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Class formations V.
(Infinite extension of the p-adic field or the rational field)

Dedicated to Professor Z. Suetuna on his sixtieth birthday

By Yukiyosi KAWADA

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In the present paper we shall continue our study on class formations (Kawada [**12**]-[**14**]). Let k be an infinite algebraic extension of the p-adic number field Q_p. M. Mori has defined the fundamental group F_k of such a field k and has proved a class field theory over k by means of F_k (Mori [**16**]). Then the author has considered the cohomology theory over F_k and has given the cohomology-theoretic treatment of Mori's theory. Moreover, by a similar method the author has established a class field theory over an infinite algebraic extension k of the rational field Q.

In Part I of this paper we shall consider the structure of the fundamental group F_k of an infinite algebraic extension k of Q_p by giving a monotone family of subgroups of F_k. Then we shall consider the ramification theory of a finite normal extension K/k by means of these subgroups of F_k. This ramification theory is different from the known theory of Herbrand [**8**]. Our new theory is especially fitted to the case where k is an H-extension of Q_p in the sense of Satake [**23**]. Finally we shall consider the relation between our ramification groups and the norm-residue symbol $(\eta, K/k)$ $(\eta \in F_k)$ for an abelian extension K/k.

In Part II we shall assume that k contains all the roots of unity both in local and in global cases. Then we have two kinds of class formations over such fields. One is our class formation derived from local or global class field theory by considering suitable inverse limit groups (Kawada [**14**]) and the other is derived from Kummer theory (Kawada [**12**, 3]). We shall prove here that these two kinds of class formations are actually isomorphic. The isomorphism can be obtained by means of Hilbert's norm-residue symbol. We have already considered the relation between our class formation theory and Moriya's theory [**20**], [**21**] (Kawada [**14**, 6]). Therefore, we might say that we have unified both Moriya's theory and Kummer theory by our class formation theory.

Part I

§ 1. Preliminaries from ramification theory.

For the investigation of the structure of the fundamental group we need the ramification theory for a non-normal finite extension K/k of a local field k of finite degree over Q_p. Such a theory has been investigated by Krasner [**15**] (cf. also Satake [**23**] and Kawada [**10**]). Here we shall summarize the results as far as we shall use them later.

Let Q_p be the p-adic number field, k be a finite extension of Q_p. We denote by $\mathfrak{o}_k$, $\mathfrak{p}_k$, U_k the ring of integers, its prime ideal and the unit group of k respectively. Let $\mathfrak{o}_k/\mathfrak{p}_k \cong GF(q_k)$, where q_k is a power of p. Then U_k contains a multiplicative group W_k of order q_k-1 which is generated by a primitive q_k-1-th root of unity. Furthermore, we shall denote by $U_k(i)$ $(i=0,1,2,\cdots)$ the subgroups of U_k consisting of all elements $\alpha\equiv 1 \pmod{\mathfrak{p}^i}$ respectively. Especially, we have $U_k=U_k(0)$. We have then the direct decomposition $U_k=W_k\times U_k(1)$. We denote the non-archimedian valuation of k with the value group Z (the ring of all integers) by ord_k and we denote by π_k an element with $\mathrm{ord}_k(\pi_k)=1$.

Let Λ be the (smallest) normal extension of k containing K. We denote by $\mathfrak{G}=G(\Lambda/k)$ and $\mathfrak{H}=G(\Lambda/K)$ the Galois group of these normal extensions respectively. Let $e=\mathrm{ord}_\Lambda(\pi_K)$. The *group of inertia* $\mathfrak{T}_{K/k}$ is defined by

$$\mathfrak{T}_{K/k}=\{\sigma\,;\ \sigma\in\mathfrak{G},\ \rho^\sigma=\rho \quad \text{for all } \rho\in W_K\}\,. \tag{1}$$

Clearly $\mathfrak{H}\subseteqq\mathfrak{T}_{K/k}$ and the subfield of K corresponding to $\mathfrak{T}_{K/k}$ is denoted by $T_{K/k}$ (the field of inertia): $k\subseteqq T_{K/k}\subseteqq K$. We define $v(\sigma)=(s/e)-1$ for $\sigma\in\mathfrak{T}_{K/k}$ where $s=\mathrm{ord}_\Lambda(\pi_K{}^\sigma-\pi_K)$, and for a real number v

$$\mathfrak{V}_{K/k}(v)=\{\sigma\,;\ \sigma\in\mathfrak{T}_{K/k},\ v(\sigma)\geqq v\} \qquad (0\leqq v<\infty)\,. \tag{2}$$

Then $\mathfrak{V}_{K/k}(v)$ is a subgroup of $\mathfrak{T}_{K/k}$ containing $\mathfrak{H}$. Especially, we have $\mathfrak{T}_{K/k}=\mathfrak{V}_{K/k}(0)$. We denote the field corresponding to $\mathfrak{V}_{K/k}(v)$ by $V_{K/k}(v)$. We shall omit the index K/k if there is no danger of misunderstanding. In general $v>v'$ implies $\mathfrak{V}(v)\subseteqq\mathfrak{V}(v')$. For various values of v from 0 to ∞ we have

$$\mathfrak{G}\supseteqq\mathfrak{T}=\mathfrak{V}(0)\supseteqq\mathfrak{V}(1/e)=\cdots=\mathfrak{V}(v_1)\supset\mathfrak{V}(v_1+(1/e))=\cdots=\mathfrak{V}(v_2)\supset\cdots\cdots$$
$$\supset\mathfrak{V}(v_{r-1}+(1/e))=\cdots=\mathfrak{V}(v_r)\supset\mathfrak{V}(v_r+(1/e))=\cdots=\mathfrak{H}\,.$$

Then we denote

$$\mathfrak{V}_0=\mathfrak{T},\ \mathfrak{V}_1=\mathfrak{V}(v_1),\ \cdots,\ \mathfrak{V}_r=\mathfrak{V}(v_r),\ \mathfrak{V}_{r+1}=\mathfrak{H}\,. \tag{3}$$

The field corresponding to $\mathfrak{V}_i$ is denoted by V_i $(i=1,\cdots,r)$. Here all v_i $(i=1,\cdots,r)$ are rational numbers, but need not be integers. Let us put $n=[K:k]$

$=[\mathfrak{G}:\mathfrak{H}]$, $n_i=[K:V_i]=[\mathfrak{V}_i:\mathfrak{H}]$, $m_i=n_0/n_i=[V_i:T]=[\mathfrak{V}_0:\mathfrak{V}_i]$ $(i=0,1,\cdots,r+1)$. These are all integers. Thus we have

$$\mathfrak{G}\supseteqq\mathfrak{T}\supseteqq\mathfrak{V}_1\supset\cdots\supset\mathfrak{V}_r\supset\mathfrak{V}_{r+1}=\mathfrak{H}\,,$$
$$k\subseteqq T\subseteqq V_1\subset\cdots\subset V_r\subset K\,,$$
$$0=v_0<v_1<\cdots<v_r<v_{r+1}=\infty\,,$$
$$n\geqq n_0\geqq n_1>\cdots>n_r>n_{r+1}=1\,,$$
$$1=m_0\leqq m_1<\cdots<m_r<m_{r+1}=n_0\,.$$

Also we have $n=ef$, $e=n_0$, $f=[T:k]$ and $e=e_0p^m$, $(e_0,p)=1$ for $e_0=[V_1:T]=m_1$, $p^m=[K:V_1]=n_1$, where f is the degree, e is the exponent and e_0 is the reduced exponent of K/k.

After Hasse [**6**] we define the function $\psi_{K/k}$ of v $(0\leqq v<\infty)$ by

$$u=\psi_{K/k}(v)=v_0+(v_1-v_0)/m_1+\cdots+(v-v_j)/m_{j+1}\quad\text{for } v_j\leqq v\leqq v_{j+1}\,. \tag{4}$$

Let us put $u_j=\psi_{K/k}(v_j)$ $(j=0,1,\cdots,r+1)$. u_j are all rational numbers, but need not be integers:

$$0=u_0<u_1<\cdots<u_r<u_{r+1}=\infty\,.$$

We call the inverse function $\varphi_{K/k}$ of $\psi_{K/k}$ the *Hasse function*:

$$v=\varphi_{K/k}(u)=u_0+m_1(u_1-u_0)+\cdots+m_{j+1}(u-u_j)\quad\text{for } u_j\leqq u\leqq u_{j+1}\,. \tag{5}$$

If $\mathfrak{p}_k$ is unramified for K/k we define

$$\psi_{K/k}(v)=v\quad(0\leqq v<\infty),\qquad \varphi_{K/k}(u)=u\qquad(0\leqq u<\infty)\,. \tag{6}$$

If $\mathfrak{p}_k$ is tamely ramified for K/k (i.e. $r=0$, $\mathfrak{V}_1=\mathfrak{H}$) then we have by (5)

$$\varphi_{K/k}(u)=n_0u\qquad\text{for}\quad 0\leqq u<\infty\,. \tag{7}$$

We define the *canonical set* $\mathfrak{U}_{K/k}$ by

$$\begin{aligned}\mathfrak{U}_{K/k}&=\{u_0,u_1,\cdots,u_r\}\qquad\text{if } \mathfrak{V}_0\neq\mathfrak{V}_1\,,\\ &=\{u_1,\cdots,u_r\}\qquad\text{if } \mathfrak{V}_0=\mathfrak{V}_1\,,\end{aligned} \tag{8}$$

and the *canonical value* by $\mathfrak{u}_{K/k}=u_r$ $(<\infty)$. If $\mathfrak{p}_k$ is unramified for K/k we define $\mathfrak{U}_{K/k}=\emptyset$ and $\mathfrak{u}_{K/k}=-1$.

The following theorems hold just as in the case of a normal extension K/k:

(I) (*Herbrand's theorem*). *For* $k\subset\Omega\subset K$ *we have*

(i) $\qquad T_{K/\Omega}=T_{K/k}\cdot\Omega\,,\qquad V_{K/\Omega}(v)=V_{K/k}(v)\cdot\Omega\,,$

(ii) $\qquad T_{\Omega/k}=T_{K/k}\cap\Omega\,,\qquad V_{\Omega/k}(v)=V_{K/k}(\varphi_{K/\Omega}(v))\cap\Omega\,.$[1)]

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1) There is a misprint in Kawada [**10**, p. 29, line 10]. $v'=\psi_{\Omega/k}(v)$ or $v=\varphi_{\Omega/k}(v')$ should be corrected as $v'=\psi_{K/\Omega}(v)$ or $v=\varphi_{K/\Omega}(v')$.

