On the inductive definition with quantifiers of second order

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(Received Sept. 15, 1960)

§ 1. Introduction.

In a former paper [4], the author defined the system of ordinal diagrams and proved that the system is well-ordered. By using ordinal diagrams he proved in [5] the consistency of a fairly impredicative theory. The theory developed in [4] was generalized in [6]. In this paper we shall generalize the result of [5] by using [6] and show the consistency of a theory which has inductive definitions with quantifiers of second order.

Let I(a) and a < *b be two primitive recursive predicates. Let us assume that the following condition is satisfied:

<* is a well-ordering of I, where I is $\{a \mid I(a)\}$.

Now we shall consider the formal system obtained as follows from G^1LC . (G^1LC is a simple type theory of second order as defined in [2], [3].)

1. Every beginning sequence is of the form $D \to D$ or of the form a = b, $A(a) \to A(b)$ or the 'mathematische Grundsequenz' in Gentzen [1] or the following form

$$I(a), \ A_i(a,b) \rightarrow G_i(a,b \ \{x,y\} (A_i(x,y) \land x < *a))$$

 $I(a), \ G_i(a,b,\{x,y\} (A_i(x,y) \land x < *a)) \rightarrow A_i(a,b)$ $i = 0,1,2,\cdots$.

Here $\{x,y\}$ is used instead of usual notations $\hat{x}\hat{y}$, λxy and A_0,A_1,A_2,\cdots are new symbols for predicates. Moreover, G_i $(i=0,1,2,\cdots)$ are arbitrary formulas satisfying the following conditions:

- a) $G_i(a, b, \alpha)$ does not contain $A_i, A_{i+1}, A_{i+2}, \cdots$.
- b) If $G_i(\alpha, b, \alpha)$ contains the figures of the form $\forall \varphi A(\varphi)$ or $\exists \varphi A(\varphi)$, then $A(\beta)$ does not contain any bound f-variable. (The bound f-variable means the quantifier of second order.)
 - 2. The following inference-schema called 'induction' is added.

$$A(a), \Gamma \rightarrow \Delta, A(a')$$

 $A(0), \Gamma \rightarrow \Delta, A(t)$

Acknowledgement: The work was done under appointment supported by the International Cooperation Administration under the Visiting Research Scientists Program administered by the National Academy of Sciences of the United States of America.

where t is an arbitary term and a is contained in none of A(0), Γ , Δ .

3. The inference \forall left and \exists right on an f-variable of the form

$$\frac{F(V), \ \Gamma \to \Delta}{\forall \varphi F(\varphi), \ \Gamma \to \Delta} \quad \text{and} \quad \frac{\Gamma \to \Delta, \ F(V)}{\Gamma \to \Delta, \ \exists \varphi F(\varphi)}$$

are restricted by the condition that $F(\alpha)$ does not contain any bound f-variable. It should be remarked that $F(\alpha)$ may contain A_0, A_1, A_2, \cdots and V may contain bound f-variables and A_0, A_1, A_2, \cdots .

Then the following theorem holds.

THEOREM. The consistency of our system can be proved by using the wellordering of the system of the ordinal diagrams of

$$((2 \cdot I) \cdot \omega + 1)^{\omega} \times N^2 + 2$$

and

$$((2 \cdot I) \cdot \omega + 1)^{\omega} \times N^2$$
.

Here N denotes the set of integers. The symbols ω , \times etc. have the ordinary meanings, the exact definitions of which will be given in 2.

REMARK. The transfinite induction over I is provable in our system. Let J(a) and $D(a,\alpha)$ be the abbreviations of $\forall \varphi(\forall x(I(x) \land \forall y(y < *x \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[x])$ $\vdash \varphi[a])$ and $\forall x(x < *a \vdash \alpha[x]) \vdash J(a)$ respectively. The following sequences are beginning sequence of our system:

$$I(i)$$
, $C(i) \rightarrow D(i, \{x\}(C(x) \land x < *i));$

$$I(i), D(i, \{x\}(C(x) \land x < *i)) \to C(i)$$
.

We see easily that the following sequences are provable in our system:

$$\forall x (x < ii \vdash C(x)), i < ii \vdash C(i) \land \forall v (v < ii \vdash C(v));$$

$$j < i, C(j) \land \forall y (y < i \vdash C(y)) \rightarrow J(j)$$
.

