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Abelian varieties attached to automorphic forms

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Introduction.

Let G be a discontinuous group acting on the upper half-plane $\mathfrak{X}$. As a subgroup of $GL(2, \boldsymbol{R})$, G admits a tensor representation M_n of degree n. One can then define the cohomology groups $H^1(M_n, G)$ after Eichler [**1**], and from Shimura [**6**], there exists a canonical isomorphism between $H^1(M_n, G)$ and the space $S_{n+2}(G)$ of cusp forms of degree $n+2$ with respect to G. Under certain "integrality" assumptions on G (for example, when $G=SL(2, \boldsymbol{Z})$, these conditions are satisfied), he defines a lattice in $H^1(M_n, G)$ and proves that the torus so obtained, admits a canonical structure of an abelian variety.

Suppose more generally, we have two discontinuous groups $G \subset G_1$ (G normal in G_1 and $(G_1 : G) < \infty$). Then, associated with a real representation R of G_1/G, we can define the cohomology groups $H^1(R \otimes M_n, G_1)$ and establish a canonical isomorphism between $H^1(R \otimes M_n, G_1)$ and the space $S_{n+2,R}(G_1)$ of vectors of cusp forms of degree $n+2$ with respect to G which remains invariant under the representation R (cf. Theorem 1). If then R is rational and G_1 satisfies the "integrality" assumption [**6**], a lattice in $H^1(R \otimes M_n, G_1)$ can be defined, and as in the case of Shimura, this torus can be endowed with a canonical structure of an abelian variety (say) $A_{n+2,R}(G_1)$. In the special case $G_1 = \Gamma(1)$, $G=\Gamma_1(q)$ (q, a prime) and $n=0$, these have been noticed by Hecke [**4**].

We note finally that these abelian varieties provide a decomposition of $A_{n+2}(H)$ for any subgroup H with $G \subset H \subset G_1$. Further in the special case $G_1 = \Gamma(1)$, $G=\Gamma_1(q)$, one can define Hecke operators τ_r (for r prime to q) as endomorphisms of these abelian varieties.

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It was noticed by the author, after the preparation of the manuscript that Gunning has also proved Theorem 1 in [**2**], but however our proof is different.

NOTATIONS.

$$\Gamma(1) = SL(2, \boldsymbol{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a,\ b,\ c,\ d \text{ integral and } ad-bc=1 \right\}$$

$\Gamma_0(q)(\subset\Gamma(1))=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma(1)\text{ with } c\equiv 0 \pmod{q}\right\}$ for q, a prime.

$\Gamma_1(q)(\subset\Gamma(1))=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma(1)\text{ with } \begin{pmatrix} a & b \\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \pmod{q}\right\}$. The tensor representation of $GL(2, \boldsymbol{C})$ is defined as follows: If $\begin{pmatrix} u \\ v\end{pmatrix}\in\boldsymbol{C}^2$ and $\sigma\in GL(2,\boldsymbol{C})$, denote by $\begin{pmatrix} u_1 \\ v_1\end{pmatrix}=\sigma\begin{pmatrix} u \\ v\end{pmatrix}$. Then if $\begin{pmatrix} u \\ v\end{pmatrix}^n$ and $\begin{pmatrix} u_1 \\ v_1\end{pmatrix}^n$ denote respectively the vectors in $\boldsymbol{C}^{n+1}$ with components $u^n, u^{n-1}v, \cdots, v^n$ and $u_1^n, u_1^{n-1}v_1, \cdots, v_1^n$, the tensor representation $\sigma\to M_n(\sigma)$ of degree n of $GL(2, \boldsymbol{C})$ is defined by $\begin{pmatrix} u_1 \\ v_1\end{pmatrix}^n=M_n(\sigma)\begin{pmatrix} u \\ v\end{pmatrix}^n$.

For simplicity, we denote $M_n\left(\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\right)$ by $L_n(z)$ for any complex variable z.

If s is a parabolic fixed point (cusp) of a discontinuous group G on the upper half plane $\mathfrak{X}$, the set of elements of G fixing s is an infinite cyclic group generated by $\tau\in G$ where $\tau=\rho\begin{pmatrix} 1 & h \\ 0 & 1\end{pmatrix}\rho^{-1}$ with ρ, an element of $SL(2,\boldsymbol{R})$ such that $\rho(\infty)=s$ and in fact $\rho=\begin{pmatrix} -s & 1 \\ -1 & 0\end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ according as s is real or ∞, and h is a positive real number. [We then denote $e^{2\pi iz/h}$ by q.] The set of all such parabolic transformations of G, i.e. $(\sigma\in G; \sigma(s)=s$ for a parabolic fixed point s of $G)$ is denoted by $Y(G)$.

§1. $\boldsymbol{R}\otimes\boldsymbol{M}_n$-forms and $\boldsymbol{R}\otimes\boldsymbol{M}_n$-vectors.

Let G be a discrete subgroup of $SL(2,\boldsymbol{R})$ such that $SL(2,\boldsymbol{R})/G$ has finite total volume. Let G_1 be another discrete subgroup of $SL(2,\boldsymbol{R})$ containing G (and in which G is normal and of finite index). Further, let $\sigma\to R(\sigma)$ be a real representation of the finite group G_1/G. If $\sigma\to M_n(\sigma)$ is the tensor representation of degree n of G_1, we shall be concerned with the representation $\sigma\to(R\otimes M_n)(\sigma)$ in the sequel. Restricted to the subgroup G, this is nothing but $M_n(\sigma)$ repeated m times, if m is the dimension of the representation $R(\sigma)$.

