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Remarks on Cantor's Absolute

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The purpose of this paper is to prove a theorem which is stated in the introduction as the main theorem. In this paper it is understood that a set theory means a set theory T in the first order predicate calculus satisfying the following conditions:

1) $\in$ is the only predicate in T. ($a=b$ is an abbreviation of $\forall x(x \in a \leftrightarrow x \in b)$).
2) T is a consistent extension of Zermelo-Fraenkel's set theory.

A model $\langle A, \overset{*}{\in}_A \rangle$ of a set theory is called 'regular', if and only if there exists no (infinite) sequence $a_0, a_1, a_2, \cdots$ of elements of A such that

$$a_1 \overset{*}{\in}_A a_0,\ a_2 \overset{*}{\in}_A a_1,\ a_2 \overset{*}{\in}_A a_2, \cdots$$

hold. Here a sequence is understood in the informal sense; it may be undefinable in any way.

We presuppose that there exists something absolute, which is a vast universe consisting of numerous concrete sets, and in which some properties (in the informal sense) are "well-defined". Such a universe C will be called *Cantor's Absolute*. It should be understood as a transcendental existence. An existencial quantifier $\exists x$ and universal quantifier $\forall x$ mean *literally* "there exists a set x such that $\cdots$" resp. "for every set x, it holds that $\cdots$". A closed formula in which $\in$ only is used as predicate, is *a priori* true or false in Cantor's Absolute.

Moreover the following propositions are assumed to hold.

(1) Let T_C be the class of all true closed formulas in Cantor's Absolute consisting solely of logical symbols, the predicate $\in$ and bound variables. T_C is called Cantor's set theory. Then T_C contains the class of all provable closed formulas in the set theory of Zermelo-Fraenkel.

(2) $\langle C, \in \rangle$ is a regular model of T_C.

(3) For any well-defined property and any set a in C, there exists a set consisting of all sets which belong to a and satisfy the property. (The word 'property' is used in the informal sense.)

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For example, there exists a set in C consisting of all the Gödel numbers of formulas in T_C; this set will be denoted by $\ulcorner T_C \urcorner$.

In this paper we shall first prove the following proposition. For every definable class of true formulas in T_C a closed formula, which means "There exists a complete model of all formulas of this class", belong to T_C.

This fomula will be given explicity as (A) in § 1, p. 201.

We shall prove the following main theorem.

MAIN THEOREM. *A contradiction follows from the following two hypotheses on Cantor's Absolute.*

HYPOTHESIS 1. *A formula $V=L$ of Gödel* [**1**] *holds on Cantor's Absolute, where $V=L$ is presupposed to be expressed by using only set variables (without using class variables).*

The second hypothesis on Cantor's Absolute expresses that "Cantor's set theory" T_C is a maximal set theory in a certain sense, in another word, that Cantor's set theory cannot be embedded in another set theory. In order to state exactly the second hypothesis, we shall first define some concepts.

A set theory T is called '*definite*', if T satisfies the following conditions.

1) T is complete (For any closed formula $\mathfrak{A}$ either $\mathfrak{A}$ or $\neg\mathfrak{A}$ belongs to T.).
2) If $\exists x\mathfrak{A}(x)$ is closed and belongs to T, then there exists a closed formula $\exists x\mathfrak{B}(x)$ such that $\exists x\mathfrak{B}(x)$, $\forall x\forall y(\mathfrak{B}(x) \wedge \mathfrak{B}(y) \vdash x=y)$ and $\exists x(\mathfrak{A}(x) \wedge \mathfrak{B}(x))$ belong to T.

Let $\mathfrak{A}$ be a formula and a be a variable. $\mathfrak{A}^a$ is obtained from $\mathfrak{A}$ by replacing all the quantifiers $\forall x, \exists y, \cdots$ by $\forall x(x \in a \vdash\ )$, $\exists y(y \in a \wedge\ )$, $\cdots$ respectively.

Let T_0 and T_1 be two set theories. We say 'T_0 can be *embedded* in T_1' if and only if there exists a closed formula $\exists x\mathfrak{A}(x)$ satisfying the following conditions.

1) $\exists x\mathfrak{A}(x)$, $\forall x\forall y(\mathfrak{A}(x) \wedge \mathfrak{A}(y) \vdash x=y)$ and $\forall x\forall y\forall z(\mathfrak{A}(x) \wedge y \in x \wedge z \in y \vdash z \in x)$ belong to T_1.
2) For every closed formula $\mathfrak{B}$, $\mathfrak{B} \in T_0$ if and only if $T_1 \ni \exists x(\mathfrak{A}(x) \wedge \mathfrak{B}^x)$.

Now we shall state the second hypothesis.

HYPOTHESIS 2. *Cantor's set theory cannot be embedded in any definite set theory* T *which contains $V=L$ and has a regular model.*

§ 1. Properties of definite set theory.

We shall use many notations in [**1**]. We always assume that every notion from [**1**] is supposed to be expressed by using only set variables (without using class variables) even if it is originally defined by using class variables in [**1**].