From above two sequences we have

$$\forall x(x < *i \vdash C(x)), j < *i \rightarrow J(j).$$

On the other hand, we see easily that the following sequence is provable in our system:

$$I(i)$$
, $\forall x(x < i \vdash J(x)) \rightarrow J(i)$.

Thus we have

From this and our beginning sequence we have

$$I(i) \rightarrow C(i)$$
,

and then

$$I(i) \rightarrow J(i)$$

which states the transfinite induction over I.

§ 2. Consistency proof of our system.

- 1. First we define a system of ordinal diagrams on which our proof is based.
- 1.1. \tilde{I} is defined to be $\{j \mid j \in I \text{ or } j \text{ is of the form } \tilde{i} \text{ where } i \in I\}$. $<_*$ is a well ordering of \tilde{I} which is defined as follows:
- 1.1.1. If $i \in I$, then $i <_* \tilde{i}$.
- 1.1.2. If i < *j, then $\tilde{i} < *\tilde{j}$.
- 1.1.3. If i < *j, then $i < *\widetilde{j}$.
- 1.2. Let n be an integer. I_n is defined to be $\{(i, n) | i \in \tilde{I}\}$. $<_n$ is a well-ordering of I_n which is defined as follows:
- 1.2.1. If $i <_* j$, then $(i, n) <_n (j, n)$.
- 1.3. I_{∞} is defined to be $\{\infty\} \cup I_0 \cup I_1 \cdots$.

 $<_{\infty}$ is a well-ordering of I_{∞} defined as follows:

- 1.3.1. If $i \in I_n$, then $i <_{\infty} \infty$ $(n = 0, 1, 2, \cdots)$,
- 1.3.2. If $i \in I_n$, $j \in I_m$ and n < m, then $i < \infty$ j.
- 1.3.3. If $i <_n j$, then $i <_{\infty} j$.
- 1.4. \hat{I} is defined to be a set consisting of elements of the form

$$[i_0, i_1, \cdots, i_n; k_1, k_2],$$

where i_0, i_1, \dots, i_n are elements of I_{∞} , $i_0 \geq_{\infty} i_1 \geq_{\infty} \dots \geq_{\infty} i_n$, and n, k_1, k_2 are integers. $\stackrel{\sim}{<}$ is a lexicographical well-ording of \hat{I} .

- 1.5. To prove the theorem, we assume that there are proof-figures to the sequence \rightarrow in our system, and we assign an o.d. (ordinal diagram) of $O(\{\infty_1,\infty_2\}\cup\hat{I},\hat{I})$, where ∞_1 and ∞_2 are the maximal elements of $\{\infty_1\}\cup\hat{I}$ and $\{\infty_1,\infty_2\}\cup\hat{I}$ respectively, to each of these proof-figures and define the reduction similarly as in [1] and in [5].
- 2. Let \$\mathbb{R}\$ be a proof-figure in our system.
- 2.1. The degree of A_n contained in \mathfrak{P} is defined as follows:
- 2.1.1. If A_n is contained in \mathfrak{P} and is of the form $A_n(j,b)$, where $j \in I$, then the degree of A_n is (\tilde{j},n) .
- 2.1.2. If A_n is contained in \mathfrak{P} and is of the form $A_n(x,b) \wedge x < *i$, where x is a variable or else ' $\neg I(x)$ or $i \leq *x$ ' is probable, then the degree is A_n is (i,n).
- 2.1.3. If A_n is contained in \mathfrak{P} and is of the form $A_n(x,b)$ with $x \in I$ and not of the form $A_n(x,b) \wedge x <^*i$, then the degree of A_n is ∞ .
- 2.2. We define the degree of a formula F in \mathfrak{P} to be

$$[i_0, i_1, \dots, i_n; k_1, k_2],$$

where i_0, i_1, \dots, i_n are the non-increasing series consisting of all the degrees of A_m contained in F ($m = 0, 1, 2, \dots$), k_1 is the number of \forall 's on f-variable in F and k_2 is the number of logical symbols except \forall on an f-variable in F.

2.3. We add the inference 'substitution' with the following restriction in our

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system (cf. [5]).

