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On certain characteristic subgroups of a finite group

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Introduction.

One often asks whether a given property $\mathscr{P}$ is preserved by a given class of group-extensions (E), provided that $\mathscr{P}$ is possessed by all the kernels and the factor groups of (E). For example, the problem to find a group R, any extension H of which by a nilpotent group is likewise nilpotent, has a trivial answer. But if we allow as H only normal (or arbitrary, etc.) subgroups of a fixed group G, the problem becomes closely related to the structure of G, and the solutions of it may serve as a sort of measure relative to that property of groups. In this connection, a theorem of Gaschütz [**3**] is quite interesting, which states that the Frattini subgroup $\Phi(G)$ of a finite group G satisfies the condition of our problem. But $\Phi(G)$ is not in general maximal among the solutions, as the characteristic subgroup $\Delta(G)(\supset\Phi(G))$ introduced in that paper of Gaschütz is also one of solutions. Since larger solution is more interesting in such a problem, one naturally asks for the largest. Unfortunately the largest solution does not always exist. A standard method to make its substitute is to form the intersection of all maximal solutions, a procedure followed for example by Baer [**2**] to define the weak hypercenter. The nature of the present problem however allows us a different approach: We require of R^σ to possess the same property in G^σ as R in G for every homomorphism $\sigma: G\rightarrow G^\sigma$. Then we can prove that there necessarily exists the largest one among R's. The requirement is certainly satisfied by $\Phi(G)$ and $\Delta(G)$. Moreover, our method is favorable in hat it goes well with the induction-arguments.

We can treat similarly several problems of the same type (e.g. concerning abelian subgroups instead of nilpotent, etc.), and obtain thus a series of characteristic subgroups of a finite group. Some of them may be explicitly determined; for example, the hypercenter may be interpreted as the largest solution of the problme concerning the nilpotency and allowing as H any subgroup of G.

Notations

$|G|$ order of a finite group G.

G_p	a p-Sylow subgroup of G.
G^p	a p-Sylow complement of G (if any) = a subgroup such that $\|G\|=\|G_p\|\cdot\|G^p\|$.
G'	commutator subgroup of G.
$\boldsymbol{Z}(G)$	center of G.
$\boldsymbol{H}(G)$	hypercenter of G = the final term of the ascending central chain of G.
$\boldsymbol{F}(G)$	Wendt-Fitting subgroup of G = the largest nilpotent normal subgroup of G.
$\Phi(G)$	Fattini subgroup of G = the intersection of all maximal subgroups of G.
$N_G(H)$	normalizer of H in G.
σ	a homomorphism of G into some group. The identity automorphism of G is denoted by 1.
$X\in\mathfrak{P}$	An object X possesses a property $\mathfrak{P}$.
$\mathfrak{P}_1\prec\mathfrak{P}_2$	If $X\in\mathfrak{P}_1$, then $X\in\mathfrak{P}_2$.

§ 1. Definition and general properties of $\boldsymbol{R}(\boldsymbol{G})$.

Let Q be a property of a pair $G\supset H$ consisting of a group G and its subgroup H, subject to the conditions:

Q 1. Let $G\supset H\supset K$. If $(G\supset K)\in Q$, then $(H\supset K)\in Q$.

Q 2. Let σ be a homomorphic mapping of G. If $(G\supset H)\in Q$, then $(G^\sigma\supset H^\sigma)\in Q$; and conversely if $(G^\sigma\supset H_1)\in Q$, then $(G\supset H_1^{\sigma^{-1}})\in Q$.

Notice that Q 2 contains the following statement

Q 2′. If $(G\supset H)\in Q$ and if N is a normal subgroup of G, then $(G\supset NH)\in Q$.

If such a property Q is given, and if $(G\supset H)\in Q$, we often say that H is a Q-subgroup of G.

Examples. $(G\supset H)\in Q_0$ H is a subgroup of G.

$(G\supset H)\in Q_n$ H is a normal subgroup of G.

$(G\supset H)\in Q_{sn}$ H is a subnormal (= nachinvariant) subgroup of G, i.e. a term of some composition series of G.

Let such a property Q be given. We then look at a property $\mathfrak{P}$ of groups subject to

P 1. If $G\in\mathfrak{P}$, then $H\in\mathfrak{P}$ for any Q-subgroup H of G.

