frac{l(l-1-2m)}{2}\equiv 0$ (mod 12), where $\bar{\mu}_1, \bar{\mu}_2$ are functions of m, l such that

$\bar{\mu}_1=\bar{\mu}_2=0$		if m, l are both even,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=1$	if m is odd, l is even,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=0$	if m is even, l is odd,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=0$	if m, l are both odd.

If $n\geqq 5, n\neq 8$ $\Omega(n)$ is in 1-1 correspondence with the pair (r, s) of integers.

$\Omega(8)$ is in 1-1 correspondence with the triple (r, s, t) of integers.

The proof is given in the Section 1. In the Section 2 we shall consider characteristic classes of $SO(n)$-bundles over Ω.

1. Ω has the cell decomposition, $S^4\overset{\nu}{\cup} e^8$, where ν denotes the Hopf map: $S^7\to S^4$. Thus the above computation is equivalent to the computation of the homotopy classes of extensions of extendable maps: $S^4\to B_n$, over Ω. As is well-known, we have $\pi_4(B_3)\cong Z$, $\pi_4(B_4)\cong Z+Z$, $\pi_4(B_n)\cong Z$. We denote by $\bar{\alpha}_n$, $(n=3,4)$, $\bar{\beta}_n$, $(n\geqq 4)$ generators of $\pi_4(B_n)$. Let $\Delta_n:\pi_k(B_n)\to\pi_{k-1}(SO(n))$ be the boundary homomorphism of the homotopy exact sequence of the universal bundle of $SO(n)$. Then we can take $\bar{\alpha}_n$ and $\bar{\beta}_n$ such that $\Delta_4(\bar{\alpha}_4)=\alpha_3$, $\Delta_4(\bar{\beta}_4)=\beta_3$,

$\Delta_3(\bar{\alpha}_3)=\{\rho\}$, $\Delta_n(\bar{\beta}_n)=i_{n*}(\beta_3)$, where i_{n*} denotes the induced homomorphism by the injection $i_n: SO(4)\to SO(n)$ and $\alpha_3, \beta_3, \{\rho\}$ have the same meanings as in § 22 of [2]. The following equalities hold

$$(m\bar{\alpha}_k+n\bar{\beta}_k)\circ\nu=(m\bar{\alpha}_k)\circ\nu+(n\bar{\beta}_k)\circ\nu+[m\bar{\alpha}_k, n\bar{\beta}_k],$$

$$(m\bar{\alpha}_k)\circ\nu=m(\bar{\alpha}_k\circ\nu)+\frac{m(m-1)}{2}[\bar{\alpha}_k, \bar{\alpha}_k],$$

$$(n\bar{\beta}_k)\circ\nu=n(\bar{\beta}_k\circ\nu)+\frac{n(n-1)}{2}[\bar{\beta}_k, \bar{\beta}_k],$$

where [,] denotes the Whitehead product.

The following Lemma 1 is an immediate consequence of these equalities.

LEMMA 1. $(m\bar{\alpha}_k+n\bar{\beta}_k)\circ\nu$

$$=m(\bar{\alpha}_k\circ\nu)+n(\bar{\beta}_k\circ\nu)+\frac{m(m-1)}{2}[\bar{\alpha}_k, \bar{\alpha}_k]$$

$$+\frac{n(n-1)}{2}[\bar{\beta}_k, \bar{\beta}_k]+mn[\bar{\alpha}_k, \bar{\beta}_k],$$

where $m=0$ if $k\geqq 5$ and $n=0$ if $k=3$.

Now $S^3\to S^7\xrightarrow{\nu}S^4$ is considered as a principal bundle with the structure group S^3. Let s_3 be the classifying space of S^3, $\beta_3': s_3\to B_4$ the mapping induced by β_3, and $\varphi: S^4\to s_3$ the characteristic map of the above bundle $S^3\to S^7\xrightarrow{\nu}S^4$. Then we have $\bar{\beta}_4=\beta_3'\circ\varphi$ and so $\bar{\beta}_4\circ\nu=\beta_3'\circ\varphi\circ\nu$. But $\varphi\circ\nu=0$ (cf. [2]). So we have

LEMMA 2. $\bar{\beta}_n\circ\nu=0$ $n\geqq 4$.