DEFINITION. A column vector of $(n+1)m$ elements $\omega=\begin{pmatrix}\omega_{01}\\ \vdots \\ \omega_{n1} \\ \vdots \\ \omega_{0m} \\ \vdots \\ \omega_{nm}\end{pmatrix}$ is an $R\otimes M_n$-**form** with respect to G_1, if the following conditions are satisfied.

a) Each component ω_{ik} is a meromorphic differential form on $\mathfrak{X}$.

b) For every $\sigma\in G_1$, $\omega\circ\sigma=(R\otimes M_n)(\sigma)\circ\omega$.

c) For every parabolic cusp s of G, the functions $f_{ij}(q)$ defined by the vector form

$$(E\otimes L_n(z))^{-1}(E\otimes M_n(\rho))^{-1}\omega\circ\rho=\begin{pmatrix} f_{01}(q)dq \\ \vdots \\ f_{n1}(q)dq \\ \vdots \\ f_{nm}(q)dq \end{pmatrix},$$

are meromorphic at $q=0$.

If they are holomorphic at $q=0$, and if ω_{ik} are holomorphic, we say that ω is a **cusp** $R\otimes M_n$**-form**.

One can define $R\otimes M_n$-vectors in a similar way.

DEFINITION. A column vector of $(n+1)m$ elements $\mathfrak{g}=\begin{pmatrix} g_{01} \\ \vdots \\ g_{0m} \\ \vdots \\ g_{nm} \end{pmatrix}$ is an $R\otimes M_n$-**vector** with respect to G_1, if it satisfies the following conditions.

a) Each component g_{ik} is a meromorphic function on $\mathfrak{X}$.

b) For every $\sigma\in G_1$, we have $\boldsymbol{g}\circ\sigma=(R\otimes M_n)(\sigma)\boldsymbol{g}$.

c) For every parabolic cusp s of G, the functions $F_{ij}(q)$ defined by the vector

$$(E\otimes L_n(z))^{-1}(E\otimes M_n(\rho))^{-1}\boldsymbol{g}\circ\rho=\begin{pmatrix} F_{01}(q) \\ \vdots \\ F_{n1}(q) \\ \vdots \\ F_{nm}(q) \end{pmatrix}$$

are meromorphic at $q=0$.

If the components g_{ik} are holomorphic and if the above defined functions $F_{ij}(q)$ are holomorphic and vanish at $q=0$, then g is defined to be a **cusp** $R\otimes M_n$**-vector**. We now deduce the following analogue of Theorem 1 in [**5**].

PROPOSITION 1. *Let n and ν be even, $n>0$, $-(n-2)\leqq\nu\leqq n+2$ and $\mu=\dfrac{n+2-\nu}{2}$. Then, if (f_i) is a vector whose components are automorphic forms of degree ν with respect to G with the property $((f_i)\circ\sigma)J(\sigma,z)^{\nu}=R(\sigma)(f_i)$ for $\sigma\in G_1$ (if $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $J(\sigma,z)=(cz+d)^{-1}$), then the vector form $\omega=(E\otimes L_n(z))\begin{pmatrix} \mathfrak{g}_1 \\ \vdots \\ \mathfrak{g}_m \end{pmatrix}dz$ (where each $\mathfrak{g}_i$ is an $(n+1)$ vector defined by $\mathfrak{g}_i=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_0 f_i \\ \vdots \\ \alpha_\mu f_i^{(\mu)} \end{pmatrix}$ with certain constants α_i and $f_i', f_i'', \cdots, f_i^{(\mu)}$ denote $\dfrac{df_i}{dz}, \cdots, \dfrac{d^\mu f_i}{dz^\mu}$) is an $R\otimes M_n$-form with respect to G_1. In order that ω be a cusp $R\otimes M_n$-form, it is necessary and sufficient that the f_i are cusp forms of degree ν, with respect to G.*

PROOF. From Theorem 1 of [**5**], we have, for elements $\sigma\in G$, $\omega\circ\sigma$

$=(E\otimes M_n)(\sigma)\omega$. We need consider only $\sigma\in G_1$ and $\notin G$. Then

$$\omega\circ\sigma=(E\otimes L_n(z)\begin{pmatrix}\mathfrak{g}_1\\ \vdots\\ \mathfrak{g}_m\end{pmatrix}\cdot dz)\circ\sigma=(E\otimes L_n(\sigma(z))\begin{pmatrix}\mathfrak{g}_1\circ\sigma\\ \vdots\\ \mathfrak{g}_m\circ\sigma\end{pmatrix}\cdot J^2dz)$$

(here $J=J(\sigma,z)$).
We now require the following lemma:

LEMMA. *If* $\boldsymbol{f}=\begin{pmatrix}\mathfrak{g}_1\\ \vdots\\ \mathfrak{g}_m\end{pmatrix}$ *(as in Proposition* 1) *and if* $\omega=(E\otimes L_n(z))\boldsymbol{f}dz$, *then* $\omega\circ\sigma=(R\otimes M_n)(\sigma)\omega$ *for* $\sigma\in G_1$ *if and only if*

$$(\boldsymbol{f}\circ\sigma)J^2=R(\sigma)\otimes M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)\boldsymbol{f}.$$

PROOF. From the relation $L_n(\sigma(z))^{-1}M_n(\sigma)L_n(z)=M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)$ by tensoring with $R(\sigma)$, we have

$$(E\otimes L_n(\sigma(z))^{-1})(R(\sigma)\otimes M_n(\sigma))(E\otimes L_n(z))=R(\sigma)\otimes M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)$$

and this gives the required.

For proving the proposition, in view of the lemma, we need verify only the following:

$$(\boldsymbol{g}_i\circ\sigma)J^2=M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)\sum_{j=1}^m r_{ij}\boldsymbol{g}_j=\sum_{j=1}^m r_{ij}M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)\cdot\boldsymbol{g}_j$$

where $R(\sigma)=(r_{ij})$.