P 2. If $G\in\mathfrak{P}$, then any homomorphic image $G^\sigma\in\mathfrak{P}$.

From among a number of examples we pick up only a few:

$G\in\mathfrak{P}_c$ G is cyclic.

$G\in\mathfrak{P}_a$ G is abelian.

$G\in\mathfrak{P}_n$ G is nilpotent.

$G\in\mathfrak{P}_{n(p)}$ G admits a normal p-Sylow complement.

$G\in\mathfrak{P}_f$ G is finite.

In these examples Q may be fairly arbitrary. (For $\mathfrak{P}_{n(p)}$, see Thompson [7].)

If $(G \supset H) \in Q$ and $H \in \mathfrak{P}$, we say that H is a $(\mathfrak{P}, Q)$-subgroup of G. For example, a $(\mathfrak{P}_n, Q_n)$-subgroup is nothing but a nilpotent normal subgroup.

Now we say that R possesses the property $\mathfrak{R}(G; \mathfrak{P}, Q)$, if the following two conditions are satisfied:

R 1. R is a normal subgroup of G.

R 2. Let any homomorphism $\sigma : G \to G^\sigma$ be given. If H is a Q-subgroup of G^σ containing R^σ and such that $H/R^\sigma \in \mathfrak{P}$, then $H \in \mathfrak{P}$.

If we replace R 2 by the weaker condition

R 2'. If H is a Q-subgroup of G containing R and such that $H/R \in \mathfrak{P}$, then $H \in \mathfrak{P}$.

we say that $R \in \mathfrak{R}'(G; \mathfrak{P}, Q)$.

$\mathfrak{R}(G; \mathfrak{P}, Q)$ (or $\mathfrak{R}'(G; \mathfrak{P}, Q)$) will be written sometimes simply as $\mathfrak{R}(G)$ (or $\mathfrak{R}'(G)$), when no confusion is feared.

It is clear that $R \in \mathfrak{R}(G)$ if and only if $R^\sigma \in R'(G^\sigma)$ for every homomorphism $\sigma : G \to G^\sigma$.

Lemma 1. i) *If $R \in \mathfrak{R}(G)$, then $R^\sigma \in \mathfrak{R}(G^\sigma)$.*

ii) *If $R \in \mathfrak{R}(G)$ and if R_1 is a normal subgroup of G contained in R, then $R_1 \in \mathfrak{R}(G)$.*

iii) *Let N be a normal subgroup of G contained in R. Then $R \in \mathfrak{R}(G)$ if and only if $R/N \in \mathfrak{R}(G/N)$ and $N \in \mathfrak{R}(G)$.*

iv) *If $R_1, R_2 \in \mathfrak{R}(G)$, then $R_1R_2 \in \mathfrak{R}(G)$.*

Proof. i) is clear from the definition.

ii) Let H be a Q-subgroup of G^σ containing $R_1{}^\sigma$ and such that $H/R_1{}^\sigma \in \mathfrak{P}$. Then $R^\sigma H$ is a Q-subgroup of G^σ (by Q 2') such that $R^\sigma H/R^\sigma \in \mathfrak{P}$ (by P 2). Hence $R^\sigma H \in \mathfrak{P}$. Since H is a Q-subgroup of $R^\sigma H$ (by Q 1), we have $H \in \mathfrak{P}$ (by P 1).

iii) Assume $R/N \in \mathfrak{R}(G/N)$. Let H be a Q-subgroup of G^σ containing R^σ and such that $H/R^\sigma \in \mathfrak{P}$. Applying the natural epimorphism $G^\sigma \to G^\sigma/N^\sigma$, we see that H/N^σ is a Q-subgroup of G^σ/N^σ and that $(H/N^\sigma)/(R^\sigma/N^\sigma) \in \mathfrak{P}$. Since R^σ/N^σ is the image of R/N by the epimorphism $G/N \to G^\sigma/N^\sigma$ induced by σ, we must have $H/N^\sigma \in \mathfrak{P}$. Assume further $N \in \mathfrak{R}(G)$, then we have $H \in \mathfrak{P}$, which proves $R \in \mathfrak{R}(G)$. The converse is a special case of i) and ii).

iv) Let σ be the natural epimorphism $G \to G/R_2$. Then we have $R_1R_2/R_2 = R_1{}^\sigma \in \mathfrak{R}(G/R_2)$ by i). Hence also $R_1R_2 \in \mathfrak{R}(G)$ by iii).