Let T be a set theory and $\exists x\mathfrak{A}(x)$ and $\exists x\mathfrak{B}(x)$ be closed formulas. $\{x\}\mathfrak{A}(x)$

is defined to belong to the same class with $\{x\}\mathfrak{B}(x)$ relative to T, if and only if $\forall x(\mathfrak{A}(x) \mapsto \mathfrak{B}(x))$ belongs to T. The class which contains $\{x\}\mathfrak{A}(x)$ is written by $(\{x\}\mathfrak{A}(x))$, and $\{x\}\mathfrak{A}(x)$ is said to represent the class.

A class $(\{x\}\mathfrak{A}(x))$ is said 'definite', if $\exists x\mathfrak{A}(x)$ and $\forall x\forall y(\mathfrak{A}(x) \wedge \mathfrak{A}(y) \vdash x=y)$ belong to T.

$A(\mathrm{T})$ is defined to be the set of all the definite classes of T.

Let $(\{x\}\mathfrak{A}(x))$ and $(\{x\}\mathfrak{B}(x))$ be two elements of $A(\mathrm{T})$. $(\{x\}\mathfrak{A}(x)) \overset{*}{\in}_{\mathrm{T}} (\{x\}\mathfrak{B}(x))$ is defined to be $\mathrm{T} \ni \exists x\exists y(\mathfrak{A}(x) \wedge \mathfrak{B}(y) \wedge x \in y)$.

Proposition 1. *Let* T *be a definite set theory. Let* $a_1, \cdots, a_n$ *be elements of* $A(\mathrm{T})$ *and be represented by* $\{x\}\mathfrak{A}_1(x), \cdots, \{x\}\mathfrak{A}_n(x)$ *respectively. Then* $\mathfrak{B}(a_1, \cdots, a_n)$ *is satisfied in* $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$ *if and only if*

$$\mathrm{T} \ni \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}(x_1, \cdots, x_n)) .$$

Proof. We shall prove this by induction on the number of logical symbols in $\mathfrak{B}(a_1, \cdots, a_n)$.

If $\mathfrak{B}(a_1, \cdots, a_n)$ has no logical symbols, then $\mathfrak{B}(a_1, \cdots, a_n)$ must be of the form $a_i \in a_j$. $a_i \in a_j$ is satisfield in $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$, if and only if $a_i \overset{*}{\in}_{\mathrm{T}} a_j$, that is, $\mathrm{T} \ni \exists x\exists y(\mathfrak{A}_i(x) \wedge \mathfrak{A}_j(y) \wedge x \in y)$, whence follows the proposition.

Let $\mathfrak{B}(a_1, \cdots, a_n)$ be of the form $\neg\mathfrak{B}_1(a_1, \cdots, a_n)$. Then $\mathfrak{B}(a_1, \cdots, a_n)$ is satisfied in $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$ if and only if

$$\mathrm{T} \ni \neg\exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}_1(x_1, \cdots, x_n)) ,$$

which is equivalent to

$$\exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \neg\mathfrak{B}_1(x_1, \cdots, x_n)) \in \mathrm{T} .$$

Hence follows the proposition.

Let $\mathfrak{B}(a_1, \cdots, a_n)$ be of the form $\mathfrak{B}_1(a_1, \cdots, a_n) \wedge \mathfrak{B}_2(a_1, \cdots, a_n)$. Then $\mathfrak{B}(a_1, \cdots, a_n)$ is satisfied in $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$, if and only if

$$\exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \cdots \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}_1(x_1, \cdots, x_n)) \wedge$$
$$\wedge \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \mathfrak{A}_n(x_n) \mathfrak{B}_2(x_1, \cdots, x_n)) \in \mathrm{T} ,$$

which is equivalent to

$$\mathrm{T} \ni \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}_1(x_1, \cdots, x_n) \wedge \mathfrak{B}_2(x_1, \cdots, x_n)) .$$

Hence follows the proposition.

Let $\mathfrak{B}(a_1, \cdots, a_n)$ be of the form $\forall x\mathfrak{B}_1(x, a_1, \cdots, a_n)$, Then $\mathfrak{B}(a_1, \cdots, a_n)$ is satisfied in $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$, if and only if $\mathfrak{B}_1(a_0, a_1, \cdots, a_n)$ is satisfied in $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}}\rangle$ for every element a_0 of $A(\mathrm{T})$, which is equivalent to the condition that

$$\exists x_0 \cdots \exists x_n(\mathfrak{A}_0(x_0) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}$$

for every definite $\{x\}\mathfrak{A}_0(x)$. Since it is easily proved that from

$$\exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \forall x_0 \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}$$

follows

$$\exists x_0 \cdots \exists x_n(\mathfrak{A}_0(x_0) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}$$

for every definite $\{x\}\mathfrak{A}_0(x)$, we have only to prove that from

$$\neg \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \forall x_0 \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}$$

follows

$$\exists x_0 \cdots \exists x_n(\mathfrak{A}_0(x_0) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \neg \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}$$

for some definite $\{x\}\mathfrak{A}_0(x)$. Since T is definite, the existence of such a definite $\{x\}\mathfrak{A}_0(x)$ follows from

$$\exists x_0 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \neg \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}\,,$$

and this follows from

$$\neg \exists x_1 \cdots \exists x_n(\mathfrak{A}_1(x_1) \wedge \cdots \wedge \mathfrak{A}_n(x_n) \wedge \forall x_0 \mathfrak{B}_1(x_0, \cdots, x_n)) \in \mathrm{T}\,.$$

Therefore the proposition holds.