For automorphic forms h_i ($1\leqq i\leqq m, m=\dim R(\sigma)$) of degree ν with respect to G, satisfying the relation, $(h_i\circ\sigma)(J(\sigma,z))^\nu=\sum_{j=1}^m r_{ij}h_j$ (for $\sigma\in G_1$), holds the identity:

$$(h_i^{(k)}\circ\sigma)J^2=\sum_{j=1}^m r_{ij}\sum_{l=0}^k\binom{k}{l}\binom{\nu+k-1}{l}l!\,c^lJ^{l+2-2k-\nu}h_j^{(k-l)}$$

for $\sigma\in G_1$. (The proof is by induction.) Using this identity and computing $M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)$ explicitly [**5**], we obtain the required relation and the proof of Proposition 1 is complete.

We have then an analogous result for cusp $R\otimes M_n$ vectors as well. Now if for a vector (f_i) of automorphic forms of degree ν with respect to G with the property that $((f_i)\circ\sigma)J^\nu=R(\sigma)(f_i)$ for $\sigma\in G_1$, we denote by ω and $\boldsymbol{f}$, the associated cusp $R\otimes M_n$-form and $R\otimes M_n$-vector respectively, then by Theorem 5 in [**5**], we have $d\boldsymbol{f}=\mu(n-\mu+1)\omega$.

If we denote by $\mathfrak{J}_{n,R}(G_1)$ the space of all cusp $R\otimes M_n$-forms, with respect to G_1, we have the following analogue of Theorem 2 in [**5**].

PROPOSITION 2. $\mathfrak{J}_{n,R}(G_1)=\sum_{\nu=2}^{n+2}\mathfrak{S}_{\nu,R}^n(G_1)$ *(ν even) where* $\mathfrak{S}_{\nu,R}^n(G_1)$ *is the space of* cusp $R\otimes M_n$ *forms associated to the space of vectors* (f_i) *of automorphic cusp forms of degree* ν *with respect to* G, *as in Proposition* 1.

PROOF: Denote by $S_{\nu,R}(G_1)$, the space of vectors (f_i) of automorphic cusp forms of degree ν with respect to G and such that $((f_i)\circ\sigma)J^{\nu}=R(\sigma)(f_i)$. Then, from Proposition (1), $S_{\nu,R}(G_1)$ is canonically isomorphic to $\mathfrak{S}^n_{\nu,R}(G_1)$ by the mapping $(f_i)\rightarrow\omega$.

Now, we have $\sum_{\nu=2}^{n+2}\mathfrak{S}^n_{\nu,R}(G_1)\subset\mathfrak{Z}_{n,R}(G_1)$. Conversely, from Theorem 2 in [5], we deduce that any vector in $\mathfrak{Z}_{n,R}(G_1)$ can be written as a sum of vectors of the form $\begin{pmatrix}\mathfrak{g}_1\\ \vdots\\ \mathfrak{g}_m\end{pmatrix}$ ($\mathfrak{g}_i$ again as defined in Proposition 1). We need only show that these summands belong to $\mathfrak{S}^n_{\nu,R}(G_1)$ respectively.

If
$$\begin{aligned}\omega=\sum_{\nu=2}^{n+2}\omega_\nu,\ \omega\circ\sigma=\sum_{\nu=2}^{n+2}\omega_\nu\circ\sigma&=(R\otimes M_n)(\sigma)\omega\\ &=(R\otimes M_n)(\sigma)(\sum_{\nu=2}^{n+2}\omega_\nu)\\ &=\sum_{\nu}(R\otimes M_n)(\sigma)\omega_\nu\end{aligned}$$

i.e. $\sum_{\nu=2}^{n+2}(\omega_\nu\circ\sigma-(R\otimes M_n)(\sigma)\omega_\nu)=0$ and this sum being a direct sum, $\omega_\nu\circ\sigma=(R\otimes M_n)(\sigma)\omega_\nu$ or $\omega_\nu\in\mathfrak{S}^n_{\nu,R}(G_1)$ for $\nu=2,4,\cdots,n+2$.

Similarly, we can obtain the decomposition of the space of cusp $R\otimes M_n$-vectors.

NOTE. If R is irreducible and if κ_ν denotes the multiplicity of the irreducible representation R in the representation of the group G_1/G in the space of cusp forms of degree ν with respect to G, then $S_{\nu,R}(G_1)$ and hence $\mathfrak{S}_{\nu,R}(G_1)$ is a complex vector space of dimension κ_ν. This can be computed explicitly and hence $\dim_C\mathfrak{Z}_{n,R}(G_1)=\sum_{\nu=2}^{n+2}\kappa_\nu$ can be computed.

§ 2. Cohomology group.

We may now define the cohomology group $H^1(R\otimes M_n, G_1)$. We call $\mathfrak{x}$, a **parabolic cocycle,** a map $\mathfrak{x}:G_1\rightarrow\boldsymbol{R}^k$ $(k=(n+1)m)$ with the following properties. (We shall denote hereafter $R\otimes M_n$ by M)

a) $\mathfrak{x}(\sigma\tau)=\mathfrak{x}(\sigma)+M(\sigma)\mathfrak{x}(\tau)$ for every $\sigma,\tau\in G_1$.

b) For each $\tau\in Y(G_1)$, there exists a vector $\mathfrak{a}\in\boldsymbol{R}^k$ with $\mathfrak{x}(\tau)=\mathfrak{a}-M(\tau)\cdot\mathfrak{a}$.

We denote by $Z^1(M,G_1)$, the parabolic cocycles and by $B^1(M,G_1)$ the coboundaries, i.e. cocycles $\mathfrak{x}\in Z^1(M,G_1)$ with the property that, for all $\sigma\in G_1$, $\mathfrak{x}(\sigma)=\mathfrak{b}-M(\sigma)\cdot\mathfrak{b}$ (for some $\mathfrak{b}$). The space $Z^1(M,G_1)/B^1(M,G_1)$ shall be denoted by $H^1(M,G_1)$.