Theorem 1. *Let Q and $\mathfrak{P}$ be given, subject to the above conditions. Then any finite group G has a characteristic subgroup $\boldsymbol{R} = \boldsymbol{R}(G; \mathfrak{P}, Q)$ such that $R \in \mathfrak{R}(G)$ if and only if R is a normal subgroup of G contained in $\boldsymbol{R}$.*

Indeed, Lemma 1 shows that the join $\boldsymbol{R} = \bigcup R$ of all subgroups $R \in \mathfrak{R}(G)$ satisfies the conditions of Theorem.

Remark. $\boldsymbol{R}$ is a characteristic subgroup even for an infinite group G, but may happen $\boldsymbol{R} \notin \mathfrak{R}(G)$ in this case. For instance, $R \in \mathfrak{R}(G)$ states for $Q=Q_0$, $\mathfrak{P}=\mathfrak{P}_f$ that R is a finite normal subgroup of G. Therefore, $\boldsymbol{R}(G\,;\mathfrak{P}_f, Q_0)$ of an abelian group G is the torsion subgroup of G which may be infinite.

From now on, *we consider only finite groups.* By Lemma 1 we have immediately

PROPOSITION 1. i) *$\boldsymbol{R}(G)^\sigma \subset \boldsymbol{R}(G^\sigma)$ for any $\sigma: G \to G^\sigma$.*

ii) *If N is a normal subgroup of G contained in $\boldsymbol{R}(G)$, then $\boldsymbol{R}(G)/N = \boldsymbol{R}(G/N)$.*

We denote by $\boldsymbol{M}=\boldsymbol{M}(G\,;\mathfrak{P}, Q)$ the intersection of all maximal $(\mathfrak{P}, Q)$-subgroups of G. This is obviously a characteristic subgroup of G.

PROPOSITION 2. *If R possesses the property $\mathfrak{R}'$, then $R \subset \boldsymbol{M}$. In particular, $\boldsymbol{R} \subset \boldsymbol{M}$.*

PROOF. Let $R \in \mathfrak{R}'$ and let M be a maximal $(\mathfrak{P}, Q)$-subgroup. Then we see $RM/R \in \mathfrak{P}$ and $(G \supset RM) \in Q$. Hence $RM \in \mathfrak{P}$. By the maximality of M we have $RM=M$, viz. $R \subset M$.

In the following, we impose a further condition on $\mathfrak{P}$, namely:

P 3. If $G_1, G_2 \in \mathfrak{P}$ and $(|G_1|, |G_2|)=1$, then $G_1 \times G_2 \in \mathfrak{P}$.

This condition is clearly satisfied by our examples above given and by many others.

PROPOSITION 3. *Let $(|G_1|, |G_2|)=1$. Then we have $\boldsymbol{R}(G_1 \times G_2)=\boldsymbol{R}(G_1) \times \boldsymbol{R}(G_2)$.*

PROOF. Put $G=G_1 \times G_2$. Any subgroup of G is of type $H_1 \times H_2$ $(H_1 \subset G_1, H_2 \subset G_2)$. Let $R_1 \in \mathfrak{R}'(G_1)$ and assume that $H_1 \times H_2$ is a Q-subgroup of G containing R_1 and such that $(H_1 \times H_2)/R_1 \in \mathfrak{P}$. By the projection $G \to G_1$, we see that H_1 is a Q-subgroup of G_1 such that $H_1/R_1 \in \mathfrak{P}$. Hence we have $H_1 \in \mathfrak{P}$. On the other hand, the projection $G \to G_2$ shows that $H_2 \in \mathfrak{P}$. Hence we have $H_1 \times H_2 \in \mathfrak{P}$ by P3. Thus $R_1 \in \mathfrak{R}'(G)$. Since these arguments hold in any homomorphic image of G, we see $\boldsymbol{R}(G_1) \subset \boldsymbol{R}(G)$. Similarly we have $\boldsymbol{R}(G_2) \subset \boldsymbol{R}(G)$. On the other hand, Prop. 1 i) shows that G_i-component of $\boldsymbol{R}(G)$ is contained in $\boldsymbol{R}(G_i)$ $(i=1,2)$, whence $\boldsymbol{R}(G) \subset \boldsymbol{R}(G_1) \times \boldsymbol{R}(G_2)$. Combining these two facts we obtain the desired equality.

