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On semi-hereditary rings

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§ 1. Introduction.

A ring R with unit element is called "*left* (*right*) *semi-hereditary*" according to [2] if any finitely generated left (right) ideal of R is projective.

The purpose of this paper is to determine completely the structure of commutative semi-hereditary rings. A. Hattori has recently given in [6] a homological characterization of Prüfer rings, i. e., semi-hereditary integral domains. This was generalized by M. Harada [5] to commutative rings whose total quotient rings are regular. The results of this paper will include those results of [5] and [6].

In § 3 we shall give a necessary and sufficient condition for a ring to be regular by using the quotient rings. Also we shall introduce a notion of quasi-regular rings and show some properties of them.

In § 4 we shall characterize semi-hereditary rings by using the quotient rings as follows: A ring R is semi-hereditary if and only if the total quotient ring K of R is regular and the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a valuation ring. Furthermore we shall introduce a notion of algebraic extensions of regular rings and show that the integral closure R' of a semi-hereditary ring R in any algebraic extension K' of the total quotient ring K of R is also semi-hereditary.

In § 5, we shall first prove that a local ring R is a valuation ring if and only if $\mathrm{w.gl.dim}\, R \leqq 1$. Secondly we shall show, as a generalization of [6], Theorem 2, that a ring R with the total quotient ring K is semi-hereditary if and only if $\mathrm{w.gl.dim}\, R \leqq 1$ and $\mathrm{w.gl.dim}\, K = 0$, or if and only if any torsion-free R-module is flat.

§ 2. Notations and terminologies.

Throughout this paper a ring will mean a commutative ring with unit element 1. Our notations and terminologies are, in general, the same as in [2] but we shall make the following modifications.

A local ring will mean a (not always Noetherian) ring with only one maximal ideal and a regular ring R will mean a ring such that for any $a \in R$ there is an element b of R with $aba=a$ (cf. von Neumann [**10**]).

Let R be a ring, M an R-module and S a multiplicatively closed subset of R. Then the quotient ring and module of R, M with respect to S are defined as in [**2**] and denoted by R_S, M_S respectively. If S is the complementary set of a prime ideal $\mathfrak{p}$ in R, then we shall use $R_{\mathfrak{p}}$, $M_{\mathfrak{p}}$ instead of R_S, M_S.

Let R be a ring and T be the set of all non zero divisors in R. Then the quotient ring K of R with respect to T will be called the "*total quotient ring*" of R. An element u of an R-module M will be called a "*torsion element*" if $tu=0$ for some $t \in T$. If we denote by $\mathrm{t}(M)$ the set of all torsion elements in M, then $\mathrm{t}(M)$ becomes an R-module and will be called a "*torsion submodule*" of M. If $\mathrm{t}(M)=M$, M will be called a "*torsion module*", and on the other hand, if $\mathrm{t}(M)=0$, it will be called a "*torsion-free*" module. Furthermore an R-module M will be called a "*divisible*" module if for any $t \in T$, $u \in M$ there is an element v of M with $u=tv$.

§ 3. Regular rings and quasi-regular rings.

First we shall prove the following[1)]

THEOREM 1. *A ring R is regular if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a field.*

PROOF. The only if part: If R is regular, then any $R_{\mathfrak{m}}$ is obviously regular, hence we have only to show that if a local ring R is regular, it is a field. Let $\mathfrak{m}$ be a maximal ideal of a local ring R. If there is a non-unit a in R, then a is contained in $\mathfrak{m}$. Since R is regular, we have $a^2b=a$ for a suitable element b of R, hence $(1-ab)a=0$. Since $ab \in \mathfrak{m}$, $1-ab$ is a unit of R. Therefore $a=0$. Thus R must be a field.

The if part: Let a be an element of R and set $\mathfrak{b}=\{b\,;ba=0,\ b \in R\}$. Since any $R_{\mathfrak{m}}$ is a field, $\mathfrak{b}$ is not contained in any maximal ideal $\mathfrak{m}$ containing a. Setting $\mathfrak{c}=(a,\mathfrak{b})$, $\mathfrak{c}$ is not contained in any maximal ideal of R and so we have $R=(a,\mathfrak{b})$. Since $(a)\mathfrak{b}=0$, (a) is a direct summand of R. Accordingly we have $(a)=(e)$ for a suitable idempotent e of R and also have $\mathfrak{b}=(1-e)$. Furthermore, if we set $d=1-e+a$, then d is clearly a unit of R and we have $de=ae=a$. So we obtain $ad^{-1}a=a$. This proves that R is regular.