Denote by $P_3(C), P_1(Q)$ the 3-dimension complex projective space and the 1-dimension quaternion projective space respectively. Define the map $\phi: P_3(C)\to P_1(Q)=S_4$ by $\phi[z_1, z_2, z_3, z_4]=[z_1+z_2j, z_3+z_4j]$, where j denotes the usual element of Q, then we have an S^2-bundle $(P_3(C), S^4, \phi)$.

Suppose that $\mathfrak{p}: S^7\to P_3(C)$ be the natural map $\mathfrak{p}(z_1, z_2, z_3, z_4)=[z_1, z_2, z_3, z_4]$, then we have $\phi_*(\{\mathfrak{p}\})=\nu$. Let Δ' be the boundary homomorphism of the homotopy exact sequence of the bundle $(P_3(C), S^4, \phi)$. Since $\Delta'(\iota_4)$ is the attaching map of 4-dimensional cell e^4 to S^2 in the cell decomposition of $P_3(C)$. ([5]), we have $\Delta'(\iota_4)=\eta$, where η denotes the Hopf map $S^3\to S^2$. Thus the associated principal bundle with the bundle $(P_3(C), S^4, \phi)$ is the $SO(3)$-bundle with the characteristic map $\{\rho\}$. Let P be the projection $SO(3)\to S^2$ and consider the following commutative diagram, where Δ'' denotes the boundary homomorphism of the homotopy exact sequence of the associated principal bundle.

$$\begin{array}{ccccccc}
\longrightarrow \pi_7(P_3(C)) & \xrightarrow{\phi_*} & \pi_7(S^4) & \xrightarrow{\Delta'} & \pi_6(S^2) & \longrightarrow & \\
 & & \uparrow id & & \uparrow P_* & & \\
 & \longrightarrow & \pi_7(S^4) & \xrightarrow[\Delta'']{} & \pi_6(SO(3)) & \longrightarrow &
\end{array}$$

Then we have $P_*\Delta''(\nu)=\Delta'(\nu)=\Delta'\phi_*\{\mathfrak{p}\}=0$. Since P_* is an isomorphism a nd $\Delta''(\nu)=\Delta_3(\bar{\alpha}_3\circ\nu)$ we have $\bar{\alpha}_3\circ\nu=0$, moreover $\bar{\alpha}_4\circ\nu=i_*(\bar{\alpha}_3)\circ\nu=i_*\{\bar{\alpha}_3\circ\nu\}=0$, where i denotes the injection $SO(3)\rightarrow SO(4)$.

LEMMA 3. $\bar{\alpha}_n\circ\nu=0$ $n\geqq 3$.

Let α_0 be the inner automorphism $SO(4)\rightarrow SO(4)$ defined by a matrix who se determinant is -1 and $\bar{\alpha}_0$ be the induced homeomorphism, $B_4\rightarrow B_4$, by α_0. Since $\alpha_{0*}(\alpha_3)=\alpha_3$, $\alpha_{0*}(\beta_3)=\alpha_3-\beta_3$ (see §22 of [2]). We have $\alpha_{0*}(\bar{\alpha}_4)=\bar{\alpha}_4$, $\bar{\alpha}_{0*}(\bar{\beta}_4)=\bar{\alpha}_4-\bar{\beta}_4$.

Thus Lemma 2 yields

$$(\bar{\alpha}_4-\bar{\beta}_4)\circ\nu=\{\bar{\alpha}_{0*}(\bar{\beta}_4)\}\circ\nu=\bar{\alpha}_{0*}(\bar{\beta}_4\circ\nu)=\bar{\alpha}_{0*}(0)=0\,.$$

On the other hand, Lemma 1, Lemma 2, Lemma 3 imply

$$(\bar{\alpha}_4-\bar{\beta}_4)\circ\nu=[\bar{\beta}_4,\bar{\beta}_4]-[\bar{\alpha}_4,\bar{\beta}_4]\,,$$

Therefore we have

LEMMA 4. $[\bar{\alpha}_4,\bar{\beta}_4]=[\bar{\beta}_4,\bar{\beta}_4]$.