By this proposition, the study of $\boldsymbol{R}$ of a nilpotent group is reduced to that of p-groups.

§ 2. The position of $\boldsymbol{R}(G)$.

2.1. In the following, we study $\boldsymbol{R}(G\,;\mathfrak{P}, Q)$ mainly in the cases $\mathfrak{P}=\mathfrak{P}_c, \mathfrak{P}_a, \mathfrak{P}_n$ and $Q=Q_0, Q_n, Q_{sn}$. We shall use simplified notations such as $\boldsymbol{R}_{c,0}=\boldsymbol{R}(\mathfrak{P}_c, Q_0)$, $\boldsymbol{R}_{n,n}=\boldsymbol{R}(\mathfrak{P}_n, Q_n)$, and similarly for $\boldsymbol{M}$-groups, etc.

If $\mathfrak{P} \prec \mathfrak{P}_n$, $\boldsymbol{R}(\mathfrak{P}, Q)$ is nilpotent. Hence the problem of the determination of $\boldsymbol{R}$ is reduced to that of p-Sylow subgroups $\boldsymbol{R}_p$.

PROPOSITION 4. *If $\mathfrak{P} \prec \mathfrak{P}_n$, $\boldsymbol{R}(G\,;\mathfrak{P}, Q_0)_p$ contains $\boldsymbol{R}(G_p\,;\mathfrak{P}, Q_0) \cap \boldsymbol{Z}(G)$.*

PROOF. Put $R=\boldsymbol{R}(G_p\,;\mathfrak{P}, Q_0) \cap \boldsymbol{Z}(G)$. This is a normal subgroup of G (and *a fortiori* of G_p). Since $\boldsymbol{R}(aG_pa^{-1})=a\boldsymbol{R}(G_p)a^{-1}$ $(a\in G)$ by Prop. 1 i), $R\subset\boldsymbol{R}(G_p')$ for every p-Sylow subgroup G_p' of G. Now let $G^\sigma\supset H\supset R^\sigma$ and let $H/R^\sigma\in\mathfrak{P}$. Since R^σ is central in G^σ, and since $\mathfrak{P}\prec\mathfrak{P}_n$, H is nilpotent. Hence H is the direct product of H_p and H^p. H^p is a $\mathfrak{P}$-group, as it is a homomorphic image of H/R^σ. Similarly, the p-group H_p/R^σ is a $\mathfrak{P}$-group. Since there is a p-Sylow subgroup G_p' of G such that $(G_p')^\sigma\supset H_p$, it follows that H_p itself is a $\mathfrak{P}$-group by the above remark $R\subset\boldsymbol{R}(G_p')$. Hence $H=H_p\times H^p\in\mathfrak{P}$. This shows $R\subset\boldsymbol{R}(G)$, or equivalently $R\subset\boldsymbol{R}(G)_p$, as R is a p-group.

Concerning $\boldsymbol{M}$, we have

LEMMA 2. *If $\mathfrak{P}\prec\mathfrak{P}_n$, $\boldsymbol{M}(G;\mathfrak{P},\mathfrak{Q}_0)_p$ is contained in $\boldsymbol{M}(G_p;\mathfrak{P},\mathfrak{Q}_0)$.*

PROOF. Any maximal $\mathfrak{P}$-subgroup H of G_p is contained in some maximal $\mathfrak{P}$-subgroup K of G, and it is then clear that $H=G_p\cap K$. Since $\mathfrak{P}\prec\mathfrak{P}_n$, $\boldsymbol{M}=\boldsymbol{M}(G;\mathfrak{P},\mathfrak{Q}_0)$ is nilpotent, and $\boldsymbol{M}_p$ is contained in every p-Sylow subgroup G_p. Hence $\boldsymbol{M}_p\subset G_p\cap\boldsymbol{M}\subset G_p\cap K=H$ for every H. This shows $\boldsymbol{M}_p\subset\boldsymbol{M}(G_p)$.