COROLLARY 1. *A ring R is regular if and only if any element of R is expressible as a product of a unit and an idempotent in R.*

COROLLARY 2. *Let R be a regular ring and $\mathfrak{a}$ be a finitely generated ideal of R. Then $\mathfrak{a}$ is generated by a single idempotent.*

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1) The contents of Theorem 1 and its corollary 1 were published in author's paper [**3**].

PROOF. By Corollary 1 we may assume that $\mathfrak{a}$ is generated by a finite number of idempotents of R. It suffices to show this in case $\mathfrak{a}=(e_1, e_2)$, where e_1, e_2 are idempotents of R. Setting $e=e_1+e_2-e_1e_2$, we obtain easily $e^2=e$ and $ee_i=e_i$ for $i=1, 2$. Therefore $\mathfrak{a}=(e)$. Thus our proof is completed.

COROLLARY 3. *Any regular ring is semi-hereditary.*

A ring R is called a "*quasi-regular*" ring if the total quotient ring K of R is regular.

PROPOSITION 1. *Let R be a quasi-regular ring and K be the total quotient ring of R. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}K\cap R=\mathfrak{a}$. Then $R/\mathfrak{a}$ is also a quasi-regular ring and $K/\mathfrak{a}K$ can be regarded as the total quotient ring of $R/\mathfrak{a}$.*

PROOF. $R/\mathfrak{a}$ can be regarded as the subring of $K/\mathfrak{a}K$ by identifying $R+\mathfrak{a}K/\mathfrak{a}K$ to $R/\mathfrak{a}$. Then $K/\mathfrak{a}K$ is obviously contained in the total quotient ring of $R/\mathfrak{a}$. Since the homomorphic image of a regular ring is also regular, $K/\mathfrak{a}K$ is regular. Then the total quotient ring of $R/\mathfrak{a}$ must coincide with $K/\mathfrak{a}K$, for the total quotient ring of a regular ring is itself.

PROPOSITION 2. *Let R be a quasi-regular ring with the total quotient ring K and S be a multiplicatively closed subset of R. Then the quotient ring R_S of R with respect to S is also a quasi-regular ring and the quotient ring K_S of K with respect to S is the total quotient ring of R_S.*

PROOF. Set $\mathfrak{a}_S=\{a; as=0 \text{ for some } s\in S, a\in R\}$. Let a be an element of $\mathfrak{a}_SK\cap R$. Then we have $a=a_1\alpha_1+a_2\alpha_2+\cdots+a_t\alpha_t$, $a_i\in\mathfrak{a}_S$, $\alpha_i\in K$. Let s_i be an element of S for each i such that $s_ia_i=0$, and set $s=\prod_{i=1}^{t}s_i$. Then we obtain $sa=0$, hence $a\in\mathfrak{a}_S$. This shows $\mathfrak{a}_S=\mathfrak{a}_SK\cap R$. So, by Proposition 1, $R/\mathfrak{a}_S$ is a quasi-regular ring with the total quotient ring $K/\mathfrak{a}_SK$. Since $K_S=K/\mathfrak{a}_SK$ and $K_S\supset R_S\supset R/\mathfrak{a}_S$, R_S is a quasi-regular ring with the total quotient ring K_S.

PROPOSITION 3. *Let R be a quasi regular ring and $\mathfrak{a}$ be a finitely generated ideal of R. Then the following statements are equivalent:*

1) *$\mathfrak{a}$ is projective.*

2) *For any maximal ideal $\mathfrak{m}$ $\mathfrak{a}R_{\mathfrak{m}}$ is zero or generated by a single non zero divisor of $R_{\mathfrak{m}}$.*

3) *$\mathfrak{a}^{-1}\mathfrak{a}$ is a direct summand of R.*

4) *$\mathfrak{a}$ is a direct summand of an invertible ideal of R.*

PROOF. The implications 1)→2) and 4)→1) are obvious.

The implication 2)→3): If we set $\mathfrak{b}=\{b; b\mathfrak{a}=0, b\in R\}$, then we have $\mathfrak{b}\not\subset\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{a}R_{\mathfrak{m}}=0$. On the other hand, in case $\mathfrak{a}R_{\mathfrak{m}}\neq 0$, $\mathfrak{a}R_{\mathfrak{m}}$ is invertible in $R_{\mathfrak{m}}$ by our assumption. Setting $\mathfrak{a}=(a_1, \cdots, a_n)$ and $\bar{\mathfrak{a}}=\mathfrak{a}R_{\mathfrak{m}}$, we have $\sum_{i=1}^{n}\bar{\alpha}_i\bar{a}_i=\bar{1}$, $\bar{\alpha}_i\in\bar{\mathfrak{a}}^{-1}$, where $\bar{1}$, $\bar{a}_i$ are the residues of 1, a_i in $R_{\mathfrak{m}}$ respectively. Set $\mathfrak{a}_{\mathfrak{m}}=\{a; as=0 \text{ for some } s\in R-\mathfrak{m}, a\in R\}$.

