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Closedness of some subgroups in linear algebraic groups

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Let M, N be closed subgroups of a linear algebraic group. It is mentioned in [1], that D. Hertzig proved that the commutator group $[M, N]$ is closed if M, N are normal. (A proof is given in [2] 3-04 Proposition 1. This fact brings about some simplification of Borel's arguments as noted in [1].) We shall give in this paper a necessary and sufficient condition for M, N to the effect that $[M, N]$ be closed, (Theorem 8 below,) from which the result of Hertzig easily follows (cf. [2], 3), and which will have also some interesting consequences. (Corollaries 9, 10, 11, below.)

In this paper we use the following conventions:
The subgroup generated by G_1, G_2 is denoted by $G_1 \vee G_2$, and the connected component of the identity of an algebraic group G is denoted by G_0.

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LEMMA 1. *Let G be an algebraic group and let $S_1, \cdots, S_m$ be its closed irreducible subsets. Let $f_\lambda(x_1, \cdots, x_m)(\lambda \in \Lambda)$ be words with $x_i \in S_i$, such that for suitable $(a_1^\lambda, \cdots, a_m^\lambda) \in S_1 \times \cdots \times S_m$, $f_\lambda(a_1^\lambda, \cdots, a_m^\lambda) = 1$ for each $\lambda \in \Lambda$. Then the subgroup H of G generated by $f_\lambda(x_1, \cdots, x_m)$, where $(x_1, \cdots, x_m)$ ranges over $S_1 \times \cdots \times S_m$ and λ ranges over Λ, is closed and connected.*

PROOF. For each $\lambda \in \Lambda$, let C_λ be the set of all $f_\lambda(x_1, \cdots, x_m)$ with $x_i \in S_i$. Then the set $C_{\lambda_1} \cdots C_{\lambda_t}$ of products $y_1 \cdots y_t$ $(y_i \in C_{\lambda_i})$ is the image of a rational map from $(S_1 \times \cdots \times S_m) \times \cdots \times (S_1 \times \cdots \times S_m)$ (t-ple product) into G, whence $C_{\lambda_1} \cdots C_{\lambda_t}$ is a thick set ('ensemble épais' cf. [1]), i.e. the closure $C(\lambda_1, \cdots, \lambda_t)$ of $C_{\lambda_1} \cdots C_{\lambda_t}$ is irreducible and $C_{\lambda_1} \cdots C_{\lambda_t}$ contains a non-empty open subset of $C(\lambda_1, \cdots, \lambda_t)$. Since $1 \in C_\lambda$, we see that $C_{\lambda_1} \cdots C_{\lambda_t} \subseteqq C_{\lambda_1} \cdots C_{\lambda_t} C_{\lambda_{t+1}}$, whence $C(\lambda_1, \cdots, \lambda_t) \subseteqq C(\lambda_1, \cdots, \lambda_t, \lambda_{t+1})$. By the fact that $C(\lambda_1, \cdots, \lambda_t)$ are irreducible subvarieties of G, we see that there is a $C(\lambda_1, \cdots, \lambda_t)$, say $C(\lambda_1, \cdots, \lambda_u)$, such that every $C(\lambda_1', \cdots, \lambda_s')$ is contained in $C(\lambda_1, \cdots, \lambda_u)$. ($C(\lambda_1, \cdots, \lambda_t)$ which has maximum dimension is a required one). Then $C_{\lambda_1} \cdots C_{\lambda_u} \subseteqq H \subseteqq C(\lambda_1, \cdots, \lambda_u)$. $C(\lambda_1, \cdots, \lambda_u)$ is the closure of H, hence is a group. Since H contains a non-empty open subset of $C(\lambda_1, \cdots, \lambda_u)$ (because $C_{\lambda_1} \cdots C_{\lambda_u}$ does), we see that $H = C(\lambda_1, \cdots, \lambda_u)$. This completes the proof.

PROPOSITION 2. *Let M and N be closed subgroups of an algebraic group G,* *whose connected components of the identity are* M_0 *and* N_0 *respectively. If* $MN=NM$, MN *is a closed subgroup of* G *with connected component of the identity* $M_0N_0=N_0M_0$.