(II) (*Chain theorem*). *For* $k \subset \Omega \subset K$ *we have*

$$\varphi_{K/k}(u) = \varphi_{K/\Omega} \circ \varphi_{\Omega/k}(u) \qquad \text{for} \quad 0 \leqq u < \infty .$$

(III) (*Monotone property of the canonical set and the canonical value*). *For* $k \subset \Omega \subset K$ *we have*

(i) $$\mathfrak{U}_{\Omega/k} \subseteqq \mathfrak{U}_{K/k}, \quad \mathfrak{u}_{\Omega/k} \leqq \mathfrak{u}_{K/k},$$

(ii) $$\mathfrak{U}_{K/k} = \mathfrak{U}_{\Omega/k} \cup \psi_{\Omega/k}(\mathfrak{U}_{K/\Omega}) .$$

(IV) (*Characterization of* V_j). $T_{K/k}$ *is characterized as the maximal unramified subfield of K. In general, let us put* $\boldsymbol{V}(u) = V_{K/k}(\varphi_{K/k}(u)) = V_j$ *for* $u_{j-1} < u \leqq u_j$ $(j = 1, \cdots, r)$. *Then* $\boldsymbol{V}(u)$ $(0 < u < \infty)$ *is characterized as the maximal subfield* Ω *of K with the property* $\mathfrak{u}_{\Omega/k} < u$.

(V) *If u is an integer, then* $\varphi_{K/k}(u)$ *is also an integer.*

(V) follows from the following lemma. Let the different of Ω/k be denoted by $\mathfrak{D}_{\Omega/k} = \mathfrak{p}_{\Omega}^{d(\Omega/k)}$ where $d(\Omega/k)$ is an integer.

LEMMA 1 (Hasse).

$$d(V_j/k) = m_j(\psi_{K/k}(v)+1)-(v+1) \quad \text{for } v_{j-1} \leqq v \leqq v_j \ (j = 1, 2, \cdots, r+1)$$

(see Hasse [**6**, p. 485]). Now let u be an arbitrary positive integer. Take v_j such that $v_{j-1} \leqq v \leqq v_j$ holds for $v = \varphi_{K/k}(u)$. Then $\varphi_{K/k}(u) = m_j(u+1) - d(V_j/k) - 1$ is an integer by Lemma 1, which proves (V).

The following results of Hasse [**6**] holds also for non-normal extensions[2].

(VI) (*Hasse's theorem*). *Let K/k be an arbitrary finite extension. Then*

(i) $N_{K/k}U_K(\varphi_{K/k}(i)) \subseteqq U_k(i) \qquad (i = 0, 1, 2, \cdots)$,

(ii) $N_{K/k}U_K(\varphi_{K/k}(i)+1) \subseteqq U_k(i+1) \qquad (i = 0, 1, 2, \cdots)$,

(iii) $U_k(i) = U_k(i+1) \cdot N_{K/k}U_K(\varphi_{K/k}(i))$ *holds if* $i \notin \mathfrak{U}_{K/k}$,

(iv) $[U_k(i) : U_k(i+1) \cdot N_{K/k}U_K(\varphi_{K/k}(i))] \leqq [\mathfrak{V}_j : \mathfrak{V}_{j+1}]$ *if* $i \in \mathfrak{U}_{K/k}$ $(i = u_j)$.

Let us denote $B_{K/k}(i) = N_{K/k}^{-1}(U_k(i)) \subseteqq U_K$ $(i = 0, 1, 2, \cdots)$. Then from (VI) follows immediately

(VI)* (i) $$B_{K/k}(0) = U_K ,$$

(ii) $$B_{K/k}(i) \supseteqq U_K(\varphi_{K/k}(i)) ,$$

(iii) $$B_{K/k}(i+1) \supseteqq U_K(\varphi_{K/k}(i)+1) .$$

Finally, let Ω/k be the maximal abelian extension contained in K. Then we have

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2) For the proof in Hasse [5] we need some modifications of the arguments in pp. 480-482 since K/k is not normal. But these do not cause any change of the rest of the proof.

(VIII) (i) $$[k^* : N_{K/k}K^*] = [\Omega : k],$$

(ii) $$[U_k(i) : U_k(i+1) \cdot N_{K/k}B_{K/k}(i)] = [\mathfrak{B}_{\Omega/k}\varphi_{\Omega/k}(i) : \mathfrak{B}_{\Omega/k}(\varphi_{\Omega/k}(i)+1)] \quad (i = 1, 2, \cdots),$$

(iii) $$[W_k : N_{K/k}W_K] = [U_k : U_k(1)N_{K/k}B_{K/k}(1)] = [\mathfrak{B}_{\Omega/k}(0) : \mathfrak{B}_{\Omega/k}(1)].$$

§ 2. The fundamental group of an infinite algebraic extension of Q_p.

Let us fix an algebraic closure $\tilde{Q}_p$ of the p-adic number field Q_p and we consider only algebraic extensions of Q_p contained in $\tilde{Q}_p$. Let k be an infinite algebraic extension of Q_p. Then we have $k = \bigcup_\lambda k_\lambda$ where k_λ are finite extensions of Q_p contained in k. Here we can choose a monotone sequence of subfields $Q_p \subseteqq k_1 \subseteqq k_2 \subseteqq \cdots$ such that

(1) $$k = \bigcup_{n=1}^{\infty} k_n$$

holds. The absolute order $N(k) = \lim[k_n : Q_p]$ is decomposed to the product

(2) $$N(k) = N_\infty(k) \cdot N_f(k)$$

where $N_\infty(k)$ is the formal product of all factors of the form p_ν^∞ (p_ν : a prime) and $N_f(k)$ is the formal product of (finite or infinite number of) finite factors $p_\nu^{r(\nu)}$. We call $N_\infty(k)$ the infinite part and $N_f(k)$ the finite part of $N(k)$. Similarly, let $[k_n : Q_p] = e_n f_n$, $e_n = \mathrm{ord}_{k_n}(p)$ and $N_{k_n/Q_p}(\pi_k) = p^{f_n}$. Then we can define the absolute exponent and the absolute degree of k by $E(k) = \lim e_n$ and $F(k) = \lim f_n$ respectively. The infinite part and the finite part of $E(k)$ and $F(k)$ can be defined similarly.

For a moment, let k be a finite extension of Q_p. By k^* we mean the multiplicative group of k, and by $\tilde{k}$ the compact completion of k^* with respect to the Artin topology (see Artin [**1**, p. 177]). Similarly, we denote by $\tilde{Z}$ the total completion of the ring of integers Z such that

(3) $$\tilde{Z} \cong \prod_p Z_p$$

holds, where Z_p means the ring of p-adic integers. We can identify canonically U_k with a subgroup of $\tilde{k}$: $U_k \subset \tilde{k}$, and every element $\alpha \in \tilde{k}$ can be represented uniquely by a fixed element $\pi \in k$ with $\mathrm{ord}_k(\pi) = 1$, $\lambda \in U_k$ and $m \in \tilde{Z}$ in the form

$$\alpha = \pi^m \lambda.$$

Hence we have the topological isomorphism

(4) $$\tilde{k} / U_k \cong \tilde{Z}$$

(where the left hand side is a multiplicative group and the right hand side is an additive group). If $k \supset k'$ we can naturally define the norm mapping

$N: \tilde{k} \to \tilde{k}'$ with the usual properties. Moreover, if k/k' is normal we can extend the automorphism of k/k' to $\tilde{k}/\tilde{k}'$.

Now let $k = \bigcup_n k_n$ be an infinite extension of Q_p as above. After Mori we define the *fundamental group* F_k of k as the inverse limit group of $\tilde{k}_n$:

$$F_k = \text{inv.}\lim \tilde{k}_n \tag{5}$$

with respect to the norm mapping $N_n : \tilde{k}_n \to \tilde{k}_{n-1}$ $(n = 2, 3, \cdots)$. Clearly F_k is a compact group. Since $N_n : U_{k_n} \to U_{k_{n-1}}$, $W_{k_n} \to W_{k_{n-1}}$, and $U_{k_n}(1) \to U_{k_{n-1}}(1)$ we can define

$$U_k = \text{inv.}\lim U_{k_n}, \quad W_k = \text{inv.}\lim W_{k_n}, \quad U_k^* = \text{inv.}\lim U_{k_n}(1)\,. \tag{6}$$

THEOREM 1. (i) *The fundamental group F_k contains the closed subgroup U_k and*

$$U_k = W_k \times U_k^* \qquad (direct)\,[3]\,. \tag{6}$$

(ii) *Let $M = M(k)$ be the set of all prime numbers p_ν such that p_ν^∞ is not a factor of $F_\infty(k)$. Then we have*

$$F_k / U_k \cong \prod_{p_\nu \in M} Z_{p_\nu}\,. \tag{7}$$

PROOF. (i) follows immediately from $U_{k_n} = W_{k_n} \times U_{k_n}(1)$. (ii) We can see easily that $F_k/U_k \cong \text{inv.}\lim (\tilde{k}_n / U_{k_n})$ with respect to the norm mapping N_n^*: $\tilde{k}_n/U_{k_n} \to \tilde{k}_{n-1}/U_{k_{n-1}}$. Let us take $\pi_n \in k_n$ such that $\text{ord}_{k_n}(\pi_n) = 1$. Then we have $N_n \pi_n \equiv \pi_{n-1}{}^{f_n/f_{n-1}} \bmod U_{k_{n-1}}$. By the isomorphism $\tilde{Z}^{(n)} = \tilde{k}_n / U_{k_n} \cong \tilde{Z}$ we may consider $F_k/U_k = \text{inv.}\lim \tilde{Z}^{(n)}$ with respect to the mapping

$$N_n^* : m \to \frac{f_n}{f_{n-1}} m \qquad (m \in \tilde{Z})\,.$$

Let us decompose each $\tilde{Z}^{(n)}$ as the direct product

$$\tilde{Z}^{(n)} = \tilde{Z}_1^{(n)} \cdot \tilde{Z}_2^{(n)}, \quad \tilde{Z}_1^{(n)} \cong \prod_{p_\nu \notin M} Z_{p_\nu}, \quad \tilde{Z}_2^{(n)} \cong \prod_{p_\nu \in M} Z_{p_\nu}\,.$$

Then we can see easily that $F_k/U_k \cong (\text{inv.}\lim \tilde{Z}_1^{(n)}) \cdot (\text{inv.}\lim \tilde{Z}_2^{(n)})$. Here the first factor reduces to $\{1\}$ and the second factor is isomorphic to $\prod_{p_\nu \in M} Z_{p_\nu}$ since N_n^* is an isomorphism of $\tilde{Z}_2^{(n)} \to \tilde{Z}_2^{(n-1)}$ $(n = 2, 3, \cdots)$, q. e. d.