PROPOSITION 2. *If a define set theory* T *has a regular model* $\langle A, \overset{*}{\in}_A \rangle$, *then* $\langle A(\mathrm{T}), \overset{*}{\in}_{\mathrm{T}} \rangle$ *is also regular.*

PROOF. Suppose that there exists a sequence $a_0, a_1, a_2, \cdots$ of elements of $A(\mathrm{T})$ such that

$$a_1 \overset{*}{\in}_{\mathrm{T}} a_0,\ a_2 \overset{*}{\in}_{\mathrm{T}} a_1,\ a_3 \overset{*}{\in}_{\mathrm{T}} a_2, \cdots .$$

Let a_i be represented by $\{x\}\mathfrak{A}_i(x)$. Since $\mathrm{T} \ni \exists x \mathfrak{A}_i(x)$ and $\mathrm{T} \ni \forall x \forall y(\mathfrak{A}_i(x) \wedge \mathfrak{A}_i(y) \vdash x = y)$, there exists just one element b_i in A such that $A_i(b_i)$ is satisfied in $\langle A, \overset{*}{\in}_A \rangle$. $b_{i+1} \overset{*}{\in}_A b_i$ holds because $a_{i+1} \overset{*}{\in}_{\mathrm{T}} a_i$ means $\exists x_i \exists x_{i+1}(\mathfrak{A}_i(x_i) \wedge \mathfrak{A}_{i+1}(x_{i+1}) \wedge x_{i+1} \in x_i)$. Hence follows a contradiction.

NOTE. Let $\mathfrak{B}$ be a closed formula. Then $\ulcorner \mathfrak{B} \urcorner$ denotes the Gödel number of $\mathfrak{B}$. In the well-known way, we can define a formula $Cd(a)$ satisfying the following conditions:

(1) $Cd(a)$ is constructed only by the predicate $\in$, logical symbols, bound variables and a free variable a.

(2) $\forall x(Cd(x) \vdash x \in \omega) \in \mathrm{T}_C$.

(3) For any given integer i, if the closed formula $Cd(i)$ belongs to T_C, then there exists a closed formula $\mathfrak{B}$ such that $i = \ulcorner \mathfrak{B} \urcorner$.

Moreover, we can define a formula $\mathfrak{D}(a, b)$ with the following properties:

(1) $\mathfrak{D}(a, b)$ is constructed only by the predicate $\in$, logical symbols, bound variables and two free variables a and b.

(2) $\forall x(\mathfrak{D}(x, \ulcorner \mathfrak{B} \urcorner) \vdash\dashv \mathfrak{B}^x) \in \mathrm{T}_C$ for any closed formula $\mathfrak{B}$.

(3) $\forall x \forall y(\mathfrak{D}(x, y) \vdash Cd(y)) \in \mathrm{T}_C$.

Let $\mathfrak{A}(a)$ be a formula with the following properties:

(1) $\forall x(\mathfrak{A}(x) \vdash Cd(x)) \in \mathrm{T}_C$.

(2) If $\mathfrak{B}$ is a closed formula and $\mathfrak{A}(\ulcorner\mathfrak{B}\urcorner) \in \mathrm{T}_C$, then $\mathfrak{B} \in \mathrm{T}_C$.

In virtue of Proposition 2, it is easily seen that there exists a set a_0 in C such that $\langle a_0, \in_{a_0}\rangle$ is isomorphic to $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$ and

$$\forall y\forall z(y \in a_0 \wedge z \in y \vdash z \in a_0)$$

is satisfied in $\langle C, \in\rangle$. By Proposition 1, we see

$$\forall y\forall z(y \in a_0 \wedge z \in y \vdash z \in a_0) \wedge \forall y(\mathfrak{A}(y) \vdash \mathfrak{D}(a_0, y))$$

is satisfied in $\langle C, \in\rangle$. Thereof

(A) $$\exists x(\forall y\forall z(y \in x \wedge z \in y \vdash z \in x) \wedge \forall y(\mathfrak{A}(y) \vdash \mathfrak{D}(x, y)))$$

is true in C.

Now consider the following hypothesis:

(B) $$\exists x(\forall y\forall z(y \in x \wedge (z \in y \vee z \subseteqq y) \vdash z \in x) \wedge \forall y(\mathfrak{A}(y) \vdash \mathfrak{D}(x, y)))\,.$$

This axiom is stronger than (A). So far, we do not know whether (B) is true or not, while (A) has an exact proof.

Proposition 3. *If $V = L$ belongs to a complete set theory* T, *then* T *is definite.*

Proof. Let $\exists x\mathfrak{B}(x) \in \mathrm{T}$. Then

$$\{x\}\exists\alpha(x = F'\alpha \wedge \mathfrak{B}(F'\alpha) \wedge \forall\beta(\mathfrak{B}(F'\beta) \vdash \alpha \leqq \beta))$$

is definite (cf. [**1**, p. 37] for F).

A set theory T is called '*positive definite*', if and only if T satisfies the following conditions.

1) T is complete.
2) $V = L$ belongs to T.
3) $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$ is regular.

Proposition 4. *Under Hypothesis* 1, *Cantor's set theory is positive definite.*

§ 2. Proof of the main theorem.