Now, every cocycle $\mathfrak{x}$ of G_1 when restricted to G gives a cocycle of G and in fact a parabolic cocycle of G_1 gives rise to a parabolic cocycle of G, since $Y(G)\subset Y(G_1)$. So, we have a map: $Z^1(M, G_1)\rightarrow Z^1(M, G)$ in which $B^1(M, G_1)$ goes to $B^1(M, G)$ so that we have a map: $H^1(M, G_1)\rightarrow H^1(M, G)$. It can then be shown that this is injective; for, choose a system of coset representatives τ_i of G_1 modulo G. Then, if $\mathfrak{x}\in Z^1(M, G_1)$ and in $B^1(M, G)$, i.e. if $\mathfrak{x}(\sigma)=M(\sigma)\cdot\mathfrak{a}-\mathfrak{a}$ for $\sigma\in G$ and $\mathfrak{a}\in \boldsymbol{R}^k$, it follows that $\mathfrak{x}(\sigma_1)=M(\sigma_1)\cdot\mathfrak{b}-\mathfrak{b}$, for every $\sigma_1\in G_1$ and $\mathfrak{b}=\dfrac{1}{(G_1:G)}[\sum_i M(\tau_i)\mathfrak{x}(\tau_i^{-1})+\sum_i M(\tau_i)\cdot\mathfrak{a}]$. In other words, $\mathfrak{x}\in B^1(M, G_1)$.

§ 3. Periods of Integrals.

Let $\omega\in\mathfrak{F}_{n,R}(G_1)$. Then, with a fixed point $z_0\in\mathfrak{X}$, set $\boldsymbol{f}(z)=\int_{z_0}^{z}Re(\omega)$. We have then $\boldsymbol{f}(\sigma(z))=M(\sigma)\boldsymbol{f}(z)+\mathfrak{x}(\sigma)$ where $\mathfrak{x}$ is a cocycle of G_1 (§ 2). $\mathfrak{x}$ is in fact, a parabolic cocycle of G_1; for the same, we note that it is enough to prove that $\underset{\substack{z\rightarrow s_1\\ in\ \mathfrak{F}_1}}{Lt}\int_{z_0}^{z}Re\omega<\infty$ where s_1 is any parabolic cusp of G_1 and $\mathfrak{F}_1$ is a fundamental domain of G_1 in $\mathfrak{X}$. We can then denote this limit by $\boldsymbol{f}(s_1)$ and if $\tau\in Y(G_1)$ fixes s_1, $\mathfrak{x}(\tau)=(E-M(\tau))\cdot\boldsymbol{f}(s_1)$ and hence $\mathfrak{x}$ is a parabolic cocycle.

Now, if $\omega=(\omega_i)$ $(1\leqq i\leqq m)$ with each $\omega_i\in\mathfrak{F}_n(G)$ we know from condition c) of the definition in § 1, that $\underset{\substack{z\rightarrow s\\ in\ \mathfrak{F}}}{Lt}\int_{z_0}^{z}Re(\omega_i)<\infty$ for every parabolic cusp s of G and $\mathfrak{F}$ is a fundamental domain of G in $\mathfrak{X}$. Since $\mathfrak{F}_1\subset\mathfrak{F}$ and the inequivalent cusps of G_1 are contained in the inequivalent cusps of G, we have the required. This parabolic cocycle $\mathfrak{x}$ is determined only upto a coboundary, for, if we change $\boldsymbol{f}(z)$ by an additive constant, $\mathfrak{x}(\sigma)$ changes by a coboundary. Hence to every vector form ω, we have associated the class $\bar{\mathfrak{x}}\in H^1(M, G_1)$ in a unique manner. We shall show that this map $\varphi:\omega\rightarrow\bar{\mathfrak{x}}$ is surjective i.e. for every class $\bar{\mathfrak{x}}\in H^1(M, G_1)$, there exists $\omega\in\mathfrak{F}_{n,R}(G_1)$ such that $\varphi(\omega)=\bar{\mathfrak{x}}$. Now, $\bar{\mathfrak{x}}$ induces a class $\iota(\bar{\mathfrak{x}})\in H^1(M, G)$ and since $H^1(M, G)=\sum_{i=1}^{m}H^1(M_n, G)$ (m copies), to the class $\iota(\bar{\mathfrak{x}})$ by Theorem 1 in [6] corresponds a vector (f_i) of cusp forms of degree $n+2$ with respect to G, i.e. $f_i\in S_{n+2}(G)$. We shall show that $(f_i)\in S_{n+2,R}(G_1)$ so that the associated vector form ω (from Proposition (1)) is in $\mathfrak{F}_{n,R}(G_1)$ with $\varphi(\omega)=\bar{\mathfrak{x}}$.

If ω_i is the vector form in $\mathfrak{F}_n(G)$ [5] associated to $f_i\in S_{n+2}(G)$, then $\omega=(\omega_i)$ $(1\leqq i\leqq m)$. Consider now the vectors $\eta=(E\otimes M_n(\tau^{-1}))\omega\circ\tau$ and $\eta^*=(R(\tau)\otimes E)\cdot\omega$, with $\tau\in G_1$. If $\eta=(\eta_i)$ and $\eta^*=(\eta_i^*)$ $(1\leqq i\leqq m)$, then η_i, $\eta_i^*\in\mathfrak{F}_n(G)$, for, $\eta_i\circ\sigma=M_n(\tau^{-1})\omega_i\circ\tau\sigma=M_n(\tau^{-1})M_n(\tau\sigma\tau^{-1})\omega_i\circ\tau=M_n(\sigma)\cdot\eta_i$ and $\eta^*\circ\sigma=(R(\tau)\otimes E)(E\otimes M_n(\sigma))\omega=(E\otimes M_n(\sigma))(R(\tau)\otimes E)\omega$ implies that $\eta_i^*\circ\sigma=M_n(\sigma)\eta_i^*$.