Since $\bar{\alpha}_i \in \mathfrak{a}^{-1}$, we have $\bar{\alpha}_i\bar{a}_j = \bar{t}_{ij}/\bar{s}_{ij}$, $\bar{s}_{ij} \in R/\mathfrak{a}_{\mathfrak{m}} - \mathfrak{m}/\mathfrak{a}_{\mathfrak{m}}$, $\bar{t}_{ij} \in R/\mathfrak{a}_{\mathfrak{m}}$ for any i and j. If we set $\bar{s} = \prod_{i,j} \bar{s}_{ij}$, we have $\bar{s}\bar{\alpha}_i\bar{a}_j \in R/\mathfrak{a}_{\mathfrak{m}}$. By Proposition 2 we can now choose a representative α_i of $\bar{\alpha}_i$ in K for any i. Furthermore, by choosing suitably an element s' of $R-\mathfrak{m}$, we obtain $s's\alpha_i a_j \in R$ for any i and j, where s is a representative of $\bar{s}$ in R. So $s's\alpha_i \in \mathfrak{a}^{-1}$ for any i. Now we have $ss's'' = \sum (ss's''\alpha_i)a_i$ for $s'' \in R-\mathfrak{m}$. The left hand side of this formula is not contained in $\mathfrak{m}$ but the right hand side is contained in $\mathfrak{a}^{-1}\mathfrak{a}$. This shows that if $\mathfrak{a}R_{\mathfrak{m}} \neq 0$, then $\mathfrak{a}^{-1}\mathfrak{a} \notin \mathfrak{m}$. Hence, if we set $\mathfrak{c} = (\mathfrak{b}, \mathfrak{a}^{-1}\mathfrak{a})$, $\mathfrak{c}$ is not contained in any maximal ideal of R, and so we have $R = (\mathfrak{b}, \mathfrak{a}^{-1}\mathfrak{a})$. Since $\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{a} = 0$, $\mathfrak{a}^{-1}\mathfrak{a}$ must be a direct summand of R.

The implication 3)$\rightarrow$4). Suppose that $\mathfrak{a}^{-1}\mathfrak{a}$ is a direct summand of R. Then there is an idempotent e of R such that $\mathfrak{a}^{-1}\mathfrak{a} = (e)$. If we set $\mathfrak{b} = (1-e, \mathfrak{a})$, then $\mathfrak{b}$ is invertible as $\mathfrak{b}^{-1} = (1-e, \mathfrak{a}^{-1}e)$. Since $\mathfrak{a}$ is a direct summand of $\mathfrak{b}$, this proves our assertion.

PROPOSITION 4. *Let R be an integrally closed quasi-regular ring with the total quotient ring K and $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} = \mathfrak{a}K \cap R$. Then $R/\mathfrak{a}$ is also integrally closed.*

PROOF. By Proposition 1 $K/\mathfrak{a}K$ is the total quotient ring of R. Now let $\bar{\alpha}$ be an element of $K/\mathfrak{a}K$ integral over $R/\mathfrak{a}$. Then we have $\bar{\alpha}^n + \bar{a}_1\bar{\alpha}^{n-1} + \cdots + \bar{a}_n = 0$, $\bar{a}_i \in R/\mathfrak{a}$. Denote by α, a_i representatives of $\bar{\alpha}, \bar{a}_i$ in K, R, respectively. Then $\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = \beta \in \mathfrak{a}K$. Since K is regular, we have $\beta = \gamma e$ for a unit γ and an idempotent e of K, by Corollary 1 to Theorem 1. Then we have $e \in R$, for R is integrally closed. So $e \in \mathfrak{a} = \mathfrak{a}K \cap R$. From $(1-e)\beta = 0$ we obtain $((1-e)\alpha)^n + a_1((1-e)\alpha)^{n-1} + \cdots + a_n(1-e) = 0$. Since R is integrally closed, we have $(1-e)\alpha \in R$. As $\overline{(1-e)\alpha} = \bar{\alpha}$, $\bar{\alpha}$ must be in $R/\mathfrak{a}$.

PROPOSITION 5. *Let R be a quasi-regular ring. Then R is integrally closed if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is integrally closed. In general, if R is integrally closed, the quotient ring R_S of R with respect to any multiplicatively closed subset S of R is integrally closed.*

PROOF. The only if part is contained in the second part and also the second part is easily obtained from Propositions 2 and 4. Hence we have only to show the if part. Let K be the total quotient ring of R and α be an element of K integral over R. If we set $S = R-\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R, then K_S can be regarded as the total quotient ring of $R_{\mathfrak{m}}$ according to Proposition 2. The residue $\bar{\alpha}$ of α in K_S is, then, integral over $R_{\mathfrak{m}}$, so, by our assumption, we have $\bar{\alpha} \in R_{\mathfrak{m}}$. Hence we have $s\alpha \in R$ for some $s \in S$. If we set $\mathfrak{c} = \{c;\ c\alpha \in R, c \in R\}$, then we must have $\mathfrak{c} = R$, i.e., $\alpha \in R$.