In this section we always assume Hypotheses 1 and 2. Let T be a positive definite set theory and a be an element of $A(\mathrm{T})$. In virtue of regularity of $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$, it is easily proved that $\mathfrak{O}(a)$ ("a is an oridinal number"; cf [**1**, p. 230]) is satisfied in $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$, if and only if a actually is an ordinal number relative to $\in^*_{\mathrm{T}}$. The ordinal number corresponding to a is expressed by $\hat{a}$ and called an ordinal number of $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$.

If we presuppose that $\mathfrak{O}(a)$ and $\mathfrak{O}(b)$ are satisfied in $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$, then we see easily the following properties.

1. '$a<b$ is satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$' is equivalent to $\hat{a}<\hat{b}$.
2. '$a=b$ is satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$' is equivalent to $\hat{a}=\hat{b}$.
3. If there exists an ordinal number $\beta<\hat{a}$, then there exists $b\in A(T)$ such that $\mathfrak{O}(b)$ and $b<a$ are satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$ and $\hat{b}=\beta$.

Let $\mathfrak{A}(\alpha_1,\cdots,\alpha_n)$ be a formula consisting of $<$, $=$, $\neg$, $\vee$, $\wedge$, $\alpha_1,\cdots,\alpha_n$. Then in virtue of the above properties we see that '$\mathfrak{A}(a_1,\cdots,a_n)$ is satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$' is equivalent to $\mathfrak{A}(\hat{a}_1,\cdots,\hat{a}_n)$ provided that $\mathfrak{O}(a_1),\cdots,\mathfrak{O}(a_n)$ are satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$. By this and the transfinite induction we can easily prove that

$$a_3=J'\langle ia_1a_2\rangle (i<9), a_2=K_1{}'a_1, a_2=K_2{}'a_1$$

are satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$, if and only if $\hat{a}_3=J'\langle i\hat{a}_1\hat{a}_2\rangle (i<9), \hat{a}_2=K_1{}'\hat{a}_1$, $\hat{a}_2=K_2{}'\hat{a}_1$ respectively provided that $\mathfrak{O}(a_1)$, $\mathfrak{O}(a_2)$ and $\mathfrak{O}(a_3)$ are satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$.

The ordinal number, which consists of all the ordinal numbers of $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$, is called the type of T and written by typ(T).

Let $a\in A(\mathrm{T})$ and a be represented by $\{x\}\mathfrak{A}(x)$ and $\mathfrak{O}(a)$ be satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$ and $b\in A(\mathrm{T})$. It is easily seen that $b=F'a$ is satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$ if and only if $\exists y(\mathfrak{A}(y)\wedge b=F'y)$ is satisfied in $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$. Since $\{x\}\exists y(\mathfrak{A}(y)\wedge x=F'y)$ is definite in T, $F'a$ is naturally defined to be $\{x\}\exists y(\mathfrak{A}(y)\wedge x=F'y)$, which also represents an element of $A(\mathrm{T})$.

PROPOSITION 5. *Let* T *be a positive definite set theory, a and b be elements of* $A(\mathrm{T})$ *and* $\mathfrak{O}(a)$ *and* $\mathfrak{O}(b)$ *be satisfied in* $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$. *Then* $F'a\in F'b$ *is satisfied in* $\langle A(\mathrm{T}), \in_{\mathrm{T}}^{*}\rangle$ *if and only if* $F'\hat{a}\in F'\hat{b}$.

PROOF. If $\hat{a}=\hat{b}$, then the proposition is trivial. Therefore we always assume $\hat{a}\neq\hat{b}$. In this proof we use '$\mathfrak{A}$ is satisfied' instead of '$\mathfrak{A}$ is satisfied in $\langle A(T), \in_{T}^{*}\rangle$'.

We may and shall assume that if $c\in A(\mathrm{T}), d\in A(\mathrm{T})$ and $\mathfrak{O}(c),\mathfrak{O}(d)$ and

$$\begin{aligned}\mathrm{Max}\,\{cd\}<\mathrm{Max}\,\{ab\}\vee(\mathrm{Max}\,\{cd\}=\mathrm{Max}\,\{ab\}\wedge c<a)\\ \vee(\mathrm{Max}\,\{cd\}=\mathrm{Max}\,\{ab\}\wedge c=a\wedge d<b)\end{aligned}$$

are satisfied, then $F'\hat{c}\in F'\hat{d}$ is equivalent to "$F'c\in F'd$ is satisfied".

Let $c_1\in A(\mathrm{T})$ and $c_2\in A(\mathrm{T})$ and $\mathfrak{O}(c_1),\mathfrak{O}(c_2)$ and $\mathrm{Max}\{c_1c_2\}\leqq a$ be satisfied. Then we have

$$\begin{aligned}F'\hat{c}_1=F'\hat{c}_2 &\rightleftarrows \forall\alpha(\alpha<\hat{a}\vdash(F'\alpha\in F'\hat{c}_1\mapsto F'\alpha\in F'\hat{c}_2))\\ &\rightleftarrows \forall\alpha(\alpha<a\vdash(F'\alpha\in F'c_1\mapsto F'\alpha\in F'c_2)) \text{ is satisfiied}\\ &\leftrightarrows F'c_1=F'c_2 \text{ is satisfied.}\end{aligned}$$

We must treat the following cases.