Next we shall consider the structure of U_k. For that purpose we shall define the Hasse function for an infinite extension k/Q_p after the idea of Satake [**23**]. First we deine the *canonical set* and the *canonical value* of k/Q_p for infinite extension k by

$$\mathfrak{U}_{k/Q_p} = \bigcup_\lambda \mathfrak{U}_{k_\lambda/Q_p}\,, \qquad Q_p \subseteqq k_\lambda \subset k\,, \tag{8}$$

$$\mathfrak{u}_{k/Q_p} = \sup_\lambda \mathfrak{u}_{k_\lambda/Q_p} = \sup_n \mathfrak{u}_{k_n/Q_p} \tag{9}$$

3) U_k^* is the p-primary component of U_k and W_k is the p-complementary component of U_k.

respectively. Also we define the ramification field with parameter u by

$$V_{k/Q_p}(u)=\bigcup_{n=1}^{\infty} V_{k_n/Q_p}(\varphi_{k_n/Q_p}(u)) \qquad 0\leqq u<\infty . \tag{10}$$

In particular, we have

$$T_{k/Q_p}=V_{k/Q_p}(0)=\bigcup_{n=1}^{\infty} V_{k_n/Q_p}(0) . \tag{11}$$

By the chain theorem (II), § 1 the sequence $\{\varphi_{k_n/Q_p}(u); n=1,2,\cdots\}$ is monotone increasing for each fixed value of u. Hence we denote its limit by

$$\varphi_{k/Q_p}(u)=\lim_{n\to\infty}\varphi_{k_n/Q_p}(u) . \tag{12}$$

In particular, $\varphi_{k/Q_p}(0)=0$. We define $u^\infty=u_{k/Q_p}{}^\infty$ by

$$u^\infty=\varphi_{k/Q_p}{}^{-1}(\infty)=\sup\{\varphi_{k/Q_p}{}^{-1}(v);\ 0\leqq v<\infty\} . \tag{13}$$

Namely, $\varphi_{k/Q_p}(u)<\infty$ for $u<u^\infty$ and $\varphi_{k/Q_p}(u)=\infty$ for $u>u^\infty$. The value $\varphi_{k/Q_p}(u^\infty)$ is $\leqq\infty$. Also the value u^∞ can be 0 or ∞. The function $\varphi_{k/Q_p}(u)$ is continuous and strictly increasing for $0\leqq u\leqq u^\infty$. This means, in particular, $\lim_{u\to u^\infty}\varphi_{k/Q_p}(u)=\infty$ if $\varphi_{k/Q_p}(u^\infty)=\infty$.

An important special case is the following case.

DEFINITION. k/Q_p is called an *H-extension* if

$$\mathfrak{U}_{k/Q_p}\leqq u_{k/Q_p}{}^\infty \tag{14}$$

holds. There are following cases of *H*-extensions.

Case A. $\mathfrak{U}_{k/Q_p}=\{(u_0), u_1, \cdots, u_r\}$ and $[V_{k/Q_p}(u_i): T_{k/Q_p}]<\infty$ $(i=1,\cdots,r)$. In this case there is an integer n_0 such that for $n\geqq n_0$ we have $V_{k/Q_p}(u_i)\subseteqq k_n\cdot T_{k/Q_p}$ $(i=1,\cdots,r)$ and $\varphi_{k/Q_p}(u)=\varphi_{k_n/Q_p}(u)$ for $0\leqq u\leqq u_r$. Hence by the chain theorem (II) § 1 we have $\varphi_{k_{n+1}/k_n}(u)=u$ for $n\geqq n_0$ and for $0\leqq u\leqq u_r$.

Case A_1: $u^\infty=\infty$. In this case there is an integer n_0 such that we have for $n\geqq n_0$

$$\varphi_{k/Q_p}(u)=\varphi_{k_n/Q_p}(u) \qquad 0\leqq u<\infty . \tag{15}$$

Hence again by the chain theorem k_{n+1}/k_n $(n\geqq n_0)$ are unramified extensions.

Case A_2: $u^\infty=u_r$. In this case $\varphi_{k/Q_p}(u)=\infty$ for $u>u_r$. By the chain theorem there is an integer n_0 such that for $n\geqq n_0$ we have

$$\varphi_{k_{n+1}/k_n}(u)=\begin{cases} u & 0\leqq u\leqq v_r , \\ v_r+m^{(n)}(u-v_r) & v_r\leqq u<\infty , \end{cases} \tag{16}$$

where $v_r=\varphi_{k/Q_p}(u_r)$, and $\lim_{n\to\infty}m^{(n)}=\infty$.

Case B. $\mathfrak{U}_{k/Q_p}=\{(u_0), u_1, u_2, \cdots, u_i,\cdots\}$ and $[V_{k/Q_p}(u_i): T_{k/Q_p}]<\infty$ $(i=1,2,\cdots)$. In this case there is an integer $n(s)$ for any given s such that $V_{k/Q_p}(u_i)\subseteqq k_n\cdot T_{k/Q_p}$ for $n\geqq n(s)$ and $u_i\leqq s$, and $\varphi_{k/Q_p}(u)=\varphi_{k_n/Q_p}(u)$ for $n\geqq n(s)$ and $u\leqq s$

hold.

Case B_1: $\lim_{i\to\infty} u_i = \infty$ and hence $u^\infty = \infty$.

Case B_2: $\lim u_i = u^\infty < \infty$. In this case $\varphi_{k/Q_p}(u) = \infty$ for $u \geqq u^\infty$.

k/Q_p is called an *H_1-extension* in cases A_1 and B_1, and k/Q_p is called an *H_2-extension* in cases A_2 and B_2. Important examples of H_1-extension are the maximal abelian extensions of $\mathfrak{p}$-adic number fields (see Tamagawa [**25**], Satake [**23**] and Kawada [**10**]).

Now let k be an arbitrary infinite algebraic extension of Q_p. Let $\mathfrak{m} = \{m_1, m_2, \cdots, m_n, \cdots\}$ be a sequence of non-negative integers such that

$$N_n(U_{k_n}(m_n)) \subseteqq U_{k_{n-1}}(m_{n-1}) \qquad n = 2, 3, \cdots \tag{17}$$

hold, where $N_n = N_{k_n/k_{n-1}}$. Then we define the closed subgroup $U_k(\mathfrak{m})$ of U_k by

$$U_k(\mathfrak{m}) = \text{inv.}\lim U_{k_n}(m_n)\,. \tag{18}$$

As special cases of $U_k(\mathfrak{m})$ we define

$$U_k(u) = \text{inv.}\lim U_{k_n}([\varphi_{k_n/Q_p}(u)]) \qquad 0 \leqq u < \infty\,, \tag{19}$$

$$U_k{}^*(u) = \text{inv.}\lim U_{k_n}([\varphi_{k_n/Q_p}(u)]+1) \qquad 0 \leqq u < \infty\,. \tag{20}$$

In particular, we have by definition

$$U_k = U_k(0), \quad U_k{}^* = U_k{}^*(0)\,. \tag{21}$$

Here we shall verify the condition (17) for the sequences of (19) and (20). By the chain theorem we have $\varphi_{k_n/Q_p}(u) = \varphi_{k_n/k_{n-1}} \circ \varphi_{k_{n-1}/Q_p}(u)$, which implies $[\varphi_{k_n/Q_p}(u)] \geqq \varphi_{k_n/k_{n-1}}([\varphi_{k_{n-1}/Q_p}(u)])$. Hence from (VI) (i) in § 1 follows

$$\begin{aligned} N_n U_{k_n}([\varphi_{k_n/Q_p}(u)]) &\subseteqq N_n(\varphi_{k_n/k_{n-1}}([\varphi_{k_{n-1}/Q_p}(u)])) \\ &\subseteqq U_{k_{n-1}}([\varphi_{k_{n-1}/Q_p}(u)])\,. \end{aligned}$$

Similarly, we can verify (17) for the case (20) by means of (VI) (ii) in § 1.

THEOREM 2. *Let k be an infinite algebraic extension of Q_p. Then we have*

(i) $U_k(u) \supseteqq U_k(u')$ *for* $u < u'$,

(ii) $U_k{}^*(u) \supseteqq U_k{}^*(u')$ *for* $u < u'$,

(iii) $U_k(u) \supseteqq U_k{}^*(u)$ *for any* u,

(iv) $U_k{}^*(u) \supseteqq U_k(u')$ *for* $\varphi_{k/Q_p}(u^\infty) \leqq u < u'$.

(v) *For integers i $(0 \leqq i < \varphi_{k/Q_p}(u^\infty))$ let*

$$\nu_i = \varphi_{k/Q_p}{}^{-1}(i) \qquad i = 0, 1, 2, \cdots\,. \tag{22}$$

Then we have

$$U_k(u) = U_k(\nu_i) \qquad \text{for } \nu_i \leqq u < \nu_{i+1}\,, \tag{23}$$

$$U_k{}^*(u) = U_k(\nu_{i+1}) \qquad \text{for } \nu_i \leqq u < \nu_{i+1}\,. \tag{24}$$

(vi) $U_k(u)/U_k^*(u)$ $(u>0)$ *is either isomorphic to a finite elementary p-group or isomorphic to an infinite product of cyclic groups of order p.*

PROOF. (i), (ii), (iii) are immediate consequences of the definition. (iv) follows from the fact $[\varphi_{k_n/Q_p}(u)]+1 \leqq [\varphi_{k_n/Q_p}(u')]$ for sufficiently argel n. (v) follows from the fact that $[\varphi_{k_n/Q_p}(u)]=i$ for $\nu_i \leqq u < \nu_{i+1}$ for sufficiently large n. (vi) follows from

$$U_k(u)/U_k^*(u) = \text{inv.}\lim \{U_{k_n}([\varphi_{k_n/Q_p}(u)])/U_{k_n}([\varphi_{k_n/Q_p}(u)]+1)\} .$$

If we put a further assumption [R] we can prove that there are no other subgroups $U_k(\mathfrak{m})$ than $U_k(u)$ or $U_k^*(u)$ $(0 \leqq u < \infty)$. Namely, for arbitrary generation of k as $k=\bigcup_{n=1}^{\infty} k_n$ we have certainly $B_{k_n/k_{n-1}}(i) \supseteqq U_{k_n}(\varphi_{k_n/k_{n-1}}(i-1)+1)$ by (VI) (ii) in § 1. Our assumption is

$$\text{[R]} \qquad U_{k_n}(\varphi_{k_n/k_{n-1}}(i-1)) \subsetneqq B_{k_n/k_{n-1}}(i) \qquad \text{for } i=1,2,\cdots$$

and for $n \geqq n_0$, where n_0 is a certain large positive integer.

This condition [R] is satisfied if $[k_n : k_{n-1}]$ are primes for $n \geqq n_0$ and $q_n = p^{f_n} > p$ for $n \geqq n_0$. For, in (VI) (iv) in § 1 we have $[\mathfrak{V}_j : \mathfrak{V}_{j+1}] \leqq [k_n : k_{n-1}] =$ a prime and $[U_{k_n}(i) : U_{k_n}(i+1)] = q_n$, which implies $N_n U_{k_n}(\varphi_{k_n/k_{n-1}}(i)) \subsetneqq U_{k_{n-1}}(i+1)$.

