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On groups of automorphisms of algebraic varieties

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Let V be a complete, non-singular variety and let $G(V)$ be the group consisting of all the automorphisms of V. Then we can define two matrix representations $M^{(V)}$ and $S^{(V)}$ of $G(V)$. The representation $M^{(V)}$ is defined by means of the l-adic representation of the ring of endomorphisms of an Albanese variety attached to V (with a fixed rational prime l different from the characteristic of the universal domain) (cf. [1] and [4]). On the other hand, the representation $S^{(V)}$ is defined also by means of the matrix representation of linear transformations of the space of the linear differential forms of the first kind on V (with respect to a fixed basis of it) (cf. [1] and [3]). While the field of coefficients of $M^{(V)}$ is always of characteristic zero, the field of coefficients of $S^{(V)}$ is contained in the universal domain under consideration and so some difficulties occur for the study of $S^{(V)}$ in the case of positive characteristics.

The purpose of this paper is to give some informations about these two representations $M^{(V)}$ and $S^{(V)}$ (or, rather, the restrictions of them to a finite subgroup of $G(V)$). Since our results are well-known when the characteristic of the universal domain is equal to zero (cf. the remark in the section 2), we shall restrict ourselves to the case of positive characteristics. First we consider the case where $V=A$ is an abelian variety, which is the case of importance as seen later. In particular, it is shown that $S^{(A)}$ gives a faithful representation of a finite multiplicative group consisting of endomorphisms of A, provided its order is prime to the characteristic of the universal domain. Secondly, we show a relation between two representations $M^{(A)}$ and $S^{(A)}$, which is suggested by a classical result. In the last section, we apply these results to the study of the representations $M^{(V)}$ and $S^{(V)}$ for an arbitrary (complete, non-singular) variety V. When V is a curve of genus greater than one, our results are already known in a more explicit form, by the theory of algebraic functions of one variable.

1. Preliminaries.

First we explain the notations, which are used throughout this paper, and give the definitions of the representations $M^{(V)}$ and $S^{(V)}$. Let V be an algebraic variety, defined over a field of positive characteristic p. Let $G(V)$ be the group consisting of all the automorphisms of V, i.e. the everywhere biholomorphic, birational transformations of V onto itself. Let A be an Albanese variety attached to V and α a canonical mapping of V into A. Let $\mathcal{A}(A)$ be the ring of endomorphisms of A. Moreover, when V is complete and non-singular, we denote by $\mathfrak{D}_0(V)$ the space consisting of all the linear differential forms of the first kind on V.

For any element T of $G(V)$, there exists, by the universal mapping property of α, an element τ of $\mathcal{A}(A)$ such that we have $\alpha\circ T(v)=\tau\circ\alpha(v)+t$, where v is a generic point of V and t is a constant point on A. Then it is easily verified that the mapping $\varphi: T\to\varphi(T)=\tau$ is a (multiplicative group) homomorphism of $G(V)$ into the unit group $\mathcal{A}(A)\frown G(A)$ of $\mathcal{A}(A)$. Let M_l be the l-adic representation of $\mathcal{A}(A)$ with a rational prime $l\neq p$. Then, with a fixed l, associating T with the matrix $M_l(\tau)=M_l(\varphi(T))$, *we get the representation* $M^{(V)}=M_l\circ\varphi$[1] *of* $G(V)$. The representation $M^{(V)}$ is of degree $=2\dim A$ and has coefficients in the l-adic number field Q_l, which is of characteristic 0. It is to be noted that, if we consider the restriction of $M^{(V)}$ to a finite subgroup G of $G(V)$, then we get a representation of G which is independent of the choice of the prime $l\neq p$ up to equivalence, because the trace of the representation M_l $(l\neq p)$ is independent of l (cf. Weil [4]).

On the other hand, when V is complete and non-singular, any element T of $G(V)$ defines a linear transformation δT of the linear space $\mathfrak{D}_0(V)$ into itself in a well-known manner. Then each linear transformation δT is represented by a matrix $S'(\delta T)$ with respect to a fixed basis of $\mathfrak{D}_0(V)$. Since we have, clearly, $\delta(T_1\circ T_2)=\delta T_2\circ\delta T_1$ for all T_1, T_2 in $G(V)$, *we get the representation* $S^{(V)}$ *of* $G(V)$ by associating T with the transposed matrix of $S'(\delta T)$. The representation $S^{(V)}$ is of degree $=\dim\mathfrak{D}_0(V)$ and has coefficients in the universal domain, which is of characteristic p.