LEMMA 3. *If $\mathfrak{P}_c\prec\mathfrak{P}\prec\mathfrak{P}_a$, $\boldsymbol{M}(G;\mathfrak{P},\mathfrak{Q}_0)$ is contained in $\boldsymbol{Z}(G)$. In particular $\boldsymbol{M}_{a,0}(G)=\boldsymbol{Z}(G)$.*

PROOF. Every element a of G is contained in some maximal $\mathfrak{P}$-subgroup M. Since M is abelian and contains $\boldsymbol{M}=\boldsymbol{M}(G;\mathfrak{P},\mathfrak{Q}_0)$ we see immediately $\boldsymbol{M}\subset\boldsymbol{Z}$. Conversely, any maximal abelian subgroup M contains $\boldsymbol{Z}$, since $\boldsymbol{Z}M$ is still abelian, and must coincide with M. Hence $\boldsymbol{M}_{a,0}\supset\boldsymbol{Z}$.

We now determine the position of $\boldsymbol{R}_{c,0}$ as follows.

THEOREM 2. *For any group G, we have $\boldsymbol{R}_{c,0}(G)=\boldsymbol{M}_{c,0}(G)=\prod_p(\boldsymbol{R}_{c,0}(G_p)\cap\boldsymbol{Z})$; and for a non-cyclic p-group G_p, $\boldsymbol{R}_{c,0}(G_p)$ reduces to 1, excepting the case of a generalized quaternion group, in which $\boldsymbol{R}_{c,0}(G_p)$ coincides with the unique subgroup of order 2.*

PROOF. It is clear that $\boldsymbol{M}(Q)$ coincides with the unique subgroup of order 2, say R, for a generalized quaternion group Q, and that $\boldsymbol{M}(G_p)=1$ for any other non-cyclic p-group G_p. Moreover, the verification of $R\subset\boldsymbol{R}_{c,0}(Q)$ is also easy. For a general G, we have to show merely $\boldsymbol{M}_p\subset\boldsymbol{R}(G_p)\cap\boldsymbol{Z}$, in virtue of Props. 2 and 4, and this last fact follows immediately from Lemmas 2 and 3.

The position of $\boldsymbol{R}_{a,0}$ is determined only for nilpotent groups. The result reads:

THEOREM 3. *For a non-abelian p-group G, we have $\boldsymbol{R}_{a,0}(G)=1$.*

PROOF by the induction on $|G|$. Let $N\neq 1$ be a normal subgroup of G such that G/N is non-abelian. Then $\boldsymbol{R}(G/N)=1$ by the induction assumption. Hence $\boldsymbol{R}\subset N$ by Prop. 1. It follows that if $\boldsymbol{R}\neq 1$ then $|\boldsymbol{R}|=p$. Furthermore, if we assume that a certain normal subgroup N does not contain $\boldsymbol{R}$, then G/N must be abelian. Hence the commutator group G' does not contain $\boldsymbol{R}$, and is a minimal normal subgroup of G. Hence G' is central, and is of order p. But then, the central subgroup $G'\boldsymbol{R}$ contains a further normal subgroup N of order

p other than G' and $\boldsymbol{R}$, and G/N is not abelian. This is a contradiction. Hence all normal subgroups of G contain $\boldsymbol{R}$. Now, let H be a minimal non-abelian normal subgroup of G. H contains an abelian normal subgroup A of G such that $(H:A)=p$, and is generated by A and an element $c \notin A$. The inner automorphism by c induces an automorphism γ of order p of $A: cac^{-1}=a^{\gamma}$ $(a \in A)$, and the center $\boldsymbol{Z}(H)$ coincides with the set of γ-invariant elements of A. Since $\boldsymbol{Z}(H)$ is normal in G, it contains $\boldsymbol{R}$. Put $Y=\{a\,;\, a \in A, a^{\gamma-1} \in \boldsymbol{Z}(H)\}$. Y is a normal subgroup of G, since it is the intersection of A and the second center of H. Further Y contains properly $\boldsymbol{Z}(H)$, since γ induces on $A/\boldsymbol{Z}(H)$ an automorphism of order p or 1. Hence there exists a normal subgroup K of G such that $Y \supset K \supset \boldsymbol{Z}(H)$ and that $(K:\boldsymbol{Z}(H))=p$. The mapping $a \to a^{\gamma-1}$ yields an isomorphism of $K/\boldsymbol{Z}(H)$ on the group $K^{\gamma-1}$ of order p. If we notice that $x^{-1}cxc^{-1} \in A$ for all $x \in G$, we see immediately that $K^{\gamma-1}$ is normal in G. Hence $K^{\gamma-1}$ must coincide with $\boldsymbol{R}$. Now let L be the group generated by K and c. Then L is certainly non-abelian. But we see that the commutator group of L coincides with $K^{\gamma-1}=\boldsymbol{R}$. This contradicts the property of $\boldsymbol{R}$. q. e. d.