THEOREM 3. *If k/Q_p satisfies the assumption* [R] *then any subgroup $U_k(\mathfrak{m})$ is either $U_k(u)$ for some u or $U_k^*(u')$ for some u'.*

PROOF. Let $\mathfrak{m}=\{m_1, m_2, \cdots\}$. By assumption [R] $N_n U_{k_n}(m_n) \subseteqq U_{k_{n-1}}(m_{n-1})$ implies $m_n \geqq \varphi_{k_n/k_{n-1}}(m_{n-1}-1)+1$. Hence let u_n be defined by the equality $\varphi_{k_n/Q_p}(u_n) = m_n - 1$ $(n=1,2,\cdots)$ $(0 \leqq u_n < \infty)$. Then we have $n_n \geqq u_{n-1}$ $(n=2,3,\cdots)$. Therefore, let $u_0 = \lim u_n$ $(0 \leqq u_0 \leqq \infty)$.

Case I. $u_0 = u_n$ for $n \geqq n_0$. In this case we have $m_n = \varphi_{k_n/Q_p}(u_0)+1$ for $n \geqq n_0$. Hence $U_k(\mathfrak{m}) = U_k^*(u_0)$ holds.

Case II. $u_0 \neq u_n$ for all n and $u_0 < \infty$. In this case let $\xi = \mathbf{lim}\, \xi_n$ [4] $\in U_k(\mathfrak{m})$, $\xi_n \in U_{k_n}(m_n)$. Then $\xi_n = N_{k_m/k_n}\xi_m$ $(n<m)$ implies that

$$\xi_n \in N_{k_m/k_n} U_{k_m}(\varphi_{k_m/Q_p}(u_m)+1) \subseteqq U_{k_n}([\varphi_{k_n/Q_p}(u_m)]+1) .$$

Let us fix n and let $m \to \infty$.

(i) $\varphi_{k_n/Q_p}(u_0) = i_0 =$ an integer for $n \geqq n_0$. Then we have $[\varphi_{k_n/Q_p}(u_m)] = i_0 - 1$ for sufficiently large m, and hence $\xi_n \in U_{k_n}(i_0)$. This means that $\xi_n \in U_{k_n}(\varphi_{k_n/Q_p}(u_0))$ for all $n \geqq n_0$ and hence $\xi \in U_k(u_0)$. Conversely, $\varphi_{k_n/Q_p}(u_0) > \varphi_{k_n/Q_p}(u_n) = m_n - 1$ implies $U_{k_n}(m_n) \supseteqq U_{k_n}(\varphi_{k_n/Q_p}(u_0))$. Hence we have $U_k(\mathfrak{m}) \supseteqq U_k(u_0)$. Therefore, $U_k(\mathfrak{m}) = U_k(u_0)$ holds in this case.

(ii) $\varphi_{k_n/Q_p}(u_0)$ is not an integer for any n. Then for any fixed n and for sufficiently large m we have $[\varphi_{k_n/Q_p}(u_0)] = [\varphi_{k_n/Q_p}(u_m)]$. Hence $\xi_n \in U_{k_n}([\varphi_{k/Q_p}(u_0)]+1)$

4) We shall distinguish two kinds of limits by lim (usual limit in a topological space) and **lim** (inverse limit).

holds for any n. This means $\xi \in U_k^*(u_0)$. Conversely, $m_n = \varphi_{k_n/Q_p}(u_n)+1 \leqq [\varphi_{k_n/Q_p}(u_0)]+1$ implies $U_k(\mathfrak{m}) \supseteqq U_k^*(u_0)$. Therefore, we have $U_k(\mathfrak{m}) = U_k^*(u_0)$ in this case.

Case III. $u_0 = \infty$. Then we have $U_k(\mathfrak{m}) = 1$, q. e. d.

§ 3. Ramification theory for a finite normal extension over an infinite extension of the p-adic number field.

Let k be an infinite algebraic extension of Q_p, and K be a finite extension of k. Let us associate with k the fundamental group F_k. Then there exists the natural injection $\varphi_{k,K}: F_k \to F_K$, which satisfies the chain condition $\varphi_{k,L} = \varphi_{K,L} \circ \varphi_{k,K}$ for $k \subset K \subset L$. Moreover, if K/k is normal with the Galois group $\mathfrak{G} = G(K/k)$, then $\sigma \in \mathfrak{G}$ operates on F_K/F_k such that the usual Galois theory holds.

LEMMA 2. *Let K/k be normal. Then U_K is invariant under $\sigma \in \mathfrak{G} = G(K/k)$, i. e. $U_K^\sigma = U_K$, and σ operates trivially on the factor group F_K/U_K. In particular, σ is the identity if and only if σ operates trivially on U_K.*

The proof of this Lemma is easy.

Now we shall define the ramification groups of K/k by means of the fundamental group of K. This definition is different from that of Herbrand [8]. For that purpose we shall consider first the usual case. Namely, for a moment let k be a $\mathfrak{p}$-adic number field and K/k be a finite normal extension. Then the group of inertia can be defined by

$$\mathfrak{T}_{K/k} = \{\sigma\,;\ \sigma \in \mathfrak{G},\ \rho^\sigma = \rho \quad \text{for all } \rho \in W_K\}\,.$$

The ramification groups can be defined by

$$\begin{aligned}\mathfrak{V}_{K/k}(i) &= \{\sigma\,;\ \sigma \in \mathfrak{T}_{K/k},\ \pi^\sigma \equiv \pi \mod \mathfrak{p}_K^{i+1} \text{ for all } \pi \in \mathfrak{p}_K\} \\ &= \{\sigma\,;\ \sigma \in \mathfrak{T}_{K/k},\ \eta^{\sigma-1} \in U_K(i+1) \text{ for all } \eta \in U_K\}\end{aligned}$$

since we have $U_K = W_K \times U_K(1)$ and for $\eta = 1+\pi \in U_K(1)$ $\eta^{\sigma-1}-1 = (\eta^\sigma - \eta)/\eta = (\pi^\sigma - \pi)/\eta \in \mathfrak{p}_K^{i+1}$ is equivalent to $\pi^\sigma \equiv \pi \mod \mathfrak{p}_K^{i+1}$.

DEFINITION. Let k be an infinite algebraic extension of Q_p and K/k be a finite normal extension with the Galois group $\mathfrak{G} = G(K/k)$. We define the *group of inertia* $\mathfrak{T}_{K/k}$ by

$$\mathfrak{T}_{K/k} = \{\sigma\,;\ \sigma \in \mathfrak{G},\ \rho^\sigma = \rho \text{ for all } \rho \in W_K\} \tag{1}$$

and the *ramification groups* by

$$\mathfrak{V}_{K/k}(u) = \{\sigma\,;\ \sigma \in \mathfrak{T}_{K/k} \text{ and } \eta^{\sigma-1} \in U_K(u) \text{ for all } \eta \in U_K\}\,, \tag{2}$$

$$\mathfrak{V}_{K/k}^*(u) = \{\sigma\,;\ \sigma \in \mathfrak{T}_{K/k} \text{ and } \eta^{\sigma-1} \in U_K^*(u) \text{ for all } \eta \in U_K\} \tag{3}$$

$$(0 \leqq u < \infty)\,.$$

In particular, we identify $\mathfrak{T}_{K/k}=\mathfrak{V}_{K/k}(0)$. That these are normal subgroups of $\mathfrak{G}$ can be seen easily. Also from Theorem 2 follows the inclusion relations:

$$\mathfrak{V}_{K/k}(u) \supseteqq \mathfrak{V}_{K/k}{}^{*}(u) \tag{4}$$

$$\mathfrak{V}_{K/k}(u) \supseteqq \mathfrak{V}_{K/k}(u'), \quad \mathfrak{V}_{K/k}{}^{*}(u) \supseteqq \mathfrak{V}_{K/k}{}^{*}(u') \quad \text{for } u<u'. \tag{5}$$

We denote the subfields of K corresponding to (1), (2) and (3) by $T_{K/k}$, $V_{K/k}(u)$ and $V_{K/k}{}^{*}(u)$ respectively.

Let $K=k(\theta)$ and θ be a root of an irreducible polynomial $f(X)$ with coefficients in k. Let $k=\bigcup_{n=1}^{\infty} k_n$, $[k_n:Q_p]<\infty$. Then all coefficients of $f(X)$ are contained in some k_{n_0}. Let $K_n=k_n(\theta)$. Then $K=\bigcup_{n=1}^{\infty} K_n$ and $[K:k]=[K_n:k_n]$ holds for $n \geqq n_0$. If K/k is normal then K_n/k_n are also normal with the same Galois group $\mathfrak{G}$ [5] for $n \geqq n_0$. Let $\eta \in F_K$, $\eta=\mathbf{lim}\,\eta_n$, $\eta_n \in \tilde{K}_n$. Then by definition $N_{K/k}\eta=\mathbf{lim}\,N_{K_n/k_n}\eta_n$ holds. We have also $\eta^{\sigma}=\mathbf{lim}\,\eta_n{}^{\sigma}$ for $\sigma \in \mathfrak{G}$.

Let us denote the group of inertia in the sense of Herbrand by $\mathfrak{T}_{K/k}{}^{(H)}$. By Herbrand [8] $\mathfrak{T}_{K/k}{}^{(H)}=\mathfrak{T}_{K_n/k_n}$ holds for sufficiently large n, and hence $\mathfrak{G}/\mathfrak{T}_{K/k}{}^{(H)}$ is a cyclic group of order f, where f is prime to $F_{\infty}(k)$. Similarly, the first ramification group $\mathfrak{V}_{K/k}{}^{(M)}$ in the sence of Moriya [**18**] is characterized by $\mathfrak{V}_{K/k}{}^{(M)}=\mathfrak{V}_{K_n/k_n}(1)$ for sufficiently large n. Now we shall compare these definitions with ours.

THEOREM 4. (i) *In general* $\mathfrak{T}_{K/k}{}^{(H)}$ *is a subgroup of* $\mathfrak{T}_{K/k}$. *If* $W_{K_n}=N_{K_m/k_n}W_{K_m}$ *hold for a sufficently large n and for all* $m>n$, *then we have* $\mathfrak{T}_{K/k}=\mathfrak{T}_{K/k}{}^{(H)}$.