PROPOSITION 6. *Let R be a local quasi-regular ring. If R is integrally closed, then it is an integral domain.*

PROOF. Let K be the total quotient ring of R. If K is not a field, then there exists an idempotent e of K which is not a unit element by Corollary 1 to Theorem 1. Since R is integrally closed, e is contained in R. Then R is expressible as a direct sum of Re and $R(1-e)$. Since R is local, this is obviously a contradiction. Consequently K is a field. Thus R is an integral domain.

PROPOSITION 7. *Let R be a quasi-regular ring. Then an R-module M is a torsion-free module if and only if the quotient module $M_{\mathfrak{m}}$ with respect to any maximal ideal $\mathfrak{m}$ of R is a torsion-free $R_{\mathfrak{m}}$-module.*

PROOF. The if part is evident, hence we have only to show the only if part. Suppose that M is a torsion-free R-module and that $\bar{a}\bar{u}=0$ for a non zero divisor $\bar{a}$ of $R_{\mathfrak{m}}$ and an element $\bar{u}$ of $M_{\mathfrak{m}}$. Set $\mathfrak{a}_{\mathfrak{m}}=\{a\,;\,as=0$ for some $s\in R-\mathfrak{m},\ a\in R\}$ and $M'=\{u\,;\,su=0$ for some $s\in R-\mathfrak{m},\ u\in M\}$. Then we may assume $\bar{a}\in R/\mathfrak{a}_{\mathfrak{m}}$ and $\bar{u}\in M/M'$ by multiplying suitably elements of $R/\mathfrak{a}_{\mathfrak{m}}-\mathfrak{m}/\mathfrak{a}_{\mathfrak{m}}$ to $\bar{a}$, $\bar{u}$. Denote by a a representative of $\bar{a}$ in R and by u a representative of $\bar{u}$ in M. Since $\bar{a}$ is a non zero divisor of $R/\mathfrak{a}_{\mathfrak{m}}$, an ideal $\mathfrak{c}=(a,\mathfrak{a}_{\mathfrak{m}})$ of R contains a non zero divisor b of R for R is quasi-regular. Setting $b=ra+a'$, $r\in R$, $a'\in\mathfrak{a}_{\mathfrak{m}}$, we have $\bar{b}\bar{u}=\bar{r}\bar{a}\bar{u}=0$. Hence, for a suitable element s of $R-\mathfrak{m}$, we have $sbu=0$. As b is a non zero divisor, we obtain $su=0$. This shows $\bar{u}=0$. Thus $M_{\mathfrak{m}}$ is a torsion-free $R_{\mathfrak{m}}$-module.

§ 4. Characterization by quotient rings and algebraic extension.

Here we shall prove our main theorem.[2)]

THEOREM 2. *A ring R is semi-hereditary if and only if the total quotient ring K of R is regular and the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a valuation ring.*

PROOF. The only if part: Assume that R is semi-hereditary. Then any $R_{\mathfrak{m}}$ is obviously semi-hereditary, hence it is a valuation ring as any finitely generated projective ideal of a local ring is a principal ideal generated by a single non zero divisor. Similarly K is also semi-hereditary. Now suppose that K is not regular. Then, by Theorem 1, there exists a maximal ideal $\mathfrak{m}'$ of K such that $K_{\mathfrak{m}'}$ is not a field but a valuation ring. If we set $\mathfrak{p}'=\{a'\,;\,a's'=0$ for some $s'\in K-\mathfrak{m}',\ a'\in K\}$, $\mathfrak{p}'$ is a prime ideal of K strictly contained in $\mathfrak{m}'$. Let a' be an element of $\mathfrak{m}'$ not contained in $\mathfrak{p}'$. Since K is semi-hereditary, a principal ideal (a') is projective over K. If we set $\mathfrak{b}=\{b'\,;\,b'a'=0,\ b'\in K\}$, then b' is a direct summand of K and therefore we have $\mathfrak{b}'=(e')$ for a suitable idempotent e' of K. Further set $c'=e'+a'$. Then c' is contained in $\mathfrak{m}'$ since $a'\notin\mathfrak{p}'$, hence c' is a non unit. On the other hand, if $c'd'=0$,

2) Mr. M. Nagata reported to the author that there exists a ring R such that K is not regular but any $R_{\mathfrak{m}}$ is a valuation ring. So we can not omit the condition that K is regular from the condition in our theorem.