- 2.3.1. To every substitution is attached an element of \hat{I} , which is called the degree of the substitution, satisfying the following condition: The degree of every implicit formula in the upper sequence of a substitution is less than that of the substitution.
- 2.3.3. The eigenvariable of a substitution is not tied by any \forall on an f-variable in the upper sequence of the substitution.
- 2.3.4. If an implicit formula in the upper sequence of a substitution contains \forall on an f-variable which ties a free f-variable, then it is \forall right in the concerned proof-figure and the degree of the substitution is ∞_1 .
- 2.4. Let $i \in \hat{I}$, \mathfrak{P} be a proof-figure and \mathfrak{S} be a sequence in \mathfrak{P} . The *i*-loader of \mathfrak{S} is the upper sequence of the uppermost substitution under \mathfrak{S} , whose degree is not greater than i in $\{\infty_1, \infty_2\} \cup \hat{I}$, if such exsists; otherwise the *i*-loader of \mathfrak{S} is the end-sequence.
- 2.5. Now we assign an o.d. of $O(\{\infty_1, \infty_2\} \cup \hat{I}, \hat{I})$ to every sequence of a proof-figure recursively as follows:
- 2.5.1. The o.d. of a beginning sequence of the form $D \rightarrow D$ is the degree of D.
- 2.5.2. The o.d. of a beginning sequence of the form a = b, $A(a) \rightarrow A(b)$ is the degree of A(a).
- 2.5.3. The o.d. of a beginning sequence of the form

$$I(i), A_n(i,a) \rightarrow G_n(i,a,\{x,y\}) (A_n(x,y) \land x < *i))$$

or
$$I(i)$$
, $G_n(i, \alpha, \{x, y\}(A_n(x, y) \land x < *i)) \rightarrow A_n(i, \alpha)$

is the degree of $A_n(i, a)$.

- 2.5.4. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequence and the lower sequence of an inference on structure, then the o.d. of \mathfrak{S}_2 is equal to that of \mathfrak{S}_1 .
- 2.5.5. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequence and the lower sequence of an inference \mathcal{T} , \wedge left, \forall on a *t*-variable or \forall right on an *f*-variable respectively, then the o.d. of \mathfrak{S}_2 is $(\infty_2, 0, \sigma)$ where 0 denotes the first element of \hat{I} and σ is the o.d. of \mathfrak{S}_1 .
- 2.5.6. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequences and \mathfrak{S} is the lower sequence of an inference \wedge right, then the o.d. of \mathfrak{S} is $(\infty_2, 0, \sigma_1 \sharp \sigma_2)$, where σ_1 and σ_2 are the o.d.'s of \mathfrak{S}_1 and \mathfrak{S}_2 respectively.
- 2.5.7. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequence and the lower sequence of an \forall left \mathfrak{F} on an f-variable respectively, then the o.d. of \mathfrak{S}_2 is

$$(\infty_2, [i_0, \cdots, i_m; k_1, k_2+2], \sigma),$$

where $[i_0, \dots, i_m; k_1, k_2]$ is the degree of the subformula of \Im and σ is the o.d. of \mathfrak{S}_1 .

2.5.8. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequences and \mathfrak{S} is the lower sequence of a cut \mathfrak{J} , then the o.d. of \mathfrak{S} is $(\infty_2, [i_0, \cdots, i_m; k_1, k_2+1], \sigma_1 \sharp \sigma_2)$, where σ_1 and σ_2 are the o.d.'s of \mathfrak{S}_1 and \mathfrak{S}_2 respectively and $[i_0, \cdots, i_m; k_1, k_2]$ is the degree

of the cut-formula of 3.

2.5.9. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequence and the lower sequence of a substitution with the degree i respectively, then the o.d. of \mathfrak{S}_2 is $(i, 0, \sigma)$ where σ is the o.d. of \mathfrak{S}_1 ,

2.5.10. If \mathfrak{S}_1 and \mathfrak{S}_2 are the upper sequence and the lower sequence of an induction, then the o.d. of \mathfrak{S}_2 is $(\infty_2, [i_0, \cdots, i_m; k_1, k_2+2], \sigma)$, where σ is the o.d. of \mathfrak{S}_1 and $[i_0, \cdots, i_m; k_1, k_2]$ is the degree of $A(\alpha)$ in the schema.