1) The case when $\hat{b}\in\mathfrak{W}(J_0)$ (cf. [**1**, pp. 36-37]), which also means '$b\in\mathfrak{W}(J_0)$ is satisfied'.

$$F'\hat{a}\in F'\hat{b}\leftrightarrows\exists\alpha(\alpha\leqq\hat{a}\wedge\alpha<\hat{b}\wedge F'\alpha=F'\hat{a})$$
$$\rightleftarrows\exists\alpha(\alpha\leqq\hat{a}\wedge\alpha<\hat{b}\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\alpha\mapsto F'\beta\in F'\hat{a})))$$
$$\rightleftarrows\exists\alpha(\alpha\leqq a\wedge\alpha<b\wedge\forall\beta(\beta<a\vdash(F'\beta\in F'\alpha\mapsto F'\beta\in F'a))$$
is satisfied

$$\rightleftarrows F'a\in F'b$$ is satisfied.

In the following we shall omit the later half of the symmetric chain of equivalences like this.

2) The case when $\hat{b}\in\mathfrak{W}(J_1)$

$$F'\hat{a}\in F'\hat{b}\rightleftarrows\forall\alpha(\alpha<\text{Max}\ \{\hat{a}K_1'\hat{b}\}\vdash(F'\alpha\in F'\hat{a}\mapsto F'\alpha\in F'K_1'\hat{b})))$$
$$\vee\ \forall\alpha(\alpha<\text{Max}\ \{\hat{a}K_2'\hat{b}\}\vdash(F'\alpha\in F'\hat{a}\mapsto F'\alpha\in F'K_2'\hat{b}))),$$

3) The case when $\hat{b}\in\mathfrak{W}(J_2)$

$$F'\hat{a}\in F'\hat{b}\rightleftarrows F'\hat{a}\in F'K_1'\hat{b}$$
$$\wedge\exists\alpha_1\exists\alpha_2\exists\alpha_3\exists\alpha_4(\alpha_1<\hat{a}\wedge\alpha_2<\hat{a}\wedge\alpha_3<\hat{a}\wedge\alpha_4<\hat{a}$$
$$\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\alpha_3\mapsto F'\beta=F'\alpha_1))$$
$$\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\alpha_4\mapsto F'\beta=F'\alpha_1\vee F'\beta=F'\alpha_2))$$
$$\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\hat{a}\mapsto F'\beta=F'\alpha_3\vee F'\beta=F'\alpha_4))$$
$$\wedge F'\alpha_1\in F'\alpha_2).$$

4) The case when $\hat{b}\in\mathfrak{W}(J_3)$

$$F'\hat{a}\in F'\hat{b}\leftrightarrows F'\hat{a}\in F'K_1'\hat{b}\wedge\neg F'\hat{a}\in F'K_2'\hat{b}.$$

5) The case when $\hat{b}\in\mathfrak{W}(J_4)$

$$F'\hat{a}\in F'\hat{b}\rightleftarrows F'\hat{a}\in F'K_1'\hat{b}\wedge\exists\alpha_1\exists\alpha_2\exists\alpha_3\exists\alpha_4(\alpha_1<\hat{a}\wedge\alpha_2<\hat{a}\wedge\alpha_3<\hat{a}\wedge\alpha_4<\hat{a}$$
$$\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\alpha_3\mapsto F'\beta=F'\alpha_1))$$
$$\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\alpha_4\mapsto F'\beta=F'\alpha_1\vee F'\beta=F'\alpha_2))$$
$$\wedge F'\alpha_2\in F'K_2'\hat{b}\wedge\forall\beta(\beta<\hat{a}\vdash(F'\beta\in F'\hat{a}\mapsto F'\beta=F'\alpha_3\vee F'\beta$$
$$=F'\alpha_4))).$$

6) The case when $\hat{b}\in\mathfrak{W}(J_5)$

$$F'\hat{a}\in F'\hat{b}\rightleftarrows F'\hat{a}\in F'K_1'\hat{b}$$
$$\wedge\exists\alpha_1\exists\alpha_2\exists\alpha_3\exists\alpha_4\exists\alpha_5(\alpha_1<K_2'\hat{b}\wedge\alpha_2<K_2'\hat{b}\wedge\alpha_3<K_2'\hat{b}$$
$$\wedge\alpha_4<K_2'\hat{b}\wedge\alpha_5<K_2'\hat{b}\wedge F'\hat{a}=F'\alpha_2$$
$$\wedge\forall\beta(\beta<K_2'\hat{b}\vdash(F'\beta\in F'\alpha_3\mapsto F'\beta=F'\alpha_1))$$
$$\wedge\forall\beta(\beta<K_2'\hat{b}\vdash(F'\beta\in F'\alpha_4\mapsto F'\beta=F'\alpha_1\vee F'\beta=F'\alpha_2))$$
$$\wedge\forall\beta(\beta<K_2'\hat{b}\vdash(F'\beta\in F'\alpha_5\mapsto F'\beta=F'\alpha_3\vee F'\beta=F'\alpha_4))$$
$$\wedge F'\alpha_5\in F'K_2'\hat{b})).$$