$d' \in K$, then $d'e' = d'a' = 0$. Since $d' \in (e') \cap (1-e')$, we obtain $d' = 0$. Thus c' is a non zero divisor. Consequently c' is a non unit and a non zero divisor of K. This contradicts the fact that K is the total quotient ring of R.

The if part: Let $\mathfrak{a}$ be a finitely generated ideal of R. Since R is quasi-regular and any $R_{\mathfrak{m}}$ is a valuation ring, $\mathfrak{a}$ satisfies the condition 2) in Proposition 3. Hence $\mathfrak{a}$ is projective. This shows that R is semi-hereditary.

COROLLARY 1. *A ring R is semi-hereditary if and only if any finitely generated ideal of R is a direct summand of an invertible ideal of R.*

PROOF. It is obvious by Proposition 3 and Theorem 2.

COROLLARY 2. *A semi-hereditary ring is integrally closed.*

PROOF. This follows from Proposition 5 immediately.

Let R be a valuation ring and K be the quotient field of R. Let K' be an algebraic extension of K and R' be the integral closure of R in K'. It is well known that the quotient ring $R'_{\mathfrak{m}'}$ of R' with respect to any maximal ideal $\mathfrak{m}'$ of R' is a valuation ring (cf. [**9**]). This fact shows, according to Theorem 2, that R' is a Prüfer ring. We shall give a generalization of this to general semi-hereditary rings.

PROPOSITION 8. *Let R be a regular ring and R' be a subring of R such that R is integral over R'. Then R' is also a regular ring.*

PROOF. By Theorem 1 it suffices to prove that for any maximal ideal $\mathfrak{m}'$ of R' $R'_{\mathfrak{m}'}$ is a field. Now put $S' = R - \mathfrak{m}'$. Then $R_{S'}$ is integral over $R'_{\mathfrak{m}'}$. Therefore any maximal ideal $\mathfrak{m}$ of $R_{S'}$ contains $\mathfrak{m}'R_{S'}$. If we set $\mathfrak{n} = \bigcap \mathfrak{m}$ where $\mathfrak{m}$ runs over all maximal ideals of $R_{S'}$, $\mathfrak{n}$ contains $\mathfrak{m}'R_{S'}$. Since $R_{S'}$ is regular, we have $\mathfrak{n} = 0$, so $\mathfrak{m}'R_{S'} = 0$. Consequently $\mathfrak{m}'R'_{\mathfrak{m}'} = 0$. This shows that $R'_{\mathfrak{m}'}$ is a field.

Let R, R' be regular rings with the common unit element such that $R \subset R'$. Then R' is called an "*algebraic extension*" of R if R' is integral over R.

THEOREM 3. *Let R be a semi-hereditary ring and K be the total quotient ring of R. Let K' be an algebraic extension of K and R'' be any intermediate ring between R and K'. Then the integral closure $\bar{R}''$ of R'' in its total quotient ring is also a semi-hereditary ring.*

PROOF. Let K'' be the total quotient ring of R''. Then, by Proposition 8, K'' is regular. Hence we may assume $K' = K''$. Now let R' be the integral closure of R in K'. Then we have obviously $R' \subset \bar{R}''$. First we shall prove that R' is semi-hereditary. By the definition of algebraic extension K' is the total quotient ring of R'. Then, by Theorem 2, it suffices to show that the quotient ring $R'_{\mathfrak{m}'}$ of R' with respect to any maximal ideal $\mathfrak{m}'$ of R' is a valuation ring. Set $\mathfrak{m} = \mathfrak{m}' \cap R$ and $S = R - \mathfrak{m}$. Then R'_S is a quasi-regular ring with the total quotient ring K'_S by Proposition 2. Since R' is integrally closed in K', R'_S is integrally closed in K'_S by Proposition 5. Also it is obvious that R'_S is integral over $R_{\mathfrak{m}}$. Hence R'_S is the integral closure of $R_{\mathfrak{m}}$ in K'_S. Since we have $R'_{\mathfrak{m}'} = (R'_S)_{\mathfrak{m}'R'_S}$, we can suppose that R is a valuation ring. If we set $\mathfrak{p}' = \{a' ; a's' = 0$, for some $s' \in R' - \mathfrak{m}'$, $a' \in R'\}$, then $\mathfrak{p}'$ is a prime ideal of R' by Proposition 6. As is easily seen we can regard $K'/\mathfrak{p}'K' \supset R'/\mathfrak{p}' \supset R$. Since R' is integrally closed, $R'/\mathfrak{p}'$ is also integrally closed in $K'/\mathfrak{p}'K'$ by Proposition 4. Since $K'/\mathfrak{p}'K'$ is an algebraic extension (in the ordinary sense) of the quotient field K of R, $R'/\mathfrak{p}'$ is a Prüfer ring, as is well-known. Now we have $R'_{\mathfrak{m}'} = (R'/\mathfrak{p}')_{\mathfrak{m}'/\mathfrak{p}'}$. Hence $R'_{\mathfrak{m}'}$ must be a valuation ring. Thus R' is semi-hereditary. From this we may assume $R = R'$, $K = K'$ and $R \subset \bar{R}'' \subset K$. Let $\bar{\mathfrak{m}}''$ be a maximal ideal of $\bar{R}''$ and set $\mathfrak{m} = \bar{\mathfrak{m}}'' \cap R$. Then $\mathfrak{m}$ is a prime ideal of R. If we set $S = R - \mathfrak{m}$, we have $K_S \supset \bar{R}''_S \supset R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$ is a valuation ring, $\bar{R}''_S$ is also a valuation ring. Accordingly $\bar{R}''_{\bar{\mathfrak{m}}''} = \bar{R}_S$. Again, by Theorem 2, $\bar{R}''$ must be semi-hereditary.