Let ω denote the generator of $\pi_6(S^3)\cong Z_{12}$ and $<,>$ denote the Samelson product. Since $\{\rho\}_*$, β_{4*}, $i_*\{\rho\}_*$ are all isomorphisms we have by [4]:

(1.1) $\Delta_3[\bar{\alpha}_3,\bar{\alpha}_3]=-<\{\rho\},\{\rho\}>=\{\rho\}_*\omega$ is an element of order 12,

(1.2) $\Delta_4[\bar{\alpha}_4,\bar{\alpha}_4]=i_*\{\rho\}_*\omega$ is an element of order 12,

(1.3) $\Delta_4[\bar{\beta}_4,\bar{\beta}_4]=-<\beta_3,\beta_3>=\beta_{3*}\omega$ is an element of order 12,

(1.4) $\Delta_k[\bar{\beta}_k,\bar{\beta}_k]=0$ if $k\geqq 5$.

LEMMA 5. *$m\bar{\alpha}_3$, $m\bar{\alpha}_4+l\bar{\beta}_4$ are extendable over Ω if and only if* $\dfrac{m(m-1)}{2}\equiv 0$, $\dfrac{l(l-1+2m)}{2}\equiv 0$ (mod 12) *respectively.*

For $(m\bar{\alpha}_3)\circ\nu=m(\bar{\alpha}_3\circ\nu)+\dfrac{m(m-1)}{2}[\bar{\alpha}_3,\bar{\alpha}_3]$ by Lemma 1

$$=\frac{m(m-1)}{2}[\bar{\alpha}_3,\bar{\alpha}_3] \qquad \text{by Lemma 3}.$$

Since $[\bar{\alpha}_3,\bar{\alpha}_3]$ has the order 12 by (1.1) we have the first case.

$$(m\bar{\alpha}_4+l\bar{\beta}_4)\circ\nu=\frac{m(m-1)}{2}[\bar{\alpha}_4,\bar{\alpha}_4]+\frac{l(l-1)}{2}[\bar{\beta}_4,\bar{\beta}_4]+lm[\bar{\alpha}_4,\bar{\alpha}_4]$$

by Lemmas 1, 2, 3

$$=\frac{m(m-1)}{2}[\bar{\alpha}_4,\bar{\alpha}_4]+\frac{l(l-1+2m)}{2}[\bar{\beta}_4,\bar{\beta}_4]$$

by Lemma 4

$$=\bar{\alpha}_4\circ\left\{\frac{m(m-1)}{2}E\omega\right\}+\bar{\beta}_4\circ\left\{\frac{l(l-1+2m)}{2}E\omega\right\}$$

by (1.2), (1.3).

Since $\pi_7(B_4)\approx\bar{\alpha}_4\circ E\pi_6(S^3)+\bar{\beta}_4\circ E\pi_6(S^3)$ and $E\omega$ has the order 12 we obtain the second case.

By the Theorem 4 in the paper of Barcus and Barratt [**1**], the homotopy classes of extension over Ω of an extendable map $f: S^4\to B_n$ are in 1-1 correspondence with $\pi_8(B_n)/A_n$, where A_n is a subgroup $\{\xi\circ E\nu\pm[f_*(\iota_4),\xi];\ \xi\in\pi_8(B_n)\}$. Since $\pi_5(B_n)$ is a torsion group and $\pi_8(B_n)$ $(n\geqq 5)$ has no torsion we obtain

LEMMA 6. *If* $n\geqq 5$ *we have* $A_n=0$.

In the case $n=3$ we have $\pi_8(B_3)\cong Z_2[\bar{\alpha}_3\circ E^2\eta\circ E\nu]$ and $\pi_5(B_3)\cong Z_2[\bar{\alpha}_3\circ E^2\eta]$. Suppose that $f=m\bar{\alpha}_3$, then we have (cf. [**6**])

$$\begin{aligned}A_3&=\{\bar{\alpha}_3\circ(E^2\eta\circ E\nu)+m[\bar{\alpha}_3,\bar{\alpha}_3\circ E^2\eta]\}\\&=\{\bar{\alpha}_3\circ E^2\eta\circ E\nu+m[\bar{\alpha}_3,\bar{\alpha}_3]\circ E^5\eta\}\\&=\{\bar{\alpha}_3\circ(E^2\eta\circ E\nu)+m\bar{\alpha}_3\circ E\omega\circ E^5\eta\}=\{(m+1)\bar{\alpha}_3\circ E^2\eta\circ E\nu\}\,.\end{aligned}$$

Thus we have

LEMMA 7. *If m is even,* $A_3=\pi_8(B_3)$ *and*
if m is odd, $A_3=0$.