The ordinal diagram of the end-sequence of a proof-figure is called the ordinal diagram of the proof-figure.

- 3. Suppose that the sequence \rightarrow is provable in our system. In the following, we shall reduce a proof-figure \mathfrak{P} to \rightarrow to a proof-figure with the o.d. less than that of \mathfrak{P} . Without loss of generality, we may assume that every free variable used as an eigenvariable in a proof-figure is different from each other. Let \mathfrak{P} be a proof-figure to \rightarrow .
- 3.1. First we substitute 0 for every free variable in \mathfrak{P} except in case it is used as an eigenvariable. In this alteration the proof-figure is still correct and the end-sequence of \mathfrak{P} and the o.d. of \mathfrak{P} are invariable.
- 3.2. We may assume that \mathfrak{P} contains no free variable other than those used as an eigenvariable in \mathfrak{P} . If the end-place of \mathfrak{P} contains an induction, apply the 'VJ-Reduktion' in Gentzen [1], where every substitution in the reduced proof-figure has the same degree as the corresponding one in \mathfrak{P} .
- 3.3. In the following we may assume besides the condition assumed in 3.2, that the end-place of \mathfrak{P} contains no induction. Let the end-place of \mathfrak{P} contain a beginning sequence of the form m=n, $A(m)\to A(n)$, where m and n are of the form $0'^{m}$. Then either $m=n\to$ or $\to m=n$ is a 'mathematische Grundsequenz.'
- 3.3.1. If $m=n \rightarrow$ is a 'mathematische Grundsequenz,' replace the beginning sequence to the proof-figure

$$m = n \rightarrow$$
Weakenings and an exchange
$$m = n, A(m) \rightarrow A(n).$$

3.3.2. If $\rightarrow m = n$ is a 'mathematische Grundsequenz,' then m is n. Replace the beginning sequence by the proof-figure

$$\frac{A(m) \to A(n)}{m = n, \ A(m) \to A(n).}$$

3.4. We may assume besides the conditions assumed in 3.3, that the end-place of \mathfrak{P} contains no beginning sequence of the form m=n, $A(m) \to A(n)$. We can reduce \mathfrak{P} to a proof-figure which contains no beginning sequence of the form $D \to D$ in the end-place in the same way as in $\lceil 5 \rceil$.

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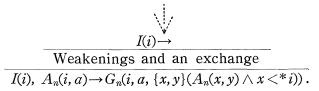
- 3.5. Then we can remove a weakening cut-formula in the end-place in the same way as in [3, 6.4].
- 3.6. We may assume that the end-place of \mathfrak{P} contains no free variable, induction, beginning sequence of the form m=n, $A(m)\to A(n)$, or $D\to D$, or weakening cut-formula. Suppose that \mathfrak{P} contains a beginning sequence of the form

(*)
$$I(i), A_n(i, a) \rightarrow G_n(i, a, \{x, y\}) (A_n(x, y) \land x < *i))$$

or
$$I(i)$$
, $G_n(i, a, \{x, y\})(A_n(x, y) \land x < *i)) \rightarrow A_n(i, a)$,

where i and α are of the form $0'^{--}$, and n is an integer. Since each case is treated similarly, we treat here only the case that \mathfrak{P} contains a beginning sequence (*). By our assumption, either $I(i) \to \text{ or } \to I(i)$ is provable without an induction, or a substition or an \forall on an f-variable.

3.6.1. In case that $I(i) \rightarrow$ is provable, replace the beginning sequence by the following proof-figure:



The ordinal diagram of the above proof-figure is less than that of (*). In the same way as in [5], we see that the ordinal diagram of the proof-figure to \rightarrow decreases by this alteration.

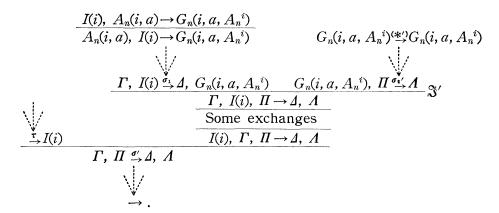
3.6.2. The case that $\rightarrow I(i)$ is provable: Since every formula in $\mathfrak P$ is implicit, there exists a cut $\mathfrak P$ where one of the cut-formulas of $\mathfrak P$ is a descendant of $A_n(i,a)$ in (*). Let $\mathfrak P$ be of the following form:

$$A_{n}(i, a) \rightarrow A_{n}(i, a) \qquad I(i), \ A_{n}(i, a) \stackrel{(*)}{\hookrightarrow} G_{n}(i, a, A_{n}^{i})$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad$$

where A_n^i is the abbreviation of $\{x,y\}(A_n(x,y) \land x < *i)$, and $A_n(i,a) \rightarrow A_n(i,a)$ in the figure may not appear.