§ 5. Homological characterization.

Now we refer to some well-known facts (cf. [**2**]).

(I) Let R be a ring and M be an R-module. Then M is R-flat, i.e., $\mathrm{w.dim}_R M = 0$ if and only if for each relation $\sum_i a_i u_i = 0$, $a_i \in R$, $u_i \in M$, there exist elements $r_{ij} \in R$, $v_j \in M$, finite in number, such that $u_i = \sum_j r_{ij} v_j$, $\sum_i r_{ij} a_i = 0$ (cf. [**2**, VI, Ex. 6]).

(II) Let R be a ring, M be an R-module and S be a multiplicatively closed subset of R. Then from (I) it follows immediately that R_S is R-flat as an R-module and we have $M_S \cong R_S \otimes M$ as R_S-modules. For any R-modules M, N and any integer $n \geqq 0$ we have $(\mathrm{Tor}_n^R(M, N))_S \cong \mathrm{Tor}_n^{R_S}(M_S, N_S)$. If M is an R_S-module, then if we regard M as an R-module, we have $M_S = M$, $\mathrm{w.dim}_R M = \mathrm{w.dim}_{R_S} M$. From these we obtain easily that for any R-module M we have $\mathrm{w.dim}_R M = \sup_{\mathfrak{m}}.\ \mathrm{w.dim}_{R_{\mathfrak{m}}} M$ and $\mathrm{w.gl.dim}\ R = \sup_{\mathfrak{m}}.\ \mathrm{w.gl.dim}\ R_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over all maximal ideals of R (cf. [**2**, VII, Ex. 9, 10, 11]).

(III) Let R be a ring with the total quotient ring K and M be an R-module. Then, by (II) we have $\mathrm{w.dim}_R K = 0$. If we set $\bar{K} = K/R$, then we have an exact sequence

$$0 \to \mathrm{Tor}_1^R(M, \bar{K}) \to M \to M \otimes_R K \to$$

and $\mathrm{t}(M) \cong \mathrm{Tor}_1^R(M, \bar{K})$. Therefore M is torsion-free if and only if $\mathrm{Tor}_1^R(M, \bar{K}) = 0$ and is a torsion module if and only if $M \otimes_R K = 0$. Again, by (II), for any torsion-free divisible R-module M, we have $\mathrm{w.dim}_R M = \mathrm{w.dim}_K M$ since M can be regarded as a K-module. Conversely any K-module can be regarded as a torsion-free divisible R-module (cf. [**2**, VII]).

We shall begin with the following

Proposition 9. *If a ring R is local, then any finite flat R-module M is always free.*