PROOF. We can express M and N as the union of a finite number of cosets of M_0 and N_0 respectively. Then we see by a trivial calculation that the group MN can be expressed as the disjoint union of a finite number of subsets of the type mM_0N_0n. We want to show that M_0N_0 is equal to the subgroup $M_0 \vee N_0$ generated by M_0 and N_0. As a subgroup of MN, $M_0 \vee N_0$ can be expressed as the disjoint union of a finite number of subsets of the type mM_0N_0n. On the other hand, mM_0, N_0n are closed and irreducible, and the multiplication in G as a mapping $mM_0 \times N_0n \rightarrow mM_0N_0n$ is a rational mapping. So mM_0N_0n is a thick set, whence it follows easily that $M_0 \vee N_0 = M_0N_0$.

REMARK: A. Borel proved in [**1**] (Proposition 5.5 p. 38) a special case of our Proposition 2 where N is in the normalizer of M. For later use in this paper, this special case will suffice.

LEMMA 3. *Let* H *be a closed connected subgroup of an algebraic group* G. *Then the smallest normal subgroup* H^* *of* G *containing* H *is closed and connected.*

PROOF. We know that $H^* = \bigvee_{g \in G} g^{-1}Hg$. By direct application of Lemma 1 to the words $f_g(x) = g^{-1}xg$ on $H (g \in G)$, we get the desired result.

We will see in Corollary 9, which will be proved at the end of this paper, that H need not be connected. (Of course in that case H^* is not necessarily connected.) But we can not prove it directly.

LEMMA 4. *Let* G *be an algebraic group,* M *a closed connected subgroup of* G *and* N *a subset of* G. *Then the commutator group* $[M, N]$ *is closed and connected.*

PROOF. $f_n(x) = x^{-1}n^{-1}xn$ $(n \in N)$ are words on M, which take the value 1 for $x=1$. By direct application of Lemma 1, we easily obtain the result.

LEMMA 5. *If an algebraic group* G *is finitely generated, it is a finite group.*

PROOF. Let $\{g_i, i=1, 2, \cdots, r\}$ be generators of G, and let k be a field of definition for G. Then $K=k(g_1, \cdots, g_r)$ is again a field of definition for G, while G contains only K-rational points. Hence $\dim G = 0$ and G is a finite group.

When A is an automorphism group of an abstract group G, we denote by G^{-1+A} the set $\{g^{-1}g^a \mid g \in G, a \in A\}$ and by $[A, G]$ the subgroup of G generated by the set.

LEMMA 6. *If* G^{-1+A} *is finite,* $[A, G]$ *is a finite group.*

This lemma is due to R. Baer. For the proof we refer the reader to his paper [**3**], Satz 3.

LEMMA 7. *Let* M *and* N *be subgroups of an abstract group* G. *Then* $[M, N]$ *is a normal subgroup of* $M \vee N$.

PROOF. By the formulas

$$x^{-1}[m, n]x=[mx, n][x, n]^{-1},\ y^{-1}[m, n]y=[m, y]^{-1}[m, ny],$$

we get the result immediately.

THEOREM 8. *Let M and N be closed subgroups of an algebraic group G. Then $[M, N]$ is a closed subgroup of G, if and only if the subgroup $M\vee N$ of G generated by M and N is closed.*