On the contrary, we can construct non-abelian p-groups G admitting a subgroup $R \neq 1$ such that $R \in \mathfrak{R}'_{a,0}(G)$. A simple construction due to referee is as follows: $G=A \times B$, where A is abelian and B is non-abelian. It is then clear $A \in \mathfrak{R}'_{a,0}(G)$. More interesting is the following example which treats essentially the same type of groups as appeared in the above proof.

Example. Let A be an abelian group of type (p, p^{n-2}) $(n \geqq 4)$ generated by z and a of respective orders p and p^{n-2}. Let γ be an automorphism of A defined by $a^{\gamma}=za$, $z^{\gamma}=z$. Then γ is of order p, and the set of γ-invariant elements of A is $Z=(z)\times(a^p)$. Now, let G be generated by A and an element c with the defining relations $c^p \in Z$ and $cx=x^{\gamma}c$ for $x \in A$. We shall show $(a^p) \in \mathfrak{R}'_{a,0}(G)$. Since $ca=zac$, (a^p) is a central subgroup of G and $G/(a^p)$ is not abelian. Hence it suffices for our purpose to show that any proper subgroup of G is abelian. Thus, let M be a maximal subgroup other than A, then $A \cap M$ is a subgroup of A of index p and containing $z=cac^{-1}a^{-1}$. But then $A \cap M$ must coincide with Z. Since Z is the center, M is certainly abelian.

By Theorem 3, $\boldsymbol{R}_{a,0}(G_p)=G_p$ or 1 according to G_p is abelian or not. Combining Prop. 2, Prop. 4 and Lemma 3, we have immediately

Proposition 5. *If all Sylow subgroups of G are abelian, we have $\boldsymbol{R}_{a,0}(G)=\boldsymbol{M}_{a,0}(G)=\boldsymbol{Z}(G)$. For general G, we only have $\boldsymbol{R}_{a,0}(G) \subset \boldsymbol{M}_{a,0}(G)=\boldsymbol{Z}(G)$.*

Remark. The center $\boldsymbol{Z}(G)$ coincides with the hypercenter $\boldsymbol{H}(G)$ for such a group G. (A proof for this fact in case G is solvable is given in Taunt [**6**].) In fact we shall obtain the equality $\boldsymbol{R}_{a,0}(G)=\boldsymbol{M}_{a,0}(G)=\boldsymbol{Z}(G)=\boldsymbol{H}(G)$, if we use Prop. 6 which follows instead of Prop. 4.

We proceed to the study of $\boldsymbol{R}_{n,0}$. Let $C(p)$ be the characteristic subgroup of G consisting of elements commuting with every p-prime-order element of G. Since the q-Sylow subgroup of $C(p)$ is central in $C(p)$ for $q \neq p$, $C(p)$ is nilpotent. Hence $C(p)_p$ is likewise a characteristic subgroup of G, and consists of all p-power-order elements commuting with every p-prime-order element of G. Clearly $C(p) \supset \boldsymbol{Z}$. Hence the following proposition covers Prop. 4.

PROPOSITION 6. *If* $\mathfrak{P} \prec \mathfrak{P}_n$, *then* $\boldsymbol{R}(G; \mathfrak{P}, \mathfrak{Q}_0)_p$ *contains* $\boldsymbol{R}(G_p; \mathfrak{P}, \mathfrak{Q}_0) \cap C(p)_p$.