Proof. Suppose that M is not free but flat. Denote by n the minimum number of elements generating M and by s the minimum number of non zero elements a_i of R such that $\sum_{i=1}^{n} a_i u_i = 0$ for not all $a_i = 0$ and a minimal base $(u_1, u_2, \cdots, u_n)$ of M. By our assumption there exists such a positive integer s. Now we may assume $\sum_{i=1}^{s} a_i u_i = 0$ for all $a_i \neq 0$ and $M = (u_1, \cdots, u_s, u_{s+1}, \cdots, u_n)$. Again, by applying (I), we obtain $u_i = \sum_{j=1}^{t} r_{ij} u'_j$, $\sum_{i=1}^{s} r_{ij} a_i = 0$, for $r_{ij} \in R$, $u'_j \in M$. If we set $u'_j = \sum_{k=1}^{n} r'_{jk} u_k$, $r'_{jk} \in R$, then we have $u_i = \sum_{j=1}^{t} \sum_{k=1}^{n} r_{ij} r'_{jk} u_k$. Since $(u_1, \cdots, u_n)$ is minimal, $\sum_{j=1}^{t} r_{sj} r'_{js}$ is a unit of R, and so at least one r_{sj_0} of r_{sj}'s is a unit of R. If $s = 1$, then $a_1 = 0$. This is a contradiction. If $s > 1$, then we have $a_s = \sum_{i=1}^{s-1} b_i a_i$, $b_i \in R$, as $\sum_{i=1}^{s} r_{ij_0} a_i = 0$. If we set $u'_i = u_i + b_i u_s$, for $1 \leqq i \leqq s-1$, then we have $\sum_{i=1}^{s-1} a_i u'_i = 0$ and $M = (u'_1, \cdots, u'_{s-1}, u_s, \cdots, u_n)$. This is also a contradiction. Thus M must be free.

Theorem 4.[3)] *A local ring R is a valuation ring if and only if* w. gl. dim $R \leqq 1$. *Especially it is a field if and only if* w. gl. dim $R = 0$.

Proof. The only if part is well known (cf. [**2**, VI, 2.9]). Hence we have only to show the if part. If w. gl. dim $R \leqq 1$, then any ideal of R is R-flat. By Proposition 9 any finitely generated ideal of R is free, hence it is generated by a single non zero divisor. Thus R is a valuation ring. Suppose that w. gl. dim $R = 0$. If R is not a field, there exists a non unit $a \neq 0$ of R. Then we have w. $\dim_R R/(a) = 0$. Again, by Proposition 9, $R/(a)$ is free. This is obviously a contradiction. Consequently R must be a field.

The following proposition is a special case of [**4**, Theorem 5].

Proposition 10. *A ring R is regular if and only if* w. gl. dim $R = 0$.

Proof. Obvious by Theorem 1, 4 and (II).

Proposition 11. *For any ring R we have* w. gl. dim $R \leqq 1$ *if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a valuation ring.*

Proof. This follows from Theorem 4 and (II).

We shall now give a characterization of semi-hereditary rings, which is a generalization of Hattori's result (cf. [**5**] and [**6**]).

Theorem 5. *For any ring R with the total quotient ring K, the following*

3) This theorem and Proposition 9 may be known. However, as these could not be found in any papers, the proofs of these are given here.

conditions are equivalent:

1) *R is a semi-hereditary ring.*

2) w. gl. dim $R \leqq 1$ *and* w. gl. dim $K = 0$.

3) *For any torsion-free R-module M, we have* w. $\dim_R M = 0$.

Proof.[4] The equivalence of 1) and 2) follows from Theorem 2 and Propositions 11 and 12. Also the implication 3) $\to$ 2) is obvious by (III). Hence it suffices to prove the implication 2) $\to$ 3). If M is a torsion-free R-module, then for any maximal ideal $\mathfrak{m}$ of R $M_{\mathfrak{m}}$ is a torsion free $R_{\mathfrak{m}}$-module by Proposition 7. Since $R_{\mathfrak{m}}$ is a valuation ring, any finite torsion-free $R_{\mathfrak{m}}$-module is projective (cf. [**2**, VII, 4.1]). Since Tor_n^R commutes with direct limites, we obtain w. $\dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = 0$. Then by applying (II) to M we obtain w. $\dim_R M = 0$.

It is shown in [**2**, VII, 4.1] that an integral domain R is a Prüfer ring if and only if any finite torsion-free R-module is projective. However a finite torsion-free module over a semi-hereditary ring which is not an integral domain is not always projective.

Corollary. *For any ring R with the total quatient ring K, the following statements are equivalent:*

1) *R is a direct sum of a finite number of Prüfer rings.*

2) w. gl. dim $R \leqq 1$ *and* gl. dim $K = 0$.