PROOF. Suppose $[M, N]$ is closed. Then by Lemma 7 it is normal in $M\vee N$. Hence M is contained in the normalizer of $[M, N]$ and $M[M, N]$ is closed by Proposition 2. $M[M, N]$ is normal in $M\vee N$, and therefore we see by the same reason that $(M[M, N])N$ is closed. On the other hand, $(M[M, N])N$ is generated by M and N as easily shown. Then we have $(M[M, N])N=M\vee N$, namely $M\vee N$ is closed. Conversely suppose $L=M\vee N$ is closed. By Lemma 4 $[M, N_0]$, $[M_0, N]$ and $[[M, N], N_0]$ are closed, connected and contained in $[M, N]$. Then $K=[M, N_0]\vee[M_0, N]\vee[[M, N], N_0]$ is closed, connected and contained in $[M, N]$, since we can apply Lemma 1 to the word $f=xy$, where $x\in[M, N_0]$, $y\in[M_0, N]$. Hence by Lemma 3, the smallest normal subgroup K^* containing K is closed and connected, and it is contained in $[M, N]$, for $[M, N]$ is normal in L. We want to show that K^* has a finite index in $[M, N]$. As $[M, N]/K^*=[MK^*, NK^*]/K^*=[MK^*/K^*, NK^*/K^*]$, we can reduce our problem to the special case where $K^*=1$. Therefore we assume $K^*=1$ and shall prove that $[M, N]$ is finite. By our assumption M_0 and N_0 are normal in L. Let $\hat{L}=L/M_0$, $\hat{M}=M/M_0$, $\hat{N}=NM_0/M_0$, $\hat{N}_0=N_0M_0/M_0$. Then $\hat{L}/\hat{N}$ is finite, because $\hat{L}/\hat{N}\cong L/M_0N_0$ and L/M_0N_0 is a finitely generated algebraic group, and so finite by Lemma 5 (as to quotient algebraic groups, see [4]). We consider $\hat{M}$ as a transformation group of $\hat{K}=[\hat{M}, \hat{N}]\hat{N}$. We know that $\hat{K}/\hat{N}_0$ is finite because it is a subgroup of $\hat{L}/\hat{N}_0$. Let $\{k_i, i=1, 2, \cdots, \alpha\}$ be the representatives of $\hat{K}$ modulo $\hat{N}_0$. For every $n_0\in\hat{N}$, $m\in\hat{M}$, we have $[n_0k_i, m]=k_i^{-1}n_0^{-1}m^{-1}n_0k_im=[k_i, m]$, since the elements of $\hat{M}$ and those of $\hat{N}_0$ commute by our assumption. Since $\hat{M}$ is finite, $\hat{K}^{-1+\hat{M}}$ is finite, and therefore by Lemma 6 $[\hat{M}, \hat{K}]=[\hat{M}, \hat{N}]=[M, N]M_0/M_0$ is a finite group. Since $[M, N]M_0$ has a finite index in $[M, N]M$, $[M, N]M/M_0$ is also a finite group. To apply Lemma 6 again we consider N as a transformation group of $[M, N]M$. Let $\{g_i, i=1, 2, \cdots, \alpha\}$ be representatives of $[M, N]M$ modulo M_0, and $\{n_j, j=1, 2, \cdots, \beta\}$ be representatives of N modulo N_0. Then for every $m_0\in M_0$ and $n_0\in N_0$,

$$[m_0g_i, n_0n_j]=g_i^{-1}m_0^{-1}n_j^{-1}n_0^{-1}m_0g_in_0n_j=[g_i, n_j],$$

since by our assumption the elements of M_0 and those of N commute, and since the elements of $[M, N]M$ and those of N_0 commute. This shows that $\{[M, N]M\}^{-1+N}$ is finite, consisting only of $[g_i, n_j](i=1, 2, \cdots, \alpha, j=1, \cdots, \beta)$.

Hence $[N, [M, N]M]$ is a finite group and therefore $[M, N]$ is finite, which completes the proof.

COROLLARY 9. *Let H be a closed subgroup of an algebraic group G. Then the smallest normal subgroup H^* of G containing H is closed.*

PROOF. An elementary calculation shows that $H^* = [G, H] \vee H$. Our theorem implies that $[G, H]$ is closed. However we know that $[G, H]$ is also normal in G. This implies that H^* is equal to the product of H and $[G, H]$. Hence by Proposition 2, H^* is closed.

COROLLARY 10. *Let G be an algebraic group and let M and N be its closed subgroups. Jf $MN = NM$, hence in particular if N is contained in the normalizer of M, then $[M, N]$ is closed.*

The proof is evident by Proposition 2.

COROLLARY 11. *Let G be an algebraic group, M its closed subgroup and N its closed connected subgroup. Then $M \vee N$ is a closed subgroup.*

PROOF. By Lemma 4 $[M, N]$ is closed. Hence by our theorem $M \vee N$ is closed.

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