In the following, we shall mainly consider a finite subgroup G of $G(V)$. Restricting $M^{(V)}$ and $S^{(V)}$ to G, we get two representations of G. We denote these two representations also by the same symbols $M^{(V)}$ and $S^{(V)}$ respectively. While $M^{(V)}$ is always an ordinary representation of G, $S^{(V)}$ is a so-called modular representation of G.

Next we list some results of the previous papers [1] and [2], which we shall need in the following sections. We consider the case where the quotient algebraic variety V_0 of V with respect to G (a finite subgroup of $G(V)$) is defined and we have a Galois covering $f: V\to V_0$. Denoting by A_0 an Albanese

1) In general, let H and H' be groups and φ a homomorphism of H into H'. If F' is a representation of H', then $H\ni\tau\to F'(\varphi(\tau))$ is a representation of H. We denote it by $F'\circ\varphi$.

variety attached to V_0, we have, by [**1**],

(1) $$\dim A_0 = \frac{1}{2}\cdot \operatorname{rank} \sum_G M^{(V)}(T),$$

where $\sum_G$ means the sum ranged over all the elements T in G. On the other hand, when V and V_0 are complete and non-singular[2] and the degree n of the covering (= the order of G) is prime to p, we have, by [**2**],

(2) $$\dim \mathfrak{D}_0(V) - \dim A \geqq \dim \mathfrak{D}_0(V_0) - \dim A_0 \geqq 0 .$$

Moreover, if the adjoint mapping $\delta\alpha$ of $\mathfrak{D}_0(A)$ into $\mathfrak{D}_0(V)$ is surjective, we have, by [**1**],

(3) $$\dim \mathfrak{D}_0(V_0) = \operatorname{rank} \sum_G S^{(V)}(T) .$$

We prove, under the above assumptions on n and $\delta\alpha$, the following lemma, whose special case was used in [**1**] in proving (3).

LEMMA. *Let ω be an element of $\mathfrak{D}_0(V)$. Then ω belongs to the subspace $\delta f(\mathfrak{D}_0(V_0))$ if and only if we have $\delta T(\omega) = \omega$ for all T in the Galois group G.*

PROOF. The 'only if' part is trivial, because we have $f \circ T = f$ for all T in G. Conversely, suppose that $\delta T(\omega) = \omega$ for all T in G. Then we have $(\sum_G \delta T)(\omega) = n\omega$. Denoting $\omega = \delta\alpha(\theta)$ with some θ in $\mathfrak{D}_0(A)$, we have $\delta T(\omega) = \delta\alpha \circ \delta(\varphi(T))(\theta)$ and so, by the injectiveness of $\delta\alpha$ and the assumption on n, we have $\theta = \delta(\sum_G \varphi(T))\left(\frac{1}{n}\cdot\theta\right)$. We may take A_0 to be a quotient abelian variety of A and then, denoting by μ the canonical homomorphism of A onto A_0, a canonical mapping α_0 of V_0 into A_0 may also be taken to satisfy the relation $\alpha_0 \circ f = \mu \circ \alpha$ (cf. [**1**]). Then, by (13) of [**2**], the above expression of θ implies that there exists an element θ_0 of $\mathfrak{D}_0(A_0)$ such that $\theta = \delta\mu(\theta_0)$. Hence we have $\omega = \delta\alpha(\theta) = \delta\alpha \circ \delta\mu(\theta_0) = \delta f \circ \delta\alpha_0(\theta_0)$ and so the 'if' part is proved.