7) The case when $\hat{b}\in\mathfrak{W}(J_6)$

$$F'\hat{a}\in F'\hat{b}\rightleftarrows F'\hat{a}\in F'K_1'\hat{b}$$
$$\wedge\exists\alpha_1\exists\alpha_2\exists\alpha_3\exists\alpha_4\exists\alpha_5\exists\alpha_6(\alpha_1<K_2'\hat{b}\wedge\alpha_2<K_2'\hat{b}$$
$$\wedge\alpha_3<K_2'\hat{b}\wedge\alpha_4<K_2'\hat{b}\wedge\alpha_5<K_2'\hat{b}\wedge\alpha_6<K_2'\hat{b}$$

$$\wedge \forall\beta(\beta < K_2{}'\hat{b} \vdash (F'\beta \in F'\alpha_3 \mapsto F'\beta = F'\alpha_1))$$
$$\wedge \forall\beta(\beta < K_2{}'\hat{b} \vdash (F'\beta \in F'\alpha_4 \mapsto F'\beta = F'\alpha_1 \vee F'\beta = F'\alpha_2))$$
$$\wedge \forall\beta(\beta < K_2{}'\hat{b} \vdash (F'\beta \in F'\alpha_5 \mapsto F'\beta = F'\alpha_2))$$
$$\wedge \forall\beta(\beta < K_2{}'\hat{b} \vdash (F'\beta \in F'\alpha_6 \mapsto F'\beta = F'\alpha_3 \vee F'\beta = F'\alpha_4))$$
$$\wedge \forall\beta(\beta < K_2{}'\hat{b} \vdash (F'\beta \in F'\hat{a} \mapsto F'\beta = F'\alpha_4 \vee F'\beta = F'\alpha_5))$$
$$\wedge F'\alpha_6 \in F'K_2{}'\hat{b}) .$$

8) The case, when $\hat{b} \in \mathfrak{W}(J_i)$ $(i = 7, 8)$, can be treated in the same way as in the last case.

PROPOSITION 6, *Let* T_1 *and* T_2 *be two positive definite set theories and* $\mathrm{typ}(\mathrm{T}_1) < \mathrm{typ}(\mathrm{T}_2)$. *Then* T_1 *can be embedded in* T_2.

PROOF. Since $\mathrm{typ}(\mathrm{T}_1) < \mathrm{typ}(\mathrm{T}_2)$, there exists $a_2 \in A(\mathrm{T}_2)$ such that $\mathfrak{O}(a_2)$ is satisfied in $\langle A(\mathrm{T}_2), \in^*_{\mathrm{T}_2}\rangle$ and $\hat{a}_2$ is $\mathrm{typ}(\mathrm{T}_1)$. We assume that a_2 is represented by $\{x\}\mathfrak{C}(x)$. In virtue of Prop. 5, it is easily seen that $\langle A(\mathrm{T}_1), \in^*_{\mathrm{T}_1}\rangle$ is isomorphic to $\langle A_2, \in^*_2\rangle$, where A_2 consists of all the elements $b_2 \in A(\mathrm{T}_2)$ such that $b_2 \in^*_{\mathrm{T}_2} c_2$ and c_2 is represented by $\{x\}\exists y(\mathfrak{C}(y) \wedge x = F'y)$, which is written by $\{x\}\mathfrak{A}(x)$, and $\in^*_2$ is the confinement of $\in^*_{\mathrm{T}_2}$ to A_2. Therefore T_1, that is, the class of all the formulas, which are satisfied in $\langle A(\mathrm{T}_1), \in^*_{\mathrm{T}_1}\rangle$, is equal to the class of all the formulas $\mathfrak{B}$ such that $\mathrm{T}_2 \ni \exists x(\mathfrak{A}(x) \wedge \mathfrak{B}^x)$.

We see also that

$$\mathrm{T}_2 \ni \forall x \forall y \forall z(\mathfrak{A}(x) \wedge y \in x \wedge z \in y \vdash z \in x) .$$

Hence follows that T_1 can be embedded in T_2.

PROPOSITION 7. *Cantor's set theory* T_C *is characterized by the condition;*

T_C *is a positive definite set theory and* $\mathrm{typ}(\mathrm{T}_C)$ *is not less than the type of any definite set theory.*

For every definite set theory $\mathrm{T}, A(\mathrm{T})$ may be considered as the subset of ω and $\mathrm{typ}(\mathrm{T})$ is less than ω_1. Therefore the characterization of T_C expressed in Prop. 7 is definable in T_C in the sense of [**2**], whence follows a contradiction fron Theorem 1, II of [**2**].

DISCUSSION. Let $A(a)$ be a formula consisting only of logical symbols, the predicate $\in$, bound variables and a. A contradiction follows if we assume that,

(1) if $A(a)$ then a is the set of all Gödel numbers of axioms of a certain positive definite set theory,

(2) $A(\ulcorner \mathrm{T}_C \urcorner)$,

(3) there exists a maximal set theory T_0 (in the sense of embedding) such that $A(\ulcorner \mathrm{T}_0 \urcorner)$.

This can be seen as follows: T_C can be embedded in T_C. That is, there exists a closed formula $\exists x\mathfrak{A}(x)$ such that, for every closed formula $\mathfrak{B}$,

$$\mathfrak{B} \in \mathrm{T}_C \rightleftarrows \exists x(\mathfrak{A}(x) \wedge \mathfrak{B}^x) \in \mathrm{T}_0 .$$

Since T_0 can be definable in T_C by the property that it is the maximum, $\exists x(A(x) \wedge B^x) \in \mathrm{T}_0$ is equivalent to

$$B(\ulcorner \exists x(\mathfrak{A}(x) \wedge \mathfrak{B}^x)\urcorner) \in \mathrm{T}_C \qquad \text{for some } B,$$

and this is equivalent to

$$\tilde{B}(\ulcorner \mathfrak{B} \urcorner) \in \mathrm{T}_C \qquad \text{for some } \tilde{B}.$$

From this we see that T_C is definable in T_C, which is a contradiction.