If $\bar{x}_i$, $\bar{y}_i$ and $\bar{y}_i^*$ denote the cohomology classes in $H^1(M_n, G)$ attached to the vector forms ω_i, η_i and η_i^* respectively, denote by $\bar{x}=(\bar{x}_i)$, $\bar{y}=(\bar{y}_i)$ and $\bar{y}^*=(\bar{y}_i^*)$ $(1\leqq i\leqq m)$. Then, from the definition, it follows that $\bar{y}(\sigma)=(E\otimes M_n(\tau^{-1}))\bar{x}(\tau\sigma\tau^{-1})$

and $\bar{y}^*(\sigma)=(R(\tau)\otimes E)\bar{x}(\sigma)$. We shall now prove that $\bar{y}(\sigma)=\bar{y}^*(\sigma)$ for every $\sigma\in G$, for,

$$x(\tau\sigma\tau^{-1})=(R\otimes M_n)(\tau)x(\sigma\tau^{-1})+x(\tau)$$
$$=(R\otimes M_n)(\tau)[(E\otimes M_n(\sigma))x(\tau^{-1})+x(\sigma)]+x(\tau)$$

so that $y(\sigma)-y^*(\sigma)$ is cohomologous to

$$(E\otimes M_n(\tau^{-1}))x(\tau\sigma\tau^{-1})-(R(\tau)\otimes E)x(\sigma)$$
$$=(R(\tau)\otimes M_n(\sigma))x(\tau^{-1})+(E\otimes M_n(\tau^{-1}))x(\tau)$$
$$=(E\otimes M_n(\sigma)-E)(R(\tau)\otimes E)x(\tau^{-1})=(E-E\otimes M_n(\sigma))\cdot\mathfrak{b}$$

where $\mathfrak{b}=-(R(\tau)\otimes E)x(\tau^{-1})$. In other words $\bar{y}(\sigma)=\bar{y}^*(\sigma)$. From Theorem 6 in [5], this means that the vector forms $\eta_i-\eta_i^*$ lie in $\mathfrak{S}_\nu^n(G)$ for $\nu<n+2$. But, by definition they lie in $\mathfrak{S}_{n+2}^n(G)$ and since these spaces are orthogonal, $\eta_i=\eta_i^*$ or $\eta=\eta^*$ in other words $\omega\circ\tau=(R(\tau)\otimes M_n(\tau))\omega$ or $\omega\in\mathfrak{J}_{n,R}(G_1)$, and in fact $\omega\in\mathfrak{S}_{n+2,R}^n(G_1)$. If $\bar{\mathfrak{x}}_1=\varphi(\omega)\in H^1(M,G_1)$, $\iota(\bar{\mathfrak{x}}_1)=\bar{x}=\iota(\bar{\mathfrak{x}})$ and ι being injective (§ 2), $\bar{\mathfrak{x}}_1=\bar{\mathfrak{x}}$.

From the decomposition of $\mathfrak{J}_{n,R}(G_1)$ in Proposition 2 and from the fact that for $\nu<n+2$, $\omega\in\mathfrak{S}_{\nu,R}^n(G_1)$ are exact differentials ($\omega=d\boldsymbol{f}$ for a cusp $R\otimes M_n$ vector $\boldsymbol{f}$) we have $\bar{\mathfrak{x}}=0$ for classes $\bar{\mathfrak{x}}=\varphi(\omega)$. Hence we have in fact a surjective homomorphism $\varphi:S_{n+2,R}(G_1)\to H^1(M,G_1)$. We shall prove later in § 4, that φ is also one-one, so that φ will then be an isomorphism. We have then

THEOREM 1. *The homomorphism $\varphi:S_{n+2,R}(G_1)\to H^1(M,G_1)$ is an isomorphism.*

If R is irreducible and if κ is the multiplicity of the representation R in the representation of G_1/G in $S_{n+2}(G)$, then from Theorem 1, *we have* $\dim_R H^1(M,G_1)=2\kappa$.

From Theorem 1, we can further deduce the following

PROPOSITION 3. *If $\mathfrak{N}_{n,R}(G_1)$ denotes the space of form vectors in $\mathfrak{J}_{n,R}(G_1)$ whose associated cocycles are coboundaries, then $\mathfrak{J}_{n,R}(G_1)/\mathfrak{N}_{n,R}(G_1)$ is canonically isomorphic to $S_{n+2,R}(G_1)$.*

PROOF. We need only to show that $\mathfrak{N}_{n,R}(G_1)$ is isomorphic to $\sum_{\nu=2}^{n}S_{\nu,R}(G_1)$, for, then from Proposition 2, it would follow that $\mathfrak{J}_{n,R}(G_1)/\mathfrak{N}_{n,R}(G_1)$ is canonically isomorphic to $S_{n+2,R}(G_1)$. Now if $\omega\in\mathfrak{J}_{n,R}(G_1)$ with $\varphi(\omega)=0$, then from Theorem 1, in the decomposition (as in Proposition 2) of ω, the $(n+2)^{\text{th}}$ component is zero, so that $\mathfrak{N}_{n,R}(G_1)\subset\sum_{\nu=2}^{n}\mathfrak{S}_{\nu,R}^n(G_1)$. But $\sum_{\nu=2}^{n}\mathfrak{S}_{\nu,R}^n(G_1)\subset\mathfrak{N}_{n,R}(G_1)$, since $\omega\in\sum_{\nu=2}^{n}\mathfrak{S}_{\nu,R}^n(G_1)$ implies that $\omega=c.\,d\boldsymbol{f}$ with a non zero constant c and a cusp $R\otimes M_n$-vector $\boldsymbol{f}$. Hence $\mathfrak{N}_{n,R}(G_1)=\sum_{\nu=2}^{n}\mathfrak{S}_{\nu,R}^n(G_1)$ which in turn is canonically isomorphic to $\sum_{\nu=2}^{n}S_{\nu,R}(G_1)$.