(ii) $\mathfrak{V}_{K/k}{}^{*}=\mathfrak{V}_{K/k}{}^{(M)}$ *holds. Hence* $\mathfrak{V}_{K/k}{}^{*}$ *is a p-group.*

(iii) $\mathfrak{V}_{K/k}(u)/\mathfrak{V}_{K/k}{}^{*}(u)$ $(0<u<\infty)$ *are elementary abelian p-groups.*

PROOF. (i) Let $\eta \in W_K$ be represented by $\eta=\mathbf{lim}\,\eta_n$ $(\eta_n \in W_{K_n})$, then $\eta^{\sigma}=\eta$ holds if and only if $\eta_n{}^{\sigma}=\eta_n$ hold for all n. Hence we have $\mathfrak{T}_{K/k} \supseteqq \mathfrak{T}_{K/k}{}^{(H)}=\mathfrak{T}_{K_n/k_n}$ (for a sufficiently large n). Conversely, assume that $N_{K_m/K_n}W_{K_m}=W_{K_n}$ $(n<m)$ holds for $n \geqq n_1$. Then $\eta^{\sigma}=\eta$, $\eta=\mathbf{lim}\,\eta_n$ $(\eta_n \in W_{K_n}, \eta \in W_K)$ holds only if $\sigma \in T_{K_n/k_n}$ for $n \geqq n_1$. Therefore, we have $\mathfrak{T}_{K/k}=\mathfrak{T}_{K/k}{}^{(H)}$ in this case.

(ii) Let $\eta=\mathbf{lim}\,\eta_n$, $\eta \in U_K$, $\eta_n \in U_{K_n}$. Then $\eta^{1-\sigma}=\mathbf{lim}\,\eta_n{}^{1-\sigma}$. Hence $\sigma \in \mathfrak{T}_{K/k}$ belongs to $\mathfrak{V}_{K/k}{}^{*}$ if $\eta_n{}^{1-\sigma} \in U_{K/k}(1)$ for all n. This proves $\mathfrak{V}_{K/k}{}^{*} \supseteqq \mathfrak{V}_{K/k}{}^{(M)}$. Conversely, assume that $\mathfrak{V}_{K/k}{}^{*}$ were larger than $\mathfrak{V}_{K/k}{}^{(M)}$. Then $\mathfrak{V}_{K/k}{}^{*}$ would contain an element σ whose order r is prime to p. From a formula

$$\sigma^{p}-1=(\sigma-1)^{p}+p(\sigma-1)^{p-1}+\cdots+p(\sigma-1)$$

follows that $\eta_n{}^{\sigma-1} \in U_{K_n}(i)$ implies $\eta_n{}^{\sigma^{p}-1} \in U_{K_n}(j)$ $(j=\min(pi, \mathrm{ord}_{K_n}(p)+i))$ (see

5) We identify an automorphism σ of K/k and its restriction on K_n/k_n.

Artin [**1**, p. 81]). Since $\sigma^r=1$ and $(r, p)=1$ we can find a suitable integer a for any given N such that $\sigma=\sigma^{ap^N}$ holds. Hence for any given j we can choose N sufficiently large such that from $\eta_n^{\sigma-1}\in U_{K_n}(1)$ follows $\eta_n^{\sigma-1}=\eta_n^{\sigma'-1}\in U_{K_n}(j)$ $(\sigma'=\sigma^{ap^N})$. Therefore we have $\eta_n^{\sigma-1}=1$, and σ operates trivially on U_K. By Lemma 2 this would imply $\sigma=1$ which contradicts with the hypothese $\sigma\notin\mathfrak{V}_{K/k}^{(M)}$. Therefore we have $\mathfrak{V}_{K/k}^*=\mathfrak{V}_{K/k}^{(M)}$.

(iii) can be proved similarly, q. e. d.

REMARK. We have similarly

$$\mathfrak{V}_{K/k}(u)\supseteqq\bigcap_{n=n_0}^{\infty}\mathfrak{V}_{K_n/k_n}([\varphi_{K_n/Q_p}(u)])\,, \tag{6}$$
$$\mathfrak{V}_{K/k}^*(u)\supseteqq\bigcap_{n=n_0}^{\infty}\mathfrak{V}_{K_n/k_n}([\varphi_{K_n/Q_p}(u)]+1)\,.$$

THEOREM 5. *Let K/Q_p be an H-extension. Then*

(i) $\mathfrak{T}_{K/k}=\mathfrak{T}_{K/k}^{(H)}$ *and hence $\mathfrak{G}/\mathfrak{T}_{K/k}$ is a cyclic group of order f.*

(ii) $\mathfrak{T}_{K/k}/\mathfrak{V}_{K/k}^*$ *is a cyclic group of order $e^{(0)}$ where $[K:k]=ef$, $e=e^{(0)}p^m$, $(e^{(0)},p)=1$.*

(iii) *For sufficiently large n K_n/k_n has the same sequence of ramification groups*

$$\mathfrak{T}\supseteqq\mathfrak{V}_1\supset\mathfrak{V}_2\supset\cdots\supset\mathfrak{V}_r\supset\mathfrak{V}_{r+1}=1 \tag{7}$$

where $[\mathfrak{T}:\mathfrak{V}_1]=e^{(0)}$ and $\mathfrak{V}_i$ $(i=1,\cdots,r)$ are p-groups.

(iv) *The sequence of ramification groups for K/k coincides with* (7), *namely there is a sequence of integers*

$$0=v_0<v_1<v_2<\cdots<v_r<v_{r+1}=\infty$$

such that

$$\mathfrak{V}_{K/k}(u)=\mathfrak{V}_i \qquad \textit{for}\quad \nu_{v_i}\leqq u<\nu_{v_{i+1}}\,, \tag{8}$$

$$\mathfrak{V}_{K/k}^*(u)=\mathfrak{V}_{i+1} \qquad \textit{for}\quad \nu_{v_i}\leqq u<\nu_{v_{i+1}} \tag{9}$$

hold, where we put $\nu_i=\varphi_{K/Q_p}^{-1}(i)$.

PROOF. (i) For an H-extension $K=\bigcup_n K_n$ the reduced exponents of K_{n+1}/K_n are all equal to 1 for sufficiently large n (say $n\geqq n_0$). Hence we have $N_{n+1}W_{K_{n+1}}=W_{K_n}$ $(n\geqq n_0)$ by (VII) in § 1. From this follows $\mathfrak{T}_{K/k}=\mathfrak{T}_{K/k}^{(H)}$ as a corollary of Theorem 4 (i). (ii) follows from Theorem 4, (ii).

(iii) Let λ be a fixed positive number and let $K_\lambda=V_{K/Q_p}(\lambda)$, $k_\lambda=k\cap K_\lambda$. By the characterization of ramification fields (IV) in § 1 $V_{K/Q_p}(\lambda)$ is characterized by the property that it contains all subfields Ω of K with $\mathfrak{u}_{\Omega/Q_p}<\lambda$. From this property follows also that $k_\lambda\supseteqq V_{k/Q_p}(\lambda)$. If we take λ sufficiently large (namely, at least $>\mathfrak{u}_{k_1/Q_p}$) we may assume that $k_\lambda\supseteqq k_1$ and also $[K_\lambda:k_\lambda]=[K:k]$. Let us denote $\mu=\varphi_{k_\lambda/Q_p}(\lambda)$.

Let $\mathfrak{T}^{(\lambda)} \supsetneqq \mathfrak{V}_1^{(\lambda)} \supset \cdots \supset \mathfrak{V}_r^{(\lambda)} \supset 1$ be the sequence of ramification groups of K_λ/k_λ and let

$$\mathfrak{V}_i^{(\lambda)} = \mathfrak{V}_{K_\lambda/k_\lambda}(v_i^{(\lambda)}), \quad v_i^{(\lambda)} = \varphi_{K_\lambda/k_\lambda}(u_i^{(\lambda)}), \quad i = 1, \cdots, r. \tag{10}$$

Then we have

$$u_i^{(\lambda)} < \mu \qquad (i = 1, \cdots, r), \tag{11}$$

because by (II), (III), (IV) of § 1 we have

$$\psi_{k_\lambda/Q_p}(u_i^{(\lambda)}) = \psi_{k_\lambda/Q_p} \circ \psi_{K_\lambda/k_\lambda}(v_i^{(\lambda)}) = \psi_{K_\lambda/Q_p}(v_i^{(\lambda)}) < \lambda .$$

Since K is an H-extension $V_{K/Q_p}(\lambda)$ is a finite extension of Q_p. Hence we can take n_0 such that $k_{n_0} \supseteqq k_\lambda$ and $K_{n_0} \supseteqq K_\lambda$. Let $n \geqq n_0$. By the characterization of K_λ and the monotone property (III) in § 1 we have

$$\begin{cases} \varphi_{k_n/k_\lambda}(u) = u & \text{for } 0 \leqq u \leqq \varphi_{k_\lambda/Q_p}(\lambda) = \mu , \\ \varphi_{K_n/K_\lambda}(u) = u & \text{for } 0 \leqq u \leqq \varphi_{K_\lambda/Q_p}(\lambda) = \varphi_{K_\lambda/k_\lambda}(\mu) . \end{cases} \tag{12}$$

Substituting (12) into the chain relation $\varphi_{K_n/k_\lambda} = \varphi_{K_n/K_\lambda} \circ \varphi_{K_\lambda/k_\lambda} = \varphi_{K_n/k_n} \circ \varphi_{k_n/k_\lambda}$ we have

$$\varphi_{K_\lambda/k_\lambda}(u) = \varphi_{K_n/k_n}(u) \qquad \text{for} \quad 0 \leqq u \leqq \mu . \tag{13}$$

Now by the definition of Hasse function we have

$$\varphi_{K_\lambda/k_\lambda}(u) = v_r^{(\lambda)} + [K: T_{K/k}](u - u_r^{(\lambda)}) \qquad \text{for} \quad u_r^{(\lambda)} \leqq u < \infty . \tag{14}$$

Then we have by (13) $\varphi_{K_n/k_n}(u)$ has the same expression for $u_r^{(\lambda)} \leqq u \leqq \mu$. Then by the property of Hasse function that its slope can be equal to $[K_n: T_{K_n/k_n}] = [K: T_{K/k}]$ only for $u > u_{K_n/k_n}$ we can conclude that $\varphi_{K_n/k_n}(u)$ has the same expression (14) also for $u_r^{(\lambda)} \leqq u < \infty$. Hence we have the same Hasse function $\varphi_{K_n/k_n}(u) = \varphi_{K_\lambda/k_\lambda}(u)$ for $n \geqq n_0$. We denote simply $\varphi_{K/k}(u) = \varphi_{K_\lambda/k_\lambda}(u)$. Since the characteristic numbers u_i, v_i, n_i, m_i are determined uniquely by Hasse function (§ 1) we can conclude that the sequence of ramification groups of K_n/k_n $(n \geqq n_0)$ is

$$\begin{gathered} \mathfrak{T}^{(n)} \supsetneqq \mathfrak{V}_1^{(n)} \supset \cdots \supset \mathfrak{V}_r^{(n)} \supset 1 , \\ \mathfrak{V}_i^{(n)} = \mathfrak{V}_{K_n/k_n}(v_i), \quad v_i = \varphi_{K/k}(u_i), \quad n_i = [\mathfrak{V}_i^{(n)} : 1] \quad (i = 1, \cdots, r), \end{gathered} \tag{15}$$

where u_i, v_i and n_i $(i = 1, \cdots, r)$ are independent of n.