In the case $n=4$ we have $\pi_5(B_4)\approx Z_2[\bar{\alpha}_4\circ E^2\eta]+Z_2[\bar{\beta}_4\circ E^2\eta]$. Suppose that $f=m\bar{\alpha}_4+l\bar{\beta}_4$. Then we have

$$A_4=a\bar{\alpha}_4\circ(E^2\eta\circ E\nu)+b\bar{\beta}_4(E^2\eta\circ E\nu)+[m\bar{\alpha}_4+l\bar{\beta}_4,a\bar{\alpha}_4\circ E^2\eta+b\bar{\beta}_4\circ E^2\eta]\,,$$

where a, b are both 0 or 1.

On the other hand,

$$\begin{aligned}&[m\bar{\alpha}_4+l\bar{\beta}_4,a\bar{\alpha}_4\circ E^2\eta+b\bar{\beta}_4\circ E^2\eta]=am[\bar{\alpha}_4,\bar{\alpha}_4\circ E^2\eta]\\&+bm[\bar{\alpha}_4,\bar{\beta}_4\circ E^2\eta]+al[\bar{\beta}_4,\bar{\alpha}_4\circ E^2\eta]+lb[\bar{\beta}_4,\bar{\beta}_4\circ E^2\eta]\\&=am[\bar{\alpha}_4,\bar{\alpha}_4]\circ E^5\eta+bm[\bar{\alpha}_4,\bar{\beta}_4]\circ E^5\eta+al[\bar{\beta}_4,\bar{\alpha}_4]\circ E^5\eta\\&+lb[\bar{\beta}_4,\bar{\beta}_4]\circ E^5\eta=am\bar{\alpha}_{4*}(E\omega\circ E^5\eta)+(bm+al+lb)\bar{\beta}_{4*}(E\omega\circ E^5\eta)\end{aligned}$$

therefore $A_n=a(m+1)\alpha_{4*}(E^2\eta\circ E\nu)+(b+bm+al+lb)\bar{\beta}_{4*}(E^2\eta\circ E\nu)$, where a,b are both 0 or 1. Since $\pi_8(B_4)\cong\bar{\alpha}_{4*}(E\pi_7(S^3))+\bar{\beta}_4(E\pi_7(S^3))$, we obtain

LEMMA 8.

If m is even and l is even,	$A_4=\pi_8(B_4)$
if m is even and l is odd,	$A_4=(\bar{\alpha}_4+\bar{\beta}_4)(E\pi_7(S^3))$
if m is odd and l is even,	$A_4=0$
if m is odd and l is odd,	$A_4=\bar{\beta}_{4*}(E\pi_7(S^3))$.

From Lemma 2 all maps: $S^4\to B_n$ are extendable over Ω if $n\geqq 5, n\neq 8$. Since $\pi_8(B_8)\cong Z+Z, \pi_8(B_n)\cong Z$ $(n\geqq 5, n\neq 8)$, we obtain by Lemma 6 the proof of

Theorem in the case $n \geqq 5$. From Lemma 5 $m\bar{\alpha}_3$ is extendable over Ω if and only if $\frac{m(m-1)}{2} \equiv 0 \pmod{12}$. Since $\pi_8(B_3) \cong Z_2$ we obtain the case $n=3$ of the Theorem by Lemma 7. From Lemma 5 $m\bar{\alpha}_4 + l\bar{\beta}_4$ is extendable over Ω if and only if $\frac{m(m-1)}{2} \equiv 0$, $\frac{l(l-1+2m)}{2} \equiv 0 \pmod{12}$. Then from Lemma 8 we have the case $n=4$ of the Theorem.

2. Let $\xi_r^s(k)$ $(k \geqq 5, k \neq 8)$, $\xi_r^{s,t}(8)$, $\xi_m^*(3)$, $\xi_{m,l}^{**}(4)$ be $SO(k)$, $SO(8)$, $SO(3)$, $SO(4)$-bundles, corresponding to the pair (r, s), (r, s, t), $(m, *)$, $(m, l, **)$ in Theorem respectively. Let $p_i(\xi(k))$ denote the i-th Pontrjagin classes of the bundle $\xi(k)$. Then we have

(2.1) $$p_1(\xi_r^s(k)) = 2re^4,\ k \geqq 5. \qquad p_1(\xi_r^{s,t}(8)) = 2re^4$$
$$p_1(\xi_m^*(3)) = 4me^4, \qquad p_1(\xi_{m,l}^{**}(4)) = 2(2m+l)e^4$$