§ 4. Petersson Metric.

We observe that there exists a positive symmetric matrix H with the property that $R(\sigma)'HR(\sigma)=H$ for all $\sigma\in G_1$. (We can take for example $H=\sum_{\bar\sigma\in\bar G_1/G} R(\bar\sigma)'R(\bar\sigma)$). We have further a matrix P_n with $M_n(\sigma)'P_nM_n(\sigma)=P_n$ [**6**], so that we have if $M(\sigma)=(R\otimes M_n)(\sigma)$, $(M(\sigma))'(H\otimes P_n)M(\sigma)=H\otimes P_n$.

Now, if $f=(f_i)\in S_{n+2,R}(G_1)$ and $g=(g_i)\in S_{n+2,R}(G_1)$, we can define $(f,g)=\sum_{i,j}\int_{\mathfrak{Z}_1} f_i h_{ij}\bar g_j y^{n+2}dv$. Then $(f,g)=\overline{(g,f)}$ and $(f,f)\geqq 0$ and $=0$ if and only if $f=0$, since H is positive definite.

On the otherhand, if ω and η are the vector forms in $\mathfrak{Z}_{n,R}(G_1)$ associated to f and g respectively, we have $\omega'\cdot H\otimes P_n\circ\eta=-(2i)^{n+1}\sum_{i,j} f_i\cdot h_{ij}\bar g_j y^{n+2}dv$ so that if we define as in [**6**], $A(f,g)=2^{n-1}i[(f,g)-(g,f)]$, then (f,g) is skew symmetric R-bilinear and $A(f,if)$ is positive definite hermitian. Further one sees that

$$A(f,g)=(-1)^{n/2+1}\int_{\mathfrak{Z}_1}(Re\omega)'(H\otimes P_n)(Re\eta)\,.$$

If $\boldsymbol{f}(z)=\int_{z_0}^{z} Re\omega$ and $\mathfrak{g}(z)=\int_{z_0}^{z} Re(\eta)$, then we have $A(f,g)=(-1)^{n/2+1}\int_{\partial\mathfrak{Z}_1}\boldsymbol{f}'(H\otimes P_n)d\mathfrak{g}$ and from (19) of [**6**] this can be expressed in terms of the parabolic cocycles x and y associated to ω and η.

We can now prove that $\varphi: S_{n+2,R}(G_1)\to H^1(M,G_1)$ is one-one, for, if $f\in S_{n+2,R}(G_1)$ whose associated class is zero, we can choose $\boldsymbol{f}$ such that the parabolic cocycle itself is zero, which means that $A(f,g)=0$ for every $g\in S_{n+2,R}(G_1)$ and in particular, $A(f,if)=0$, but this implies that $f=0$.

§ 5. Abelian varieties attached to $S_{n+2,R}(G_1)$.

For defining abelian varieties associated with the representation $M=R\otimes M_n$, we assume that G_1 satisfies the integrality assumption (A) of [**6**], namely that there exists a non-singular real matrix U such that $U'^{-1}P_nU^{-1}$ and $UM_n(\sigma)U^{-1}$ are integral for all $\sigma\in G_1$. We may assume without loss of generality that P_n and $M_n(\sigma)$ are integral for all $\sigma\in G_1$ (for, if $\boldsymbol{f}$ is an M_n-form, $U\boldsymbol{f}$ is an $UM_n(\sigma)U^{-1}$-form). For example, this is satisfied if $G_1\subset SL(2,\boldsymbol{Z})$. We shall further assume that $R(\sigma)$ is rational for all $\sigma\in G_1$. Then $R(\sigma)$ being the representation of a finite group, has an equivalent representation $R_0(\sigma)$ with integral elements [7]. On taking R to be this R_0 we have $(R\otimes M_n)(\sigma)$ integral for all $\sigma\in G_1$.

Under this hypothesis, we define integral cocycles and we denote the group of parabolic integral cocycles as $\tilde{Z}^1(M,G_1)$ and the integral coboundaries as

$\tilde{B}^1(M, G_1)$. Then the group $\tilde{Z}^1/\tilde{B}^1=\tilde{H}^1(M, G_1)$ is a lattice in $H^1(M, G_1)$ of maximal rank. Under the isomorphism $\varphi: S_{n+2,R}(G_1)\to H^1(M, G_1)$ the inverse image $\varphi^{-1}(\tilde{H}^1(M, G_1))$ is a lattice in $S_{n+2,R}(G_1)$ and from (19) of [**6**], the Petersson metric takes rational values for form vectors in this lattice so that $\lambda A(f, g)$ (for a constant λ) gives a Riemann form on this torus and hence it is an abelian variety, which we denote by $A_{n+2,R}(G_1)$. From Theorem 1, we see that the dimension of this abelian variety is κ, where κ is the sum of multiplicities κ_i of the irreducible representations R_i (contained in R) in the representation of G_1/G by cusp forms of degree $n+2$ with respect to G.

§ 6. Applications.

We shall obtain in this section, a decomposition of the abelian varieties $A_{m'}(H)$ associated with an even integer m' and a subgroup H with $G\subset H\subset G_1$ in terms of the abelian varieties $A_{m',R}(G_1)$ of § 5.

We have now the following relation between induced characters of subgroups and rational characters namely, that if $G\subset H\subset G_1$ and if ψ_1 denotes the identity character of H and χ_{ψ_1}, the induced character of G_1/G, then $\chi_{\psi_1}=\sum_{j=1}^{t} c_j\chi_j=\sum_{i=1}^{s} c_i\Xi_i$, where Ξ_i are rational characters (composed of conjugate characters χ_j) and c_i, non-negative integers, and in fact, the same is true of the induced representation $R_{\chi_{\psi_1}}$, namely that it is equivalent to a direct sum of the rational representations R_{Ξ_i} each with multiplicity c_i.