PROOF. Put $R = \boldsymbol{R}(G_p; \mathfrak{P}, \mathfrak{Q}_0) \cap C(p)_p$. Then R is a normal subgroup of G, since the normalizer of R contains G_p and all p-prime-order elements of G. Let $G^\sigma \supset H \supset R^\sigma$, and let $H/R^\sigma \in \mathfrak{P}$. Since R^σ is a p-group and H/R^σ is nilpotent, H_p is normal in H. Hence it admits a complement H^p. Let x be an arbitrarily fixed element of H_p, and put $xyx^{-1} = z_y \cdot y$ for every $y \in H^p$. Then $z_y \in R^\sigma$. Since any p-prime-order element of G^σ is the image by σ of a p-prime-order element of G, every element of R^σ commutes with every element of H^p. Hence the mapping $y \rightarrow z_y$ yields a homomorphism $H^p \rightarrow R^\sigma$. Since $(|H^p|, |R^\sigma|) = 1$, we must have $xy = xy$. It follows that $H = H_p \times H^p$. The rest of the proof is completely the same as in Prop. 4.

THEOREM 4. $\boldsymbol{R}_{n,0}(G) = \boldsymbol{M}_{n,0}(G) = \prod_p C(p)_p = \boldsymbol{H}(G)$ (*hypercenter*).

PROOF. By Prop. 6, $C(p)_p \subset \boldsymbol{R}_p$, and by Prop. 2, $\boldsymbol{R}_p \subset \boldsymbol{M}_p$, where $\boldsymbol{R} = \boldsymbol{R}_{n,0}(G)$ and $\boldsymbol{M} = \boldsymbol{M}_{n,0}(G)$. Now, every q-Sylow subgroup G_q for $q \neq p$ is contained in some maximal nilpotent subgroup, which contains necessarily $\boldsymbol{M}$, and *a fortiori* $\boldsymbol{M}_p$. Hence every element of $\boldsymbol{M}_p$ commutes with every element of G_q, $q \neq p$. This shows $\boldsymbol{M}_p \subset C(p)_p$. Thus the equality of first three groups of Theorem is shown. Since $\boldsymbol{H}(G)$ is the last term of the ascending central chain of G, it possesses the property $R'_{n,0}$. Now, let a subgroup R of G possess the property $R'_{n,0}$. Then, since R_pG_q/R_p is nilpotent, R_pG_q is also nilpotent. Hence every element of R_p commutes with every element of G_q, $q \neq p$, and we see $R_p \subset C(p)_p$. Hence $R \subset \prod_p C(p)_p$. We refer to Baer [**1**] Th. 1 for the final step $C(p)_p \subset \boldsymbol{H}(G)$.

REMARK. The equality of the last three groups of Theorem 4 is shown in Baer [**1**].

In the proof of the Theorem, we observe

COROLLARY. *If R possesses the property* $\mathfrak{R}'_{n,0}(G)$, *then also the property* $\mathfrak{R}_{n,0}(G)$.

2.2. It follows from Gaschütz [**3**, Satz 10 and Satz 3], that $\boldsymbol{R}_{n,n}$ contains Φ. $\boldsymbol{R}_{n,n}$ contains also $\boldsymbol{H} = \boldsymbol{R}_{n,0}$, since $\boldsymbol{R}(G; \mathfrak{P}, \mathfrak{Q}_1) \supset \boldsymbol{R}(G; \mathfrak{P}, \mathfrak{Q}_2)$ whenever $\mathfrak{Q}_1 \prec \mathfrak{Q}_2$. Now, Baer [**2**] introduced a notion of the weak hypercenter H_ω, and likewise proved that H_ω is a characteristic subgroup contained in $\boldsymbol{F}$ and containing Φ and $\boldsymbol{H}$ and further that any weakly hypercentral subgroup possesses the property $\mathfrak{R}'_{n,n}$ (but not necessarily $\mathfrak{R}_{n,n}$). Hence we may say that $\boldsymbol{R}_{n,n}$ and H_ω are considerably near to each other, and it will be an interesting problem to study more closely the relationship between them. Note also that $\boldsymbol{F}/\Phi$ is the direct product of elementary abelian groups (Gaschütz [**3**, Satz 13]).