§ 3. Elementary properties of "embedding".

PROPOSITION 8. *Let* T *be a definite set theory, which has a regular model. Then* T *cannot be embedded in* T.

PROOF. Suppose that T can be embedded in T itself. Then there exists a closed formula $\exists x \mathfrak{A}_0(x)$ satisfying the following condition.

1) $\mathrm{T} \ni \exists x \mathfrak{A}_0(x)$, $\mathrm{T} \ni \forall x \forall y(\mathfrak{A}_0(x) \wedge \mathfrak{A}_0(y) \vdash x = y)$ and
   $\mathrm{T} \ni \forall x \forall y \forall z(\mathfrak{A}_0(x) \wedge y \in x \wedge z \in y \vdash z \in x)$.
2) For every closed formula $\mathfrak{B}$, $\mathrm{T} \ni \mathfrak{B}$ is equivalent to $\mathrm{T} \ni \exists x(\mathfrak{A}_0(x) \wedge \mathfrak{B}^x)$.

$\mathfrak{A}_{i+1}(x_{i+1})$ is defined to be

$$\exists x_i(\mathfrak{A}_0(x_i) \wedge x_{i+1} \in x_i \wedge \mathfrak{A}_i^{x_i}(x_{i+1})) .$$

First we have

$$\begin{aligned}
&\exists u \exists v(\mathfrak{A}_0(u) \wedge \mathfrak{A}_1(v) \wedge v \in u) \\
&\rightleftarrows \exists u \exists v(\mathfrak{A}_0(u) \wedge \exists x(\mathfrak{A}_0(x) \wedge v \in x \wedge \mathfrak{A}_0^x(v)) \wedge v \in u) \\
&\rightleftarrows \exists u \exists v \exists x(\mathfrak{A}_0(u) \wedge \mathfrak{A}_0(x) \wedge v \in x \wedge \mathfrak{A}_0^x(v) \wedge v \in u) \\
&\rightleftarrows \exists x \exists v(\mathfrak{A}_0(x) \wedge v \in x \wedge \mathfrak{A}_0^x(v)) \\
&\rightleftarrows \exists x(\mathfrak{A}_0(x) \wedge \exists v(v \in x \wedge \mathfrak{A}_0^x(v))) \\
&\rightleftarrows \exists v \mathfrak{A}_0(v) .
\end{aligned}$$

Now we shall prove by the induction on $i+1$:

1) $\{x_{i+1}\}\mathfrak{A}_{i+1}(x_{i+1})$ is definite.
2) $\exists u \exists v(\mathfrak{A}_{i+1}(u) \wedge \mathfrak{A}_{i+2}(v) \wedge v \in u) \in \mathrm{T}$.

$$\begin{aligned}
&\exists x_{i+1} \mathfrak{A}_{i+1}(x_{i+1}) \\
&\rightleftarrows \exists x_i(\mathfrak{A}_0(x_i) \wedge \exists x_{i+1}(x_{i+1} \in x_i \wedge \mathfrak{A}_i^{x_i}(x_{i+1}))) \\
&\rightleftarrows \exists x_{i+1} \mathfrak{A}_i(x_{i+1}) .
\end{aligned}$$

$$\begin{aligned}
&\forall u \forall v(\mathfrak{A}_{i+1}(u) \wedge \mathfrak{A}_{i+1}(v) \vdash u = v) \\
&\leftrightarrows \forall u \forall v(\exists x(\mathfrak{A}_0(x) \wedge u \in x \wedge \mathfrak{A}_i^x(u)) \wedge \exists x(\mathfrak{A}_0(x) \wedge v \in x \wedge \mathfrak{A}_i^x(v)) \vdash u = v) \\
&\rightleftarrows \exists x(\mathfrak{A}_0(x) \wedge \forall u(u \in x \vdash \forall v(v \in x \vdash (\mathfrak{A}_i^x(u) \wedge \mathfrak{A}_i^x(v) \vdash u = v)))) \\
&\rightleftarrows \forall u \forall v(\mathfrak{A}_i(u) \wedge \mathfrak{A}_i(v) \vdash u = v) .
\end{aligned}$$

$$\begin{aligned}
&\exists u \exists v(\mathfrak{A}_{i+1}(u) \wedge \mathfrak{A}_{i+2}(u) \wedge v \in u) \\
&\rightleftarrows \exists u \exists v(\exists x(\mathfrak{A}_0(x) \wedge u \in x \wedge \mathfrak{A}_i^x(u)) \wedge \exists x(\mathfrak{A}_0(x) \wedge v \in x \wedge \mathfrak{A}_{i+1}^x(v)) \wedge v \in u) \\
&\rightleftarrows \exists x(\mathfrak{A}_0(x) \wedge \exists u(u \in x \wedge \exists v(v \in x \wedge \mathfrak{A}_i^x(u) \wedge \mathfrak{A}_{i+1}^x(v) \wedge v \in u)))
\end{aligned}$$

$\rightleftarrows \exists u \exists v(\mathfrak{A}_i(u) \wedge \mathfrak{A}_{i+1}(v) \wedge v \in u)$.

a_i is defined to be $(\{x\}\mathfrak{A}_i(x))$. Clearly $a_i \in A(\mathrm{T})$ and $a_{i+1} \in^*_{\mathrm{T}} a_i$ in contradiction to the regularity of $\langle A(\mathrm{T}), \in^*_{\mathrm{T}}\rangle$.