We have then the following decomposition of the cohomology groups; $H^1(R_{\chi_{\psi_1}}, G_1)=\sum_{i=1}^{s} c_i H^1(R_{\Xi_i}, G_1)$ and the same holds good also for the lattices, so that we have an isogeny

$$H^1(R_{\chi_{\psi_1}}, G_1)/\tilde{H}^1(R_{\chi_{\psi_1}}, G_1)\cong\prod_{i=1}^{s}(A_{m',R_{\Xi_i}}(G_1))^{c_i}$$

(meaning thereby c_i copies of $A_{m',R_{\Xi_i}}(G_1)$).

We shall see that $H^1(R_{\chi_{\psi_1}}, G_1)$ and $H^1(R_{\psi_1}, H)$ are isomorphic and the same holds for the lattices, so that it would follow from the above that there is an isogeny

$$A_{m'}(H)\cong\prod_{i=1}^{s}(A_{m',R_{\Xi_i}}(G_1))^{c_i}.$$

PROPOSITION 4: *$H^1(R_{\chi_{\psi_1}}, G_1)$ and $H^1(R_{\psi_1}, H)$ are isomorphic.*

PROOF: From the Theorem 1, there corresponds to a class $\bar{x}\in H^1(R_{\psi_1}, H)$ an automorphic form f of degree m' belonging to H. Let $G_1=\bigcup_{i=1}^{p} H\sigma_i$ be a coset decomposition of G_1 modulo H. Then the vector of forms $(f\circ\sigma_i)J(\sigma_i, z)^{m'}$ belongs to the induced representation $R_{\chi_{\psi_1}}$ so that it corresponds to a class $\bar{y}\in H^1(R_{\chi_{\psi_1}}, G_1)$.

This is a monomorphism, for if $\bar{y}=0$, then from the isomorphism theorem, $f\circ\sigma_i=0$ which implies $f=0$ or $\bar{x}=0$. We shall prove that it is an epimorphism by showing that they are of the same dimension. Now, from $\chi_{\psi_1}=\sum_{i=1}^{s} c_i \Xi_i$ we have

$$\dim_{\boldsymbol{R}} H^1(R_{\chi_{\psi_1}}, G_1)=\sum_{i=1}^{s} c_i\cdot \dim_{\boldsymbol{R}} H^1(R_{\Xi_i}, G_1)$$

$$=\sum_{i=1}^{s} c_i 2\kappa_i \text{ where } \kappa_i \text{ is the sum of}$$

multiplicities ρ_j of the primitive characters χ_j (contained in Ξ_i) in the representation M of G_1/G by $S_{m'}(G)$. If μ is the character of M, then $\mu=\sum_{j=1}^{t}\rho_j\chi_j$ and $\kappa_i=\sum_{\chi_j\subset\Xi_i}\rho_j$. Let $\chi_j/H=\sum_{k=1}^{l}\lambda_{jk}\psi_k$, where ψ_k are all the primitive characters of H/G and $\psi_1=1$, so that $\mu/H=\sum_{j=1}^{t}\rho_j\chi_j/H=\sum_{j=1}^{t}\rho_j(\sum_{k=1}^{l}\lambda_{jk}\psi_k)$. Now, $\dim_{\boldsymbol{R}} H^1(R_{\psi_1}, H)$ $=2$ (multiplicity of 1 in $\mu/H)=2\sum_{j=1}^{t}\rho_j\lambda_{j1}$, and $\lambda_{j1}=$ multiplicity of ψ_1 in χ_j/H $=$ multiplicity of χ_j in $\chi_{\psi_1}=c_j$ and is the same for all conjugate χ_j. Hence

$$\dim_{\boldsymbol{R}} H^1(R_{\psi_1}, H)=2\sum_{j=1}^{t}\rho_j\lambda_{j1}=2\sum_{i=1}^{s}c_i(\sum_{\chi_j\subset\Xi_i}\rho_j)$$

$$=2\sum_{i=1}^{s}c_i\kappa_i$$

$$=\dim_{\boldsymbol{R}} H^1(R_{\chi_{\psi_1}}, G_1)$$

COROLLARY 1. 1) *If $H=G$, then $c_i=\chi_i(1)$ so that there is an isogeny*

$$A_{m'}(G)\cong\prod_{i=1}^{s}(A_{m',R_{\Xi_i}}(G_1))^{\chi_i(1)}.$$

When $m'=2$, $G_1=\Gamma(1)$, $G=\Gamma_1(7)$, we have $s=1$ and $\chi(1)=3$, so that $A_2(G)$ is isogenous to a product of three copies of the elliptic curve corresponding to $Q(\sqrt{-7})$.

2) *In the case $G=\Gamma_1(q)$, $H=\Gamma_0(q)$, $G_1=\Gamma(1)$ we have $\chi_{\psi_1}=\chi_1+\chi_q$, χ_q being the character of the q-dimensional representation of $\Gamma(1)/\Gamma_1(q)$. Then there is an isogeny:*

$$A_{m'}(\Gamma_0(q))\cong A_{m'}(\Gamma(1))\times A_{m',R_{\chi_q}}(\Gamma(1)).$$

When $m'=2,4,6,8,10$, $A_{m'}(\Gamma(1))=0$, so that $A_{m'}(\Gamma_0(q))\cong A_{m',R_{\chi_q}}(\Gamma(1))$ and for $q=11,17,19$, they are elliptic curves without complex multiplications [**4**].

NOTE. If H/G is a cyclic subgroup of order t, generated by $\rho\in G_1/G$ then in the decomposition,

$$\chi_{\psi_1}=\sum_{i=1}^{s}c_i\Xi_i,\ c_i=\frac{1}{tp_i}\sum_{\nu=1}^{t}\Xi_i(\rho^\nu)$$

where p_i is the order of the primitive characters contained in Ξ_i.