2. The representation $S^{(A)}$.

First we consider the case where $V = A$, i.e. V is an abelian variety. Let $\tau \neq \varepsilon_A$[3] be an element of $G(A) \cap \mathcal{A}(A)$ (i.e. a unit of $\mathcal{A}(A)$) of finite order n. We assume that n is prime to p. Then we can easily find a point t on A such that the order of t (as an element of the additive group A) is exactly equal to n. Putting $A^* = A \times A$, we define an automorphism T^* of the abelian variety A^* by

$$T^*(x, y) = (\tau(x), y+t) = (\tau, \varepsilon_A)(x, y) + (0, t),$$

2) It is known that the completeness of V_0 always follows from that of V and, if the covering is unramified, the non-singularity of V_0 also follows from that of V

3) For an abelian variety B, ε_B denotes the identity element of $\mathcal{A}(B)$.

where (τ, ε_A) is an endomorphism of A^* obtained as the product of τ and ε_A. Let G be a cyclic subgroup of $G(A^*)$ generated by T^*. Then it is easily verified that T^* is of order n and has no fixed point on A^*. Now we apply the results listed in **1** to the complete, non-singular variety A^* and the finite subgroup G of $G(A^*)$ of order prime to p. Since A^* may be regarded as embedded in some projective space, we can define the quotient algebraic variety V_0 of A^* with respect to G and so we get a Galois covering $f: A^* \to V_0$, which is unramified over every point on V_0 and has the degree prime to p. Then V_0 is complete and non-singular (cf. the footnote 2)). Clearly A^* is an Albanese variety attached to itself and, as $\tau \neq \varepsilon_A$, we have $\varphi(T^*) = (\tau, \varepsilon_A) \neq \varepsilon_{A^*}$. Since the l-adic representation of $\mathcal{A}(A^*)$ is faithful (cf. Weil [4]), the matrix $M^{(A^*)}(T^*) = M_l((\tau, \varepsilon_A))$ is different from the unit matrix. Therefore the representation $M^{(A^*)}$ of G contains at least one representation different from the identity; so we have, by (1) and the orthogonality relation of group-characters (cf. Lemma 3 of [**1**]), the strict inequality

$$\dim A_0 < \frac{1}{2} \cdot \deg M^{(A^*)} = \dim A^* ,$$

where A_0 is an Albanese variety attached to V_0. On the other hand, as $\dim \mathfrak{D}_0(A^*) = \dim A^*$, we have, by (2),

$$\dim \mathfrak{D}_0(V_0) = \dim A_0 = \dim \mathfrak{D}_0(A_0)$$

and so the strict inequality

$$\dim \mathfrak{D}_0(V_0) < \dim A^* = \dim \mathfrak{D}_0(A^*) .$$

Therefore $\delta f(\mathfrak{D}_0(V_0))$ is a proper subspace of $\mathfrak{D}_0(A^*)$ and so, by Lemma in **1**, we can find an element θ^* in $\mathfrak{D}_0(A^*)$ such that $\delta T^{*i}(\theta^*) \neq \theta^*$ for some exponent i and we have consequently $\delta T^*(\theta^*) \neq \theta^*$. By Koizumi [**3**], θ^* is expressed as $\delta p_1(\theta_1) + \delta p_2(\theta_2)$ with some θ_1, θ_2 in $\mathfrak{D}_0(A)$, where p_1 (resp. p_2) is the projection of A^* onto the first (resp. second) factor. Then we have, from the definition of T^*,

$$\delta T^*(\theta^*) = \delta T^* \circ \delta p_1(\theta_1) + \delta T^* \circ \delta p_2(\theta_2) = \delta p_1 \circ \delta\tau(\theta_1) + \delta p_2(\theta_2) . \tag{4}$$

Since we have $\delta T^*(\theta^*) \neq \theta^*$, $\delta\tau(\theta_1)$ must not be equal to θ_1 and so $\delta\tau$ is not the identity transformation on $\mathfrak{D}_0(A)$.