(2.2) $$\begin{aligned} p_2\{\xi_r^s(k)\} - p_2\{\xi_r^0(k)\} &= 48\, se^8 \quad \text{if } k=5 \\ &= 24\, se^8 \quad \text{if } k=6 \\ &= 12\, se^8 \quad \text{if } k=7 \\ &= 6\, se^8 \quad \text{if } k \geqq 9 \end{aligned}$$

(2.3) $$p_2(\xi_r^{s,t}(8)) - p_2(\xi_r^{0,0}(8)) = 6(2s-t)e^8$$

(2.4) $p_2(\xi_{m,l}^{**}(4))$ is independent on $**$, that is, $p_2(\xi_{m,l}^{**}(4))$ is determined only by the first Pontrjagin class and Euler class, where e^4, e^8 denote generators of $H^4(\Omega, Z)$, $H^8(\Omega, Z)$ respectively. For (2.1) follows from Theorem 4.1 of [3]. Let $\Omega_1 \cup \Omega_2$ be the space obtained by identifying S^4 of Ω_1 with S^4 of Ω_2, where Ω_i is a copy of Ω. Define a map $F: S^8 \to \Omega_1 \cup \Omega_2$ as follows

$F \mid E_+^8 =$ the characteristic map of the cell e^8 of Ω_1

$F \mid E_-^8 =$ the characteristic map of the cell e^8 of Ω_2

where E_+^8, E_-^8 are the upper, and the lower-hemisphere of S^8 respectively. Let f, g be two extensions of an extendable map: $S^4 \to B_k$ over Ω. We define a map $f \cup g: \Omega_1 \cup \Omega_2 \to B_k$ such that $f \cup g \mid \Omega_1 = f$ and $f \cup g \mid \Omega_2 = g$. Then we can identify $\{f \cup g\}_*(\{F\}) \in \pi_8(B_k)$ with the integer s if $k \geqq 5, k \neq 8$ and the pair of integers (s, t) if $k=8$. Let $\xi(f \cup g)$ be the bundle over S^8 whose characteristic map is $\{f \cup g\}_*(\{F\})$ and we put $ae^8 = p_2(\xi_m(k)) = p_2(\xi_r^{s,t}(8))$ and $be^8 = p_2(\xi_m^0(k)) = p_2(\xi_r^{s,t}(8))$, $(k \geqq 5, k \neq 8)$.

Then $p_2\{\xi(f \cup g)\} = F^*(ae^8) - F^*(be^8) = a(s^8) - b(s^8)$, where s^8 is the generator of $H^8(S^8, Z)$. On the other hand we can compute $p_2(\xi(f \cup g))$ from Theorem 4.4 of [3]. Thus we obtain (2.2), (2.3), (2.4) from $p_2(\xi(f \cup g))$.

By the above result we have

(2.5) If $k \geqq 5, k \neq 8$, two $SO(k)$-bundles over Ω are equivalent if and only if their first and second Pontrjagin classes coincide with each other.

If $k=8$, two $SO(k)$-bundles over Ω are equivalent if and only if their first and second Pontrjagin classes and Euler class coincide with each other.

In the case of $SO(3), SO(4)$-bundles, a bundle over Ω is trivial if and only if the first and second Pontrjagin classes, and Euler class are all trivial.

Next we consider realizable cohomology classes of Ω as Pontrjagin classes. Let B be the S^3 bundle, $S^{11} \to \Omega$ as usual. We may take B as $B_{0,1}^{**}(4)$. Since $p_1(B) = 2(e^4), p_2(B) = 1(e^8)$, we have $p_2(B_{0,2}^{**}(4)) = 1$, and $p_2(B_1^0(k)) = 1(e^8), p_2(B_1^{0,0}(8))$, $(k \geqq 5, k \neq 8)$. If $\frac{m(m-1)}{2} = 0 \pmod{12}$, there exists a map $\lambda : \Omega \to \Omega$ such that $\lambda \mid S^4$ is of degree m. Hence we have

$$p_2(B_{0,m}^{**}(4)) = m^2, \quad p_2(B_m^0(k)) = m^2 \ (k \geqq 5, k \neq 8), \quad p_2(B_m^{0,0}(8)) = m^2 .$$

Combining these results with (2.2), (2.3), (2.4) we can know in some case which cohomology classes of Ω are realizable as second Pontrjagin classes.

Musashi Institute of Technology
University of Tokyo

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