PROPOSITION 9. *Let* T_1, T_2 *and* T_3 *be set theories. If* T_1 *can be embedded in* T_2 *and* T_2 *can be embedded in* T_3, *then* T_1 *can be embedded in* T_3.

PROOF. We may assume

1) $\{x\}\mathfrak{A}(x)$ is definite in T_2 and
$\forall x \forall y \forall z(\mathfrak{A}(x) \wedge y \in x \wedge z \in y \vdash z \in x) \in \mathrm{T}_2$.
2) $\{x\}\mathfrak{B}(x)$ is definite in T_3 and
$\forall x \forall y \forall z(\mathfrak{B}(x) \wedge y \in x \wedge z \in y \vdash z \in x) \in \mathrm{T}_3$.
3) $\mathrm{T}_1 \ni \mathfrak{C} \rightleftarrows \mathrm{T}_2 \ni \exists x(\mathfrak{A}(x) \wedge \mathfrak{C}^x)$ and
$\mathrm{T}_2 \ni \mathfrak{C} \rightleftarrows \mathrm{T}_3 \ni \exists x(\mathfrak{B}(x) \wedge \mathfrak{C}^x)$.

We have

$$\mathrm{T}_1 \ni \mathfrak{C} \rightleftarrows T_2 \ni \exists y(\mathfrak{A}(y) \wedge \mathfrak{C}^y)$$
$$\rightleftarrows \mathrm{T}_3 \ni \exists x(\mathfrak{B}(x) \wedge \exists y(y \in x \wedge \mathfrak{A}^x(y) \wedge (\mathfrak{C}^y)^x))$$
$$\rightleftarrows \mathrm{T}_3 \ni \exists x(\mathfrak{B}(x) \wedge \exists y(y \in x \wedge \mathfrak{A}^x(y) \wedge \mathfrak{C}^y))$$
$$\rightleftarrows \mathrm{T}_3 \ni \exists y(\exists x(\mathfrak{B}(x) \wedge y \in x \wedge \mathfrak{A}^x(y)) \wedge \mathfrak{C}^y).$$

$\tilde{\mathfrak{B}}(y)$ is defined to be $\exists x(\mathfrak{B}(x) \wedge y \in x \wedge \mathfrak{A}^x(y))$. Then

$$\exists y \tilde{\mathfrak{B}}(y) \in \mathrm{T}_3 \rightleftarrows \exists x(\mathfrak{B}(x) \wedge \exists y(y \in x \wedge \mathfrak{A}(y))) \in \mathrm{T}_3$$
$$\rightleftarrows \exists y \mathfrak{A}(y) \in \mathrm{T}_2.$$

$$\forall u \forall v(\tilde{\mathfrak{B}}(u) \wedge \tilde{\mathfrak{B}}(v) \vdash u = v) \in \mathrm{T}_3$$
$$\rightleftarrows \forall u \forall v(\exists x(\mathfrak{B}(x) \wedge u \in x \wedge \mathfrak{A}^x(u)) \wedge \exists x(\mathfrak{B}(x) \wedge v \in x \wedge \mathfrak{A}^x(v)) \vdash u = v) \in \mathrm{T}_3$$
$$\rightleftarrows \exists x(\mathfrak{B}(x) \wedge \forall u(u \in x \vdash \forall v(v \in x \vdash (\mathfrak{A}^x(u) \wedge \mathfrak{A}^x(v) \vdash u = v)))) \in \mathrm{T}_3$$
$$\rightleftarrows \forall u \forall v(\mathfrak{A}(u) \wedge \mathfrak{A}(v) \vdash u = v) \in \mathrm{T}_2.$$

$$\forall y \forall u \forall v(\tilde{\mathfrak{B}}(y) \wedge u \in y \wedge v \in u \vdash v \in y) \in \mathrm{T}_3$$
$$\rightleftarrows \forall y \forall u \forall v(\exists x(\mathfrak{B}(x) \wedge y \in x \wedge \mathfrak{A}^x(y)) \wedge u \in y \wedge v \in u \vdash v \in y) \in \mathrm{T}_3$$
$$\rightleftarrows \exists x(\mathfrak{B}(x) \wedge \forall y(y \in x \vdash \forall u \forall v(u \in y \wedge v \in u \wedge \mathfrak{A}(y) \vdash v \in y))) \in \mathrm{T}_3$$
$$\rightleftarrows \forall y \forall u \forall v(\mathfrak{A}(y) \wedge u \in y \wedge v \in u \vdash v \in y) \in \mathrm{T}_2.$$

PROPOSITION 10. *Let* T_1 *and* T_2 *be positive definite set theories. If* T_1 *is embedded in* T_2, *then* $\mathrm{typ}(\mathrm{T}_1) < \mathrm{typ}(\mathrm{T}_2)$.

PROOF. This follows from Props. 6, 8 and 9.

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