§ 7. Examples.

In the following, we shall restrict our attention to the case $G_1=\Gamma(1)$ and $G=\Gamma_1(q)$. Then the absolutely irreducible representations of G_1/G are of dimensions 1, q, $\frac{q+1}{2}$, $\frac{q-1}{2}$, $q+1$ and $q-1$. All of them are real except those of dimension $\frac{q-1}{2}$ (when $q\equiv 3 \pmod 4$) in which case the two complex representations are conjugates [3].

There is only one representation of dimension 1 and only one of dimension q and both are rational. The representations of dimension $\frac{q+1}{2}$ are 2 in number, which are conjugate to each other over $Q(\sqrt{q})$ so that the direct sum of these two representations is rational. The representations of dimension $\frac{q-1}{2}$ (when $q\equiv 3 \pmod 4$) are conjugates over $Q(\sqrt{-q})$ and their direct sum is again rational. About dimension $q+1$, for every divisor $t\big/\frac{q-1}{2}$ $(t>2)$ there are $\frac{1}{2}\varphi(t)$ conjugate representations over the real field $Q(\rho+\rho^{-1})$ (ρ being a primitive t^{th} root of unity) so that the direct sum of these is again a rational representation. The same is true of dimension $q-1$, but t runs over divisors of $\frac{q+1}{2}$ $(t>2)$.

In all the above mentioned cases, associated with these rational representations, we obtain abelian varieties $A_{m',R}(\Gamma(1))$ of the appropriate dimension. In the case $m'=2$, these have been indicated by Hecke [4].

§ 8. Endomorphisms of the abelian varieties $A_{n+2,R}(G_1)$.

We shall continue to consider the case when $G_1=\Gamma(1)$ and $G=\Gamma_1(q)$. Then every element $\tau\in G_1$ induces an endomorphism of $A_{n+2,R}(G_1)$ as follows: If $\bar{x}\in H^1(M,G_1)$, we define $\bar{y}=\bar{x}^{\tau}$ where $\bar{y}(\sigma)=M(\tau^{-1})\bar{x}(\tau\sigma\tau^{-1})$. It is easily seen that if $\bar{x}$ is associated to a vector $(f_i)\in S_{n+2,R}(G_1)$, then $\bar{y}$ is associated to $R(\tau^{-1})((f_i)\circ\tau)J(\tau,z)^{n+2}\in S_{n+2,R}(G_1)$. The map $\bar{x}\to\bar{y}$ takes $\tilde{H}^1(M,G_1)$ into itself so that τ induces an endomorphism of $A_{n+2,R}(G_1)$.

Now, we shall consider the Hecke operators. Let ρ be a (2, 2) integral matrix of determinant r prime to q. Then we can decompose $G\rho G=\bigcup_{\mu} G\rho_{\mu}$ where the representatives ρ_{μ} can be chosen in a canonical way.

We may then define, after Shimura [6], for $(f_i)\in S_{n+2,R}(G_1)$

$$(g_i)=((f_i)\cdot\tau_r)=r^{n+1}\sum_{\mu=1}^{s}(f_i(\rho_{\mu}(z)))J(\rho_{\mu},z)^{n+2} \qquad (i=1,\cdots,m).$$

It can then be shown that $g_i\in S_{n+2}(G)$, but $(g_i)\notin S_{n+2,R}(G_1)$. On the other hand, for $\sigma \in G_1$,

$$(g_i)\circ\sigma=(f_i)\circ\tau_r\sigma=(f_i)\circ\sigma_r\tau_r=R(\sigma_r)((f_i)\circ\tau_r)J(\sigma, z)^{-(n+2)}$$

where $\rho_\mu\sigma=\sigma_r\rho_{\kappa(\mu)}$ and $\sigma_r\in G_1$ is independent of μ and $\mu\to\kappa(\mu)$ is a permutation of $(1, \cdots, s)$.

Then, under our hypothesis on G, G_1 and R, it follows from [**3**] that $R(\sigma_r)$ is equivalent to $R(\sigma)$ i.e. $R(\sigma_r)=A_rR(\sigma)A_r^{-1}$ with A_r rational. If we denote by $(h_i)=B_r(f_i)\circ\tau_r$ where $B_r=\lambda A_r^{-1}$ is integral (for a suitable integer λ), and if x is a cocycle attached to (f_i) and y, to (h_i), it can be verified as in [**6**] that

$$y(\sigma)=r^n(\sum_\mu B_r\otimes M_n(\rho_\mu^{-1})x(\sigma_r))+t(\sigma),$$

where $$t(\sigma)=(M(\sigma)-E)\cdot\mathfrak{b} \quad \text{with} \quad \mathfrak{b}=r^n\sum_\mu(B_r\otimes M_n)\rho_\mu^{-1}(\boldsymbol{f}_i(\rho_\mu(z_0)))$$

($\boldsymbol{f}_i$ being the integral attached to x_i and z_0 is a fixed point of $\mathfrak{X}$), $t(\sigma)$ is a coboundary. Hence the map $\bar{x}\to\bar{y}$ gives an endomorphism of $A_{n+2,R}(G_1)$, since it takes $\widetilde{H}^1(M, G_1)$ into itself. Consequently, we have the following

PROPOSITION 5. *The characteristic roots of τ_r as an endomorphism of $A_{n+2,R}(G_1)$ are algebraic integers belonging to a field of degree $\leqq 2\kappa$ (where $\kappa=\dim A_{n+2,R}(G_1)$).*

One can also define the transpose endomorphism τ_r^* as in [**6**] and then show that τ_r and τ_r^* are conjugate with respect to the Riemann form and if $\tau_r=\tau_r^*$, the characteristic roots of τ_r are totally real and belong to a field of degree $\leqq\kappa$.

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