We shall prove, moreover, that the groups $\mathfrak{V}_i^{(n)}$ are the same for all $n \geqq n_0$:

$$\mathfrak{V}_i^{(n)} = \mathfrak{V}_i \qquad (i = 1, \cdots, r). \tag{16}$$

For that purpose we use the relation

$$(N_{n+1} U_{K_{n+1}}) U_{K_n}(v) = U_{K_n} \qquad 0 < v < \varphi_{K_\lambda/k_\lambda}(\mu) \tag{17}$$

which follows from (VI) in § 1 and from the fact that all the ramification groups $\mathfrak{V}_{K_{n+1}/K_n}(v)$ are the same for $0 < v < \varphi_{K_\lambda/k_\lambda}(\mu)$. Now let $\sigma \in \mathfrak{V}_i^{(n+1)}$.

Then we have $\eta_{n+1}^{1-\sigma} \in U_{K_{n+1}}(v_i+1)$ for all $\eta_{n+1} \in U_{K_{n+1}}$. Since every $\eta_n \in U_{K_n}$ can be represented by $\eta_n = \xi_n \cdot N_{n+1}\eta_{n+1}$, $\xi_n \in U_{K_n}(v_i+1)$, $\eta_{n+1} \in U_{K_{n+1}}$ we have $\eta_n^{1-\sigma} = \xi_n^{1-\sigma} \cdot N_{n+1}\eta_{n+1}^{1-\sigma} \in U_{K_n}(v_i+1)$ for $\sigma \in \mathfrak{V}_i^{(n+1)}$. Hence $\mathfrak{V}_i^{(n+1)} \subseteqq \mathfrak{V}_i^{(n)}$ holds. Since these two groups have the same order we have $\mathfrak{V}_i^{(n+1)} = \mathfrak{V}_i^{(n)}$, which proves (16).

(iv) By Theorem 2 we have only to consider the ramification groups $\mathfrak{V}_{K/k}(\nu_i)$, $i=1,2,\cdots$ (and in case A_2 also $\mathfrak{V}_{K/k}(u)$ for $\varphi_{K/Q_p}(u^\infty) \leqq u < \infty$). Now let us consider $\mathfrak{V}_{K/k}(\nu_i)$. Since $\varphi_{K_n/Q_p}(u) = \varphi_{K/Q_p}(u)$ for $0 \leqq u \leqq \nu_i+1$ for sufficiently large n (say for $n \geqq n_1 \geqq n_0$) and $\mathfrak{V}_{K_n/k_n}(v)$ are the same for all these n, we have by the remark (6)

$$\mathfrak{V}_{K/k}(\nu_i) \supseteqq \mathfrak{V}_{K_n/k_n}(i) \qquad \text{for} \quad n \geqq n_1 . \tag{18}$$

Now we shall prove the equality in (18). By the same reason as in (iii) it is enough for that purpose to see that

$$U_{K_n} = (N_n^* U_K) U_{K_n}(u) \qquad \text{for} \quad 0 < u \leqq \nu_i+1 \ (n \geqq n_1) . \tag{19}$$

holds, where we denote by $N_n^* U_K$ the range of η_n in the expression $\eta = \mathbf{lim}\, \eta_n$, $\eta \in U_K$, $\eta_n \in U_{K_n}$.

Let η_n be an arbitrary element in U_{k_n}. Then by (17) we have for $m > n > n_2$ $\eta_n = (N_m' \eta_m) \xi_n^{(m)}$, $\eta_m \in U_{K_m}$, $\xi_n^{(m)} \in U_{K_m}(u)$ where N_m' means the norm N_{K_m/K_n}. Since U_{K_n} and $U_{K_m}(u)$ are compact we can select a subsequence $\{m_i\}$ of $\{m\}$ such that $\lim_{i\to\infty} N_{m_i}' \eta_{m_i} = \eta_n^*$ in U_{K_n} and $\lim \xi_n^{(m_i)} = \xi_n$ in $U_{K_m}(u)$ exist. Next take a suitable subsequence of $\{m_i\}$ such that $\lim N_{K_m/K_{n+1}} \eta_m = \eta_{m+1}^*$ exists for this subsequence, etc. Finally by the diagonal procedure we can choose a subsequence $\{m(i)\}$ of $\{m_i\}$ such that in $U_{K_{n+r}}$

$$\lim_{i\to\infty} N_{K_{m(i)}/K_{n+r}} \eta_{m(i)} = \eta_{n+r}^* \qquad r=1,2,\cdots$$

exist. Then $\eta = \lim_{r\to\infty} \eta_{n+r}^*$ exists in U_K and we have $\eta_n = (N_n^* \eta) \xi_n$, which proves (19).

In case A_2 we have $\mathfrak{V}_{K/k}(u) = 1$ for $u \geqq \varphi_{K/Q_p}(u^\infty)$ by the reason $v_r < \varphi_{K/Q_p}(u^\infty)$, q. e. d.

Here we shall give a simple example of H-extension. Let $k_n = Q_p(\zeta_n)$, $p \neq 2$, $n=1,2,\cdots$ where ζ_n is a primitive p^n-th root of unity, and let $k = \bigcup_n k_n$. Then we know that $[k_n : Q_p] = p^{n-1}(p-1)$, $\mathfrak{U}_{k_n/Q_p} = \{0,1,\cdots,n-1\}$ and

$$\varphi_{k_n/Q_p}(u) = \begin{cases} (p^i-1)+p^i(p-1)(u-i) & \text{for } i \leqq u \leqq i+1, \ (i=0,1,\cdots,n-2), \\ (p^{n-1}-1)+p^{n-1}(p-1)(u-(n-1)) & \text{for } n-1 \leqq u < \infty . \end{cases}$$

Hence we have $u^\infty = \infty$, $\mathfrak{U}_{k/Q_p} = \{0,1,\cdots,n,\cdots\}$, $\varphi_{k/Q_p}(u) = \varphi_{k_n/Q_p}(u)$ for $0 \leqq u \leqq n$ and

$$T_{k/Q_p} = Q_p,\ V_{k/Q_p}(u) = k_n \qquad \text{for} \quad n-1 < u \leqq n \ (n=1,2,\cdots) .$$

Let K be an arbitrary normal extension of k with $[K: k]=s$. Let $K=k(\theta)$, $K_n=k_n(\theta)$ and $[K_n:k_n]=s$ for $n\geqq n_0$. We can apply Theorem 5 for K/k. Let u_i, v_i, n_i $(i=1,\cdots,r)$ be the characteristic numbers of $\varphi_{K/k}(u)$, and let us put $\mu_i=\varphi_{k/Q_p}{}^{-1}(u_i)$ $(i=1,\cdots,r)$. We have $\mathfrak{U}_{K_n/Q_p}=\mathfrak{U}_{k_n/Q_p}\cup\psi_{K_n/Q}\mathfrak{U}_{K_n/k_n}=\{0,1,\cdots,n-1\}\cup\{\mu_1,\cdots,\mu_r\}$ and $\varphi_{K/Q_p}(u)=\varphi_{K_n/Q_p}(u)$ holds for $0\leqq u\leqq n$ if we take $n\geqq N$. Here we choose N such that $N-1\geqq\mu_r$. Finally we can see that

$$\varphi_{K/k}(u)=\varphi_{K_n/k_n}(u) \qquad \text{for} \quad n\geqq N. \tag{20}$$

Now we shall compare our definition of ramification groups with that of Herbrand in two cases.

(I) Let k/Q_p be the extension in the above example. Let $\mathfrak{A}$ be an arbitrary primary ideal of $\mathfrak{o}_K$. Let $\mathfrak{A}\frown\mathfrak{o}_{K_n}=\mathfrak{p}_{K_n}{}^{m_n}$. Then we have $p\,m_n\geqq m_{n+1}\geqq p(m_n-1)+1$. Let u_n be defined by $u_n=\psi_{K_n/Q_p}(m_n)$. Then we can see easily from (20) that $m_n-1>\mu_r$ for sufficiently large n, say $n\geqq n_0$. This implies that $\mathfrak{V}_{K_n/k_n}(m_n-1)=1$ for $n\geqq n_0$. Therefore, the ramification groups of K/k in the sense of Herbrand are all trivial. The above argument can be applied to every H-extension k/Q_p.

(II) Let now K/k be completly ramified (Herbrand [8], p. 493). Then by Theorems 23, 24, 25 in his paper the ramification theory of Herbrand is effective in this case. We shall show that our ramification theory is also effective in this case. Namely, let us take N large such that K_{n+1}/K_n and k_{n+1}/k_n are tamely ramified for $n\geqq N$. Then we have $\varphi_{K_n/Q_p}(u)=\varphi_{K_n/K_N}\circ\varphi_{K_N/Q_p}(u)=e_{K_n/K_N}\varphi_{K_N/Q_p}(u)$ where e_{K_n/K_N} means the exponent of K_n/K_N. By (VI) in §1 we have also $N_{K_n/K_N}U_{K_n}(\varphi_{k_n/Q_p}(u))=U_{K_N}(\varphi_{K_N/Q_p}(u))$ for $\varphi_{K_N/Q_p}(u)=0,1,2,\cdots$. From these follows

$$\mathfrak{V}_{K/k}(u)=\mathfrak{V}_{K_n/k_n}(\varphi_{K_n/Q_p}(u)) \qquad \text{for} \quad n\geqq N. \tag{21}$$

This proves that the sequence of ramification groups for K/k are the same with that of K_N/k_N.

§4. Norm-residue symbol and ramification groups.

Let k_0 be an infinite algebraic extension of Q_p. The theory of abelian extensions over k_0 can be described by a class formation theory (Kawada [14]). Namely, let us associate with every finite extension k of k_0 the fundamental group F_k. Let K/k be a finite normal extension. As before let $K=\bigcup_n K_n$ and $k=\bigcup_n k_n$, $[K: k]=[K_n: k_n]$ $(n=1,2,\cdots)$. The fundamental 2-cocycle $\xi_{K/k}$ of the normal extension K/k is a 2-cocycle of the Galois group $\mathfrak{G}$ of K/k with coefficients in F_K, which is derived from the fundamental 2-cocyles ξ_{K_n/k_n} of K_n/k_n by

$$\xi_{K/k}(\sigma,\tau)=\lim\xi_{K_n/k_n}(\sigma,\tau) \qquad \sigma,\tau\in\mathfrak{G}. \tag{1}$$

Hence the norm-residue symbol for $\eta=\lim \eta_n \in F_k$ is represented by

(2) $$(\eta, K/k)=(\eta_n, K_n/k_n) \in \mathfrak{G}\,.$$

Now we shall consider on some properties on the relation between norm-residue symbol and ramification groups. Let us assume that K/k is abelian, and $K=\bigcup_n K_n$, $k=\bigcup_n k_n$ as before. The conductor $\mathfrak{f}_n=\mathfrak{p}_{k_n}{}^{c_n}$ of K_n/k_n is given by the formula

(3) $$c_n=\mathfrak{u}_{K_n/k_n}+1$$

and $\mathfrak{f}_n$ is characterized by the following properties:

(i) $(\eta_n, K_n/k_n)=1$ for all $\eta_n \in U_{k_n}(c_n)$,

(ii) there exists an $\eta_n \in U_{k_n}(c_n-1)$ such that $(\eta_n, K_n/k_n) \neq 1$.