In the preceding arguments, we have assumed that the order n of τ is prime to p. But we can show, more generally, that if n is not equal to a power of p, then $\delta\tau$ is also not the identity transformation on $\mathfrak{D}_0(A)$. In fact, then, with a suitable exponent m, τ^m has the order greater than 1 and prime to p and so, from the above arguments, it follows that $(\delta\tau)^m = \delta\tau^m$ is not the identity transformation on $\mathfrak{D}_0(A)$; consequently $\delta\tau$ is not the identity on $\mathfrak{D}_0(A)$. Now we shall compute the order of $\delta\tau$ considered as a linear transformation of the linear space $\mathfrak{D}_0(A)$. Let the order of τ be equal to $n=n'\cdot p^a$ with $(n',p)=1$ and let the order of $\delta\tau$ on $\mathfrak{D}_0(A)$ be equal to m; then clearly m is a divisor of n. Since $\delta\tau^m=(\delta\tau)^m$ is the identity transformation on $\mathfrak{D}_0(A)$, it follows also from the above arguments that the order of τ^m must be equal to some power p^b of p. (b may be equal to 0.) On the other hand, the order of τ^m is clearly equal to n/m and so we have $n=m\cdot p^b$. Therefore *the order m of $\delta\tau$ on $\mathfrak{D}_0(A)$ divides n and is divisible by n'.* In particular, if n is prime to p, then we have $n'=n$ and so $m=n$, i.e. the order of $\delta\tau$ on $\mathfrak{D}_0(A)$ is equal to the order of τ. Summarizing the results just obtained, we have the following

THEOREM 1. *Let $V=A$ be an abelian variety and let τ be a unit of $\mathcal{A}(A)$ (i.e. an element of $G(A)\cap\mathcal{A}(A)$) of finite order n greater than 1. If n is not equal to a power of p, then $\delta\tau$ is not the identity transformation on $\mathfrak{D}_0(A)$. Moreover, if n is prime to p, then $\delta\tau$ has the same order n on $\mathfrak{D}_0(A)$ as τ.*

COROLLARY. *Let G be a finite subgroup of the unit group of $\mathcal{A}(A)$. If the order of G is prime to p, then $S^{(A)}$ is a faithful representation of G.*

REMARK. In the classical case, we know that $S^{(A)}$ is a faithful representation of $\mathcal{A}(A)$ itself, i.e. the mapping $\lambda\to\delta\lambda$ is injective ($\lambda\in\mathcal{A}(A)$). This is a consequence of the two well-known facts that M_l is a faithful representation of $\mathcal{A}(A)$ and that, for any λ in $\mathcal{A}(A)$, $M_l(\lambda)$ has the same characteristic roots as the direct sum of $S(\lambda)$ and the complex conjugate of it, where S is the representation of $\mathcal{A}(A)$ defined similarly as $S^{(A)}$.

As for the statement of Theorem 1, if the order of τ is a power of p, then there occur the two possible cases actually, i.e. 1) $\delta\tau$ is the identity transformation on $\mathfrak{D}_0(A)$ and 2) $\delta\tau$ is not the identity transformation on $\mathfrak{D}_0(A)$ (and, moreover, has the same order as τ). We give some examples of these two cases: Let $p=2$.

1) Let τ be $-\varepsilon_A$. Then τ has the order 2 and clearly $\delta\tau$ is the identity on $\mathfrak{D}_0(A)$.

2) Let $A=B\times B$, where B is an abelian variety, and let τ be the endomorphism of A defined by $\tau(x,y)=(y,x)$. Then the order of τ is equal to 2. Taking an element $\theta_0\neq 0$ of $\mathfrak{D}_0(B)$, we put $\theta=\delta p_1(\theta_0)$, where p_1 is the projection of A onto the first factor B. Then θ is an element $\neq 0$ of $\mathfrak{D}_0(A)$ and we can easily prove that $\delta\tau(\theta)\neq\theta$.

Now let $T(A)$ be the subgroup of $G(A)$ consisting of all the translations of A by points on A. Then it is easily verified that $T(A)$ is a normal subgroup of $G(A)$ and is the kernel of the homomorphism φ of $G(A)$ into the unit group of $\mathcal{A}(A)$. Moreover, since it is known that any element of $T(A)$ induces the identity transformation on $\mathfrak{D}_0(A)$, the linear transformation δT of $\mathfrak{D}_0(A)$ associated to an element T of $G(A)$ is uniquely determined by the coset $\tilde{T}$ of $G(A)$ modulo $T(A)$ which contains T. Hence the representation $S^{(A)}$ of