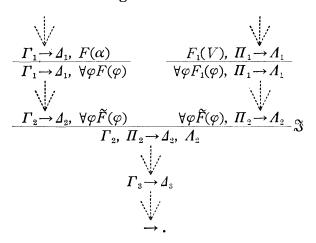
Consider the following proof-figure \mathfrak{P}' :



Every substitution in \mathfrak{P}' has the same degree as the corresponding one in \mathfrak{P} . $\sigma = (\infty_2, \lceil \widetilde{(i,n)}; 0, 0 \rceil, \sigma_1 \sharp \sigma_2)$ and $\sigma' = (\infty_2, \lceil 0; 0, k \rceil, \tau \sharp (\infty_2, \lceil (i,n), \cdots; k_1, k_2 \rceil, \sigma_1 \sharp \sigma_2'))$ where $\lceil 0; 0, k \rceil$ and $\lceil (i,n), \cdots; k_1, k_2 \rceil$ denote the degrees of I(i) and $G_n(i,a,A_n^i)$ respectively. By an analogous method as in $\lceil 5 \rceil$, we see that $\sigma' <_0 \sigma$. Then the ordinal diagram of \mathfrak{P}' is less than that of \mathfrak{P} .

4. Now we may assume that the end-place of \mathfrak{P} contains no free variable, induction, weakening cut-formula or beginning sequence except a 'mathematische Grundsequenz.' If all the sequences in \mathfrak{P} are in the end-place, we treat in the same way as in [1]. Then we may assume that \mathfrak{P} contains an inference on logical symbols and that the end-place contains a suitable cut in the same way as in [3, §6]. To define the essencial reduction, we treat separately several cases according to the form of the outermost logical symbol of the cut-formulas of the suitable cut of \mathfrak{P} . Since other cases are treated in the same way as in [5], we treat here only the case that the outermost logical symbol of the suitable cut \mathfrak{P} of \mathfrak{P} is \forall on an f-variable.

Thus let \$\mathbb{P}\$ be of the following form:

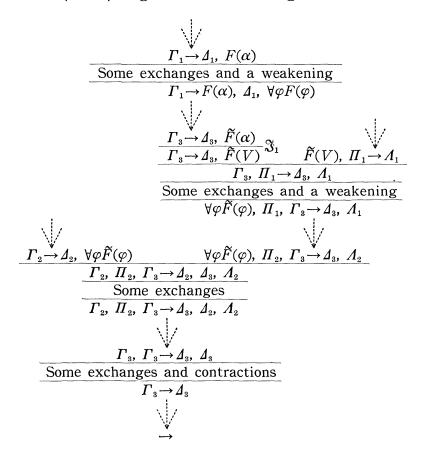


Let $[i_0, \dots, i_n; k_1, k_2]$ be the degree of $\widetilde{F}(\alpha)$. Let i mean ∞_1 or $[i_0, \dots, i_n; k_1, k_2+1]$

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according as $\forall \varphi \widetilde{F}(\varphi)$ contains a free f-variable or not. Let $\Gamma_3 \to \Delta_3$ be the i-loader of Γ_2 , $\Pi_2 \to \Delta_2$, Λ_2 . We can prove easily that $\forall \varphi F_1(\varphi)$ is $\forall \varphi \widetilde{F}(\varphi)$.

A reduction \mathfrak{P}' of \mathfrak{P} is given in the following form:



where \mathfrak{F}_1 is a substitution whose eigenvariable is α and whose degree is defined to be *i*. Every substitution in \mathfrak{F}' except \mathfrak{F}_1 has the same degree as the corresponding one in \mathfrak{F} . Following 4.2 in [5], we can show that substitutions in \mathfrak{F}' satisfy 1.3.1-2.3.4. and that the ordinal diagram of \mathfrak{F}' is less than that of \mathfrak{F} .

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