3) *Any finite torsion-free R-module is projective.*

Proof. The implications 1) $\leftrightarrow$ 2) $\to$ 3) are obvious by Theorem 4. Hence we have only to prove the implication 3) $\to$ 2). Assume that R satisfies the condition 3). Then, by Theorem 5, we have w. gl. dim $R \leqq 1$ and w. gl. dim $K = 0$. Hence it suffices to show gl. dim $K = 0$, that is, that K is semi-simple. If we set $\mathfrak{p}_{\mathfrak{m}} = \{a;\ as = 0 \text{ for some } s \in R - \mathfrak{m},\ a \in R\}$ for any maximal ideal $\mathfrak{m}$ of R, then $\mathfrak{p}_{\mathfrak{m}}$ is a prime ideal of R. Since any $R/\mathfrak{p}_{\mathfrak{m}}$ is a torsion-free R-module generated by a single element, it is projective by our assumption, and so $\mathfrak{p}_{\mathfrak{m}}$ is a direct summand of R. Accordingly we have $\mathfrak{p}_{\mathfrak{m}} = (e_{\mathfrak{m}})$ for a suitable idempotent $e_{\mathfrak{m}}$ of R. If we set $\bar{e}_{\mathfrak{m}} = 1 - e_{\mathfrak{m}}$ for any $\mathfrak{m}$ and denote by $\mathfrak{a}$ the ideal generated by all $\bar{e}_{\mathfrak{m}}$'s, then $\mathfrak{a}$ is not contained in any maximal ideal $\mathfrak{m}$ of R, hence we have $\mathfrak{a} = R$. Then we have $1 = a_1\bar{e}_{\mathfrak{m}_1} + a_2\bar{e}_{\mathfrak{m}_2} + \cdots + a_n\bar{e}_{\mathfrak{m}_n}$, $a_i \in R$, by choosing suitably a finite number of $\bar{e}_{\mathfrak{m}}'s$. Since $\bar{e}_{\mathfrak{m}}$ is contained in $\mathfrak{p}_{\mathfrak{m}'}$ such that $\mathfrak{p}_{\mathfrak{m}'} \neq \mathfrak{p}_{\mathfrak{m}}$, this shows that there is only a finite number of $\mathfrak{p}_{\mathfrak{m}}$ in R. Consequently K must be semi-simple, as any $\mathfrak{p}_{\mathfrak{m}} K$ is a maximal ideal of K.

The following proposition is a slight generalization of [**8**, Theorem 1].

Proposition 12. *Let R be a semi-hereditary ring whose total quotient ring*

4) This theorem can be proved by using the similar method as in [**6**]. However, in this case, we need to use the proof of the only if part of Theorem 2 to show the implication 1) $\to$ 2). The condition d) in [**6**], Theorem 2 can not be generalized without assuming any condition for a ring.

K is semi-simple. Then any finite R-module M is expressible as a direct sum of a torsion-free module and a torsion module.

Proof. By (III) we have an exact sequence

$$0 \to \operatorname{Tor}_1^R(M, K/R) \to M \to M \otimes_R K \to \cdots$$

If we set $M' = \text{Image } M$ in $M \otimes_R K$, then we have also an exact sequence:

$$0 \to \operatorname{Tor}_1^R(M, K/R) \to M \to M' \to 0.$$

Since M' is a finite torsion-free R-module, it is projective by Corollary to Theorem 5. Then the above exact sequence splits and we have $M \cong \operatorname{Tor}_1^R(M, K/R) \oplus M'$. This proves our assertion.

Finally we shall give a characterization of quasi-regular rings.[5)]

Proposition 13. *For any ring R the following conditions are equivalent:*

1) *R is a quasi-regular ring.*

2) *Any torsion-free divisible R-module M is R-flat.*

3) *For any R-modules M, N and any* $n \geqq 1$ $\operatorname{Tor}_n^R(M, N)$ *is a torsion R-module.*

Proof. Let K be the total quotient ring of R. By (III) we have $\text{w.dim}_R M = \text{w.dim}_K M$ for any torsion-free divisible R-module M. If R is quasi-regular, then we have $\text{w.gl.dim}\, K = 0$ by Proposition 10. Hence $\text{w.dim}_R M = \text{w.dim}_K M = 0$. This shows 1) → 2). Let M, N be any R-modules. Applying (II), we have $(\operatorname{Tor}_n^R(M, N))_T \cong \operatorname{Tor}_n^K(M_T, N_T) \cong (\operatorname{Tor}_n^R(M_T, N_T))_T$. If R satisfies the condition 2), then $\operatorname{Tor}_n^R(M_T, N_T) = 0$ for $n \geqq 1$ since M_T, N_T are regarded as torsion-free divisible R-modules. Therefore we have also $(\operatorname{Tor}_n^R(M, N))_T = 0$. Since $(\operatorname{Tor}_n^R(M, N))_T \cong K \otimes \operatorname{Tor}_n^R(M, N)$ by (II), $\operatorname{Tor}_n^R(M, N)$ is a torsion R-module by (III). Thus 2) → 3) is shown. Let M_K, N_K be any K-module. If we regard M_K, N_K as R-modules and $\operatorname{Tor}_n^R(M_K, N_K)$ is a torsion R-module, we obtain $\operatorname{Tor}_n^K(M_K, N_K) \cong K \otimes \operatorname{Tor}_n^R(M_K, N_K) = 0$ by (II), (III). This proves 3) → 1).

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5) The implications 1) → 2) and 1) → 3) in this proposition were first proved by M. Harada [**5**].

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