$G(A)$ may be considered as the representation of the factor group $\tilde{G}(A) = G(A)/T(A)$. Then, by Theorem 1 and its Corollary, we have the following assertions: Let T be an element of $G(A)$ and $\tilde{T}$ the coset of $G(A)$ modulo $T(A)$ containing T. Suppose that $\tilde{T}$ is of finite order n greater than 1 in the factor group $\tilde{G}(A) = G(A)/T(A)$. If n is not equal to a power of p, then δT is not the identity transformation on $\mathfrak{D}_0(A)$. In particular, if n is prime to p, then δT has the same order n on $\mathfrak{D}_0(A)$ as $\tilde{T}$. Moreover, let G be a subgroup of $G(A)$ and suppose that the order of the factor group $\tilde{G} = G/G \cap T(A)$ is finite and prime to p. Then $S^{(A)}$ gives a faithful representation of $\tilde{G}$.

3. A relation between $M^{(A)}$ and $S^{(A)}$.

THEOREM 2. *Let $V = A$ be an abelian variety and let T be an element of $G(A)$ of finite order prime to p. We denote by m_t and d_t the numbers of the primitive t-th roots of unity (counting multiplicities) among the characteristic roots of $M^{(A)}(T)$ and $S^{(A)}(T)$ respectively. Then we have*

$$m_t = 2d_t\,.$$[4]

PROOF. Let $\tau = \varphi(T)$ be the unit of $\mathcal{A}(A)$ associated to T. Then, from the definition and $\delta T = \delta\varphi(T)$ on $\mathfrak{D}_0(A)$, we have

$$M^{(A)}(T) = M^{(A)}(\tau) \quad \text{and} \quad S^{(A)}(T) = S^{(A)}(\tau)\,;$$

moreover, as the order of τ divides that of T, it is also prime to p. Hence we may assume, without any loss of generality, that $T = \tau$ is a unit of $\mathcal{A}(A)$. If the order n of τ is equal to 1, our assertion is trivial; so let n be greater than 1 and prime to p. We use the same notations A^*, T^*, V_0 and A_0 as in the proof of Theorem 1. Then we have the equivalence of two matrices:

$$M^{(A^*)}(T^*) = M^{(A^*)}((\tau, \varepsilon_A)) \sim \begin{pmatrix} M^{(A)}(\tau) & 0 \\ 0 & M^{(A)}(\varepsilon_A) \end{pmatrix},$$

where $M^{(A)}(\varepsilon_A)$ is the unit matrix of degree $= 2\dim A$; and so, by (1) and the orthogonality relation of group-characters, we have

$$\dim A_0 = \frac{1}{2}\cdot m_1 + \dim A\,.$$

On the other hand, using the expression (4) of $\delta T^*(\theta^*)$ (which is valid for any element θ^* in $\mathfrak{D}_0(A^*)$), we can easily show that there holds also the equivalence of two matrices:

$$S^{(A^*)}(T^*) \sim \begin{pmatrix} S^{(A)}(\tau) & 0 \\ 0 & S^{(A)}(\varepsilon_A) \end{pmatrix},$$

4) In the classical case, this is a simple consequence of the remark stated in **2**.

where $S^{(A)}(\varepsilon_A)$ is the unit matrix of degree $=\dim A$. Since the order of the cyclic group generated by τ is prime to p, we can apply the ordinary theory of group-characters to the representation $S^{(A)}$ of this group and so, by (3), we have similarly as above

$$\dim \mathfrak{D}_0(V_0)=d_1+\dim A\,.$$

Hence we have, by (2), $\frac{1}{2}\cdot m_1+\dim A=d_1+\dim A$ and so the equality

$$(5) \qquad m_1=2d_1\,.$$

Now let t be a divisor of the order n of τ. Then the multiplicities of 1 as the characteristic roots of $M^{(A)}(\tau^t)$ and $S^{(A)}(\tau^t)$ are equal to $\sum_{t'|t}m_{t'}$ and $\sum_{t'|t}d_{t'}$ respectively, where the sums range over all the divisors t' of t. Applying the relation (5) to τ^t instead of τ, we have

$$\sum_{t'|t}m_{t'}=2\sum_{t'|t}d_{t'}\,.$$

Therefore, from the induction on t with (5), it follows that we have $m_t=2d_t$. If t is not a divisor of n, then clearly we have $m_t=2d_t=0$.