By the translation theorem we have $(\eta_{n+1}, K_{n+1}/k_{n+1})=(N_{n+1}\eta_{n+1}, K_n/k_n)$ for $\eta_{n+1} \in \tilde{k}_{n+1}$. From this follows that

$$N_{n+1}U_{k+1}(c_{n+1}) \subseteqq U_{k_n}(c_n)\,.$$

Therefore, $U_k(\mathfrak{f})$ for the sequence $\mathfrak{f}=(c_1, c_2, \cdots)$ defines a subgroup of U_k which we shall call the *conductor* of K/k.

Especially if $k=\bigcup_n k_n$ satisfies the condition [R] in §3 we have

$$c_{n+1}-1 \leqq \varphi_{k_{n+1}/k_n}(c_n-1)\,.$$

Let us put $c_n-1=\varphi_{k_n/Q_p}(u_n)$. Then we have $u_1 \geqq n_2 \geqq u_3 \geqq \cdots$. Hence put $u_\infty=\lim u_n$. Then by Theorem 3 we have either $U_k(\mathfrak{f})=U_k^*(u_\infty)$ or $=U_k(u_\infty)$.

Next we shall consider the range of norm-residue symbol for an abelian extension K/k. Let H be a subgroup of F_k. We denote by $(H, K/k)$ the set of all the elements $(\eta, K/k)$ for $\eta \in H$.

Theorem 6. *For an abelian extension K/k we have*

$$(U_k, K/k) \subseteqq \mathfrak{T}_{K/k},\ (U_k(u), K/k) \subseteqq \mathfrak{V}_{K/k}(u),\ (U_k{}^*(u), K/k) \subseteqq \mathfrak{V}_{K/k}{}^*(u)$$
$$(0 \leqq u < \infty)\,.$$

In particular, we have $(U_k(\mathfrak{f}), K/k)=1$.

Proof. Let $\eta=\lim \eta_n$, $\eta_n \in \tilde{k}_n$. If η belongs to $U_k(u)$, then $\eta_n \in U_{k_n}([\varphi_{k_n/Q_p}(u)])$. Hence we have $(\eta, K/k)=\lim(\eta_n, K_n/k_n)$ belongs to

$$\bigcap_{n \geqq n_0} \mathfrak{V}_{K_n/k_n}([\varphi_{k_n/Q_p}(u)]) \subseteqq \mathfrak{V}_{K/k}(u)$$

by the remark (6) in §3. The others can be proved similarly.

Theorem 7. *Let k/Q_p be an H-extension, and K/k be an abelian extension.*

(i) *For the conductor $\mathfrak{f}_n=\mathfrak{p}_{k_n}{}^{c_n}$ of K_n/k_n c_n are the same for all sufficiently large n. We put $c=c_n$ (say for $n \geqq n_0$). Then the conductor of K/k is equl to $U_k(\varphi_{K/k}{}^{-1}(c))$.*

(ii) *We have*

$$(U_k, K/k) = \mathfrak{T}_{K/k},\ (U_k(u), K/k) = \mathfrak{V}_{K/k}(u),\ (U_k^*(u), K/k) = \mathfrak{V}_{K/k}^*(u)$$
$$(0 \leqq u < \infty).$$

PROOF. (i) Since k/Q_p is an H-extension we can apply the results of Theorem 5. Hence $\mathfrak{u}_{K_n/k_n}$ are the same for sufficiently large n (say for $n \geqq n_0$). This proves that c_n are the same for $n \geqq n_0$. Also we have $\varphi_{K/k}(u) = \varphi_{K_n/k_n}(u)$ $(0 \leqq u \leqq c)$ for sufficiently large n. From this follows that $U_k(\mathfrak{f}) = U_k(\varphi_{K/k}^{-1}(c))$.

(ii) We have proved in Theorem 5 that $\mathfrak{V}_{K/k}(\nu_i) = \mathfrak{V}_{K_n/k_n}(i)$ for sufficently large n. Also we have seen that the range of η_n in the expression of $\eta = \lim \eta_k$ for $\eta \in U_k$ is equal to U_{k_n} modulo $U_{k_n}(c_n)$ for sufficiently large n (say $n \geqq n_0$). Hence the range of $(\eta, K/k)$ for $\eta \in U_k$ is equal to the range of $(\eta_n, K_n/k_n)$ for $\eta_n \in U_{k_n}$ for $n \geqq n_0$. Also we know by the local class field theory that $(U_{k_n}, K_n/k_n) = \mathfrak{T}_{K_n/k_n}$ (see e. g. Hasse [4]). Moreover, we know that $\mathfrak{T}_{K/k} = \mathfrak{T}_{K_n/k_n}$ for sufficiently large n (Theorem 5). Combining these facts we have $(U_k, K/k) = \mathfrak{T}_{K/k}$. Similarly we have the rest of our formulas from the known result in local class field theory (see Iyanaga [9]), q. e. d.

Part II

§ 5. Class formations over a local field which contains all the roots of unity.

Let k be an infinite algebraic extension of Q_p. We shall prove first

LEMMA 3. *Let us assume that the infinite part of the absolute order $N_\infty(k)$ is divisible by factor p^∞ for every prime p. Then for any finite normal extension K/k we have*

(1) $$N_{K/k}K^* = k^*.$$

PROOF. Since K/k is solvable it is enough to prove (i) for a cyclic extension. Let $k = \bigcup_n k_n$, $K = \bigcup_n K_n$, K_n/k_n be normal and $[K: k] = [K_n : k_n] = r$. By local class field theory $H_n = N_{K_n/k_n}K_n^*$ is a subgroup of k_n^* of index r. Let $m > n$ be such that $[K_m : K_n]$ is divisible by r. Then by the translation theorem we have $H_m = N_{K_m/k_m}K_m^* = N_{k_m/k_n}^{-1}H_n$ which includes k_n^*. Hence k_n^* is contained in $H_m \subseteqq N_{K/k}K^*$. Since this inclusion holds for all n we have $k^* = N_{K/k}K^*$, q. e. d.

Suppose now that k *contains all the roots of unity*. Then the assumption in Lemma 3 is certainly satisfied. Hence k satisfies the condition:

K 1. the characteristic of k is 0,

K 2. k contains all the roots of unity,

K 3. for every finite normal extension K/k $k^* = N_{K/k}K^*$ holds.

Hence the class formation theory of Kummer extensions can be applied over such a field k (see Kawada [14, § 3]). Namely let us associate with each k the group

(2) $$E(k)=(k^*\otimes(Q/Z))^\wedge ,$$

where ^ means the (compact) character group, $\otimes$ means the tensor product over Z, and k^* is considered as a multiplicative group and Q/Z is considered as an additive group. Hence $\chi\in E(k)$ is a character of the abelian group $k^*\otimes(Q/Z)$:

$$\chi=\chi(\xi\otimes(r/s)) \qquad \xi\in k^*,\ r,s\in Z.$$

$(\mathrm{I})_K$ Let $K\supseteqq k$ and $[K:k]$ be finite. Then the injection $\varphi_{k,K}: E(k)\to E(K)$ is given by

(3) $$\varphi_{k,K}(\chi)(A)=\chi(N_{K/k}A),$$

where $A=\xi\otimes(r/s)\in K^*\otimes(Q/Z)$, $N_{K/k}A=N_{K/k}\xi\otimes(r/s)$. That $\varphi_{k,K}$ is an into-isomorphism follows from K 3.

$(\mathrm{II})_K$ If K/k is normal with the Galois group $\mathfrak{G}=G(K/k)$, then $\sigma\in\mathfrak{G}$ operates on $E(K)$

(4) $$\chi^\sigma(A)=\chi(A^{\sigma^{-1}}) \qquad A^\sigma=(\xi\otimes(r/s))^\sigma=\xi^\sigma\otimes(r/s) \quad (\xi\in K^*).$$

We have then $\varphi_{k,K}E(k)=E(K)^{\mathfrak{G}}$.

$(\mathrm{III})_K$ Let Ω^a/k be the maximal abelian extension and $\Gamma(k)$ be the Galois group of Ω^a/k with the compact Krull topology. Let W be the multiplicative group of all the roots of unity contained in k^* and let us fix an isomorphism $\psi: W\to Q/Z$. Then the generalized norm-residue symbol

(5) $$\Phi_k: E(k)\to\Gamma(k) \qquad \text{(topological isomorphism)}$$

is given as follows. Φ_k is characterized by its adjoint isomorphism: $\Phi_k{}^*$: $\Gamma(k)^\wedge\to E(k)^\wedge=k^*\otimes(Q/Z)$. Namely, for any given element $A=\xi\otimes(r/s)\in E(k)^\wedge$ the image $\lambda=(\Psi_k{}^*)^{-1}(A)\in\Gamma(k)^\wedge$ is defined by

(6) $$\lambda(\sigma)=\exp(2\pi i\,\psi(\xi^{r/s})^{\sigma-1}) \qquad (\sigma\in\Gamma(k)).$$

Therefore, the image $\sigma=\Phi_k(\chi)\in\Gamma(k)$ of $\chi\in E(k)$ is determined by the relation

(7) $$\lambda(\sigma)=\chi(\xi\otimes(r/s)).$$

Now we shall compare this class formation theory for Kummer extensions with our class formation theory (Kawada [**14**]) which we have applied in § 4. For that purpose we need Hilbert's norm-residue symbol.

Let now k be a finite extension of Q_p which contains the set W_n of all the n-th roots of unity. Then Hilbert's norm-residue symbol $\left(\frac{\alpha,\beta\,|\,k}{\mathfrak{p}}\right)_n$ ($\mathfrak{p}=\mathfrak{p}_k$ and $\alpha\in\tilde{k}$, $\beta\in k^*$) is defined by means of the norm-residue symbol

$$(\beta^{1/n})^\sigma=\left(\frac{\alpha,\beta\,|\,k}{\mathfrak{p}}\right)_n\beta^{1/n}, \quad \sigma=\left(\frac{\alpha,k(\beta^{1/n})}{\mathfrak{p}}\right).$$

If there is no danger of misunderstanding we omit k in the symbol. We use the following properties:

N 1. $\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)_n$ belongs to W_n.