Now, Gaschütz [**3**] introduced a characteristic subgroup $\Delta(G)$ as the intersection of all non-normal maximal subgroups of G. As a refinement of inclusion $\boldsymbol{R}_{n,n}\supset\Phi$, we shall prove

PROPOSITION 7. *$\boldsymbol{R}_{n,n}(G)$ contains $\Delta(G)$.*

A proof for this assertion is obtained using some results of Gaschütz as follows: Both $\boldsymbol{R}_{n,n}$ and Δ contain Φ. According to Prop. 1, we may therefore assume that G is Φ-free. But in this case, Δ coincides with the center (Gaschütz [**3**, Satz 15]) and hence is contained in $\boldsymbol{R}_{n,n}$. Now, we shall give here another proof which is independent of Gaschütz's results, and thus including a direct proof of Satz 16 (and of Satz 10) of [**3**]. By the definition of Δ, it follows immediately that $\Delta(G)^{\sigma}\subset\Delta(G^{\sigma})$ for any homomorphism $G\to G^{\sigma}$. Hence we may argue by induction, and have merely to prove $\Delta\in\mathfrak{R}'_{n,n}$. Thus, let H be a normal subgroup of G containing Δ and such that H/Δ is nilpotent. Since ΔH_p is normal in G, we have $G=\Delta H_p\cdot N_G(H_p)=\Delta N_G(H_p)$. If $N_G(H_p)\neq G$, then there exists a maximal subgroup M containing $N_G(H_p)$, and M must be normal in G (since $M\not\supset\Delta$). $H\frown M$ is a proper normal subgroup of H and contains $N_H(H_p)=H\frown N_G(H_p)$. But this is impossible as is well known.

Since any nilpotent subnormal subgroup is contained in a nilpotent normal subgroup (Itô [**5**]), we see readily $\boldsymbol{R}_{n,sn}(G)=\boldsymbol{R}_{n,n}(G)$. (See also Inagaki [**4**].)

In general, we have

PROPOSITION 8. *If $\mathfrak{P}_c\prec\mathfrak{P}\prec\mathfrak{P}_n$, we have $\boldsymbol{R}(G\,;\mathfrak{P},Q_{sn})\subset\boldsymbol{R}_{n,n}(G)$.*

PROOF. Put $R=\boldsymbol{R}(G\,;\mathfrak{P},Q_{sn})$. Let H be a normal subgroup of G^{σ} containing R^{σ} and such that H/R^{σ} is nilpotent. Then every subgroup of H containing R^{σ} is subnormal in G^{σ}. In particular $R^{\sigma}\{h\}$ is subnormal for every $h\in H$. Since $R^{\sigma}\{h\}/R^{\sigma}\in\mathfrak{P}_c\prec\mathfrak{P}$, $R^{\sigma}\{h\}$ itself is a $\mathfrak{P}$-group, and *a fortiori* nilpotent. Hence $\{h\}$ is subnormal in $R^{\sigma}\{h\}$, and hence in H. h being arbitrary, H must be nilpotent.

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References

[1] R. Baer, Group elements of prime power index, Trans. Amer. Math. Soc., **75** (1953), 20-47.

[2] R. Baer, Nilpotent characteristic subgroups of finite groups, Amer. J. Math., **75** (1953), 633-664.

[3] W. Gaschütz, Über die Φ-Untergruppe endlicher Gruppen, Math. Z., **58** (1953), 160-170.

[4] N. Inagaki, On some generalizations of a theorem of Gaschütz on finite groups, forthcoming.

[5] N. Itô, Über die Frattini-Gruppe einer endlichen Gruppe, Proc. Japan Acad., **31** (1955), 327-328.

[6] D. R. Taunt, On A-groups, Proc. Cambridge Phil. Soc., **45** (1949), 24-42.

[7] J. G. Thompson, Normal p-complements for finite groups, Math. Z., 72 (1960), 332-354.
[8] H. Wielandt, Eine Verallgemeinerung der invarianten Untergruppen, Math. Z., 45 (1939), 209-244.
[9] H. Zassenhaus, Lehrbuch der Gruppentheorie, 1937.