REMARK. Let T be an element of $G(A)$, which is of finite order $n=n'\cdot p^a$ with $(n',p)=1$. Then it is easily verified that there exist some powers T_r and T_s of T such that we have $T=T_r\circ T_s=T_s\circ T_r$ and the orders of T_r, T_s are equal to n', p^a respectively. Since we have $S^{(A)}(T)=S^{(A)}(T_r)\cdot S^{(A)}(T_s)$ and the characteristic roots of $S^{(A)}(T_s)$ are all equal to the p^a-th roots of unity, i. e. all equal to 1, $S^{(A)}(T)$ and $S^{(A)}(T_r)$ have the same characteristic roots. Therefore, applying Theorem 2 to T_r, we can show that the number of the primitive t-th roots of unity among the characteristic roots of $S^{(A)}(T)$ is equal to the half of the corresponding number of $M^{(A)}(T_r)$.

4. The representation $S^{(V)}$.

Now we return to the general case where V is an arbitrary complete, non-singular variety. From the definition, we have the relation, for the representation $M^{(V)}$ of $G(V)$,

$$M^{(V)}=M^{(A)}\circ\varphi\,,$$

where we use the same prime l. First we intend to show an analogous relation for the representation $S^{(V)}$ of $G(V)$. For any element T of $G(V)$, we have, from the definition of φ,

$$(6) \qquad \delta T\circ\delta\alpha=\delta\alpha\circ\delta(\varphi(T)) \quad \text{on} \quad \mathfrak{D}_0(A)\,.$$

Hence the subspace $\delta\alpha(\mathfrak{D}_0(A))$ of $\mathfrak{D}_0(V)$ is invariant by δT for all T in $G(V)$. Moreover, denoting by $\theta_1,\theta_2,\cdots,\theta_g$ a basis of $\mathfrak{D}_0(A)$ ($g=\dim A$), we can show, by (6) and the injectiveness of $\delta\alpha$, that the relation

$$\delta T \circ \delta\alpha(\theta_j) = \sum_{i=1}^{g} a_{ij}\delta\alpha(\theta_i)$$

implies the relation

$$\delta(\varphi(T))(\theta_j) = \sum_{i=1}^{g} a_{ij}\theta_i$$

with the same coefficients a_{ij} and conversely. Consequently we have the following

THEOREM 3. *The representation $S^{(V)}$ of $G(V)$ can be transformed equivalently into the following form*:

$$S^{(V)} \sim \begin{pmatrix} S^{(A)} \circ \varphi & 0 \\ * & F \end{pmatrix},$$

where F is a representation of $G(V)$ of degree $= \dim \mathfrak{D}_0(V) - \dim \mathfrak{D}_0(A)$. *In particular, if we have $\delta\alpha(\mathfrak{D}_0(A)) = \mathfrak{D}_0(V)$, i.e.* $\dim \mathfrak{D}_0(V) = \dim A$, *then the representation $S^{(V)}$ and $S^{(A)} \circ \varphi$ are equivalent.*

Then, combining with the results in **2** and **3**, we get several results on the representations $M^{(V)}$ and $S^{(V)}$ of $G(V)$. We shall state some of them as theorems.

THEOREM 4. *Let T be an element of $G(V)$ of finite order prime to p. If we have $\varphi(T) \neq \varepsilon_A$, then δT is not the identity transformation on $\mathfrak{D}_0(V)$. Under the assumption $\delta\alpha(\mathfrak{D}_0(A)) = \mathfrak{D}_0(V)$, the converse is also true and δT has the same order on $\mathfrak{D}_0(V)$ as $\varphi(T)$.*

THEOREM 5. *Let T be an element of $G(V)$ of finite order prime to p. We denote by m_t and d_t the numbers of the primitive t-th roots of unity among the characteristic roots of $M^{(V)}(T)$ and $S^{(V)}(T)$ respectively. Then we have*

$$m_t \leqq 2d_t.$$

In particular, if we have $\delta\alpha(\mathfrak{D}_0(A)) = \mathfrak{D}_0(V)$, then we have $m_t = 2d_t$.

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