N 2. $\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)_n = 1$ in case $\beta \in k^{*n}$ or $\alpha \in N\tilde{k}(\beta^{1/n})$.

N 3. $\left(\frac{\alpha_1\alpha_2, \beta}{\mathfrak{p}}\right)_n = \left(\frac{\alpha_1, \beta}{\mathfrak{p}}\right)_n \left(\frac{\alpha_2, \beta}{\mathfrak{p}}\right)_n$.

N 4. $\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)_n = \left(\frac{\beta, \alpha}{\mathfrak{p}}\right)_n^{-1}$ for $\alpha, \beta \in k^*$

N 3*. $\left(\frac{\alpha, \beta_1\beta_2}{\mathfrak{p}}\right)_n = \left(\frac{\alpha, \beta_1}{\mathfrak{p}}\right)_n \left(\frac{\alpha, \beta_2}{\mathfrak{p}}\right)_n$.

N 5. If $W_n \subset k$ and m divides n then

$$\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)_m = \left(\frac{\alpha, \beta}{\mathfrak{p}}\right)_n^{n/m} = \left(\frac{\alpha, \beta^{n/m}}{\mathfrak{p}}\right)_n.$$

N 6. Let τ be an automorphism of k then

$$\left(\frac{\alpha^\tau, \beta^\tau \mid k^\tau}{\mathfrak{p}^\tau}\right)_n = \left(\frac{\alpha, \beta \mid k}{\mathfrak{p}}\right)_n^\tau \qquad \text{for } \alpha \in \tilde{k}, \beta \in k^*.$$

N 7. Let $K \supset k$ and $k \supset W_n$ then

$$\left(\frac{\alpha, \beta \mid K}{\mathfrak{p}_K}\right)_n = \left(\frac{\alpha, \beta \mid k}{\mathfrak{p}_k}\right)_n \qquad \text{for } \alpha \in \tilde{k}, \beta \in k^*.$$

N 8. Let Ω/k be any finite extension and $k \supset W_n$. Then

$$\left(\frac{\alpha, \beta \mid \Omega}{\mathfrak{p}_\Omega}\right)_n = \left(\frac{N_{\Omega/k}\alpha, \beta \mid k}{\mathfrak{p}_k}\right)_n \qquad \text{for } \alpha \in \tilde{\Omega}, \beta \in k^*.$$

$$\left(\frac{\alpha, \beta \mid \Omega}{\mathfrak{p}_\Omega}\right)_n = \left(\frac{\alpha, N_{\Omega/k}\beta \mid k}{\mathfrak{p}_k}\right)_n \qquad \text{for } \alpha \in \tilde{k}, \beta \in \Omega^*.$$

From these follows

LEMMA 4. *Let k contain W_n (the set of all the n-th roots of unity). Then $\tilde{k}/\tilde{k}^n$ and k^*/k^{*n} are dual with respect to the inner product* $\left(\frac{\alpha, \beta \mid k}{\mathfrak{p}_k}\right)$.

For the proof see e. g. Hasse [**4**, § 14].

Let us change our notation. Let now k be an infinite algebraic extension of Q_p such that k contains all the roots of unity, which we denote by W. Let $k = \bigcup_n k_n$ as before, but we may assume that k_n contains W_n. Let us denote the fundamental group F_k by $F(k)$ which is a compact group. Let $E(k)^\wedge = k^* \otimes (Q/Z)$ as before.

(I) We shall define the function

$$F(k) \times E(k)^\wedge \to W \tag{8}$$

as follows: let $\eta = \mathbf{lim}\,\eta_n$, $\eta \in F(k)$, $\eta_n \in \tilde{k}_n$ and $A = \xi \otimes (r/s) \in E(k)^\wedge$. Let k_n be chosen such that $\xi \in k_n$ and n is divisible by s. Then we define the symbol

$$(9) \qquad (\eta, A) = \left(\frac{\eta_n, \xi^{rn/s} \mid k_n}{\mathfrak{p}_{k_n}}\right)_n \in W.$$

Firstly we must see that the valus of (9) is independent of the choice of n. It is enough to compare the cases of m and n where m is divisible by n. Then we have by N 5, N 7, N 8

$$\left(\frac{\eta_m, \xi^{rm/s} \mid k_m}{\mathfrak{p}_{k_n}}\right)_m = \left(\frac{\eta_m, \xi^{rn/s} \mid k_m}{\mathfrak{p}_{k_n}}\right)_n = \left(\frac{N_{k_m/k_n}\eta_m, \xi^{rn/s} \mid k_n}{\mathfrak{p}_{k_n}}\right)_n = \left(\frac{\eta_n, \xi^{rn/s} \mid k_n}{\mathfrak{p}_{k_n}}\right)_n.$$

Secondly let $A = \xi \otimes (r/s) = \eta \otimes (t/u)$ $(r, s, t, u \in Z)$, then we have $\xi^{ru} = \eta^{ts}$. Therefore let us take n such that n is divisible by su. Then we have $\xi^{rn/s} = \eta^{tn/u}$. This shows that the value of (9) is independent of the representation of A.

(II) (η, A) is a continuous bilinear function on $F(k) \times E(k)^\wedge$ with respect to the compact topology of $F(k)$ and the discrete topology of $E(k)^\wedge$.

That (η, A) is bilinear follows from N 3 and N 3* immediately. To prove that (η, A) is continuous it is enough to see that for any given $A \in E(k)^\wedge$ there exists a neighbourhood $U_k(u)$ of unity in $F(k)$ such that $(U_k(u), A) = 1$ holds. This follows from the fact that for any given β in k_n there exists $U_{k_n}(u)$ (i. e. the conductor of the extension $k_n(\beta^{1/n})/k_n$) such that $\left(\frac{U_{k_n}(u), \beta \mid k_n}{\mathfrak{p}_{k_n}}\right)_n = 1$ holds.

(III) $(F(k), A) = 1$ for $A \in E(k)^\wedge$ if and only if $A = 0$ and $(\eta, E(k)^\wedge) = 1$ for $\eta \in F(k)$ if and only if $\eta = 1$.

To prove the first statement let us take an element $A \neq 0$ in $E(k)^\wedge$. Then we may represent $A = \xi \otimes (1/n)$ and $\xi^{1/n}$ is not contained in k. We can assume here that $[k(\xi^{1/n}) : k] = n$. Let $k_m (\supset W_m)$ be such that $\xi \in k_m^*$ and m is divisible by n. Then by local class field theory the set of all η_m in $\tilde{k}_m$ which satisfy $\left(\frac{\eta_m, \xi^{m/n}}{\mathfrak{p}_{k_m}}\right)_m = 1$ makes a subgroup H_m of k_m^* of index n. By the translation theorem we see as in § 4 that the set of all $\eta \in F(k)$, $\eta = \mathbf{lim}\,\eta_n$ with $\left(\frac{\eta_m, \xi^{m/n}}{\mathfrak{p}_{k_m}}\right)_m = 1$ makes a subgroup H of $F(k)$ of index n. Hence $(F(k), A) = 1$ implies $A = 0$.

To prove the second statement let us assume that $(\eta, E(k)^\wedge) = 1$ for an η in $F(k)$. Let $\eta = \mathbf{lim}\,\eta_n$, $\eta_n \in \tilde{k}_n$. Then $\left(\frac{\eta_n, \beta \mid k_n}{\mathfrak{p}_{k_n}}\right)_n = 1$ must hold for all $\beta \in k_n^*$. By Lemma 4 this implies $\eta_n \in (\tilde{k}_n)^n$. This must hold for all n. Hence from $\eta_m \in (\tilde{k}_m)^m$ where m is divisible by n follows that $\eta_n = N_{k_m/k_n}\eta_m \in (\tilde{k}_n)^m$. Therefore, $\eta_n \in \bigcap_m (\tilde{k}_n)^m = 1$. This proves $\eta = 1$, q. e. d.

THEOREM 8. *Let (η, A) be the continuous bilinear function $F(k)\times E(k)^{\wedge}\rightarrow W$ defined above. Then $\eta\in F(k)$ defines uniquely an element $\chi_\eta\in E(k)$ by*

$$\chi_\eta(A)=(\eta, A) \qquad (A\in E(k)^{\wedge}). \tag{10}$$

By this mapping $\Psi_k: \eta\rightarrow\chi_\eta$ we have the topological isomorphism

$$\Psi_k:\ F(k)\cong E(k)=(k^*\otimes(Q/Z))^{\wedge}. \tag{11}$$

(i) *Let $k\subset K$. Then the following diagram is commutative, where $\iota_{k,K}$ is the natural injection.*

$$\begin{array}{ccc} F(k) & \xrightarrow{\iota_{k,K}} & F(K) \\ \downarrow \Psi_k & & \downarrow \Psi_K \\ E(k) & \xrightarrow{\varphi_{k,K}} & E(K) \end{array} \tag{12}$$

(ii) *Let K/k be normal with the Galois group $\mathfrak{G}=G(K/k)$. Then $\Psi_K: F(K)\rightarrow E(K)$ is an $\mathfrak{G}$-isomorphism, i. e.*

$$\sigma\circ\Psi_K=\Psi_K\circ\sigma \qquad \textit{for}\quad \sigma\in\mathfrak{G}. \tag{13}$$

PROOF. (I), (II), (III) show that $F(k)$ and $E(k)^{\wedge}$ are orthogonally paired by the bilinear function (η, A). Since $F(k)$ is compact and $E(k)^{\wedge}$ is discrete we have the topological isomorphism $\Psi_k: F(k)\rightarrow E(k)$ (see e. g. S. Lefscetz, Algebraic topology, p. 67, (20.6)).

To prove (i) let $\eta\in F(k)$. Then we may identify $\iota_{k,K}\eta=\eta\in F(K)$. Put $\Psi_k(\eta)=\chi_\eta$ and $\Psi_K(\eta)=X_\eta$. Then for $\eta=\mathbf{lim}\,\eta_n\in\tilde{k}_n$ and $A=\xi\otimes(r/s)$, $\xi\in K_n{}^*$ we have by N 4, N 8

$$X_\eta(\xi\otimes(r/s))=(\eta, \xi\otimes(r/s))=\left(\frac{\eta_n,\ \xi^{nr/s}\,|\,K_n}{\mathfrak{p}_{K_n}}\right)_n=\left(\frac{\eta_n,\ N_{K/k}\xi^{nr/s}\,|\,k_n}{\mathfrak{p}_{k_n}}\right)_n$$
$$=(\eta,\ N_{K/k}A)=(\varphi_{k,K}\chi_\eta)(A).$$

This proves (i).

To prove (ii) let $\eta\in F(K)$, $\eta=\mathbf{lim}\,\eta_n$, $\eta_n\in\tilde{k}_n$, $A=\xi\otimes(r/s)$, $\xi\in K^*$. Put $\Psi_K(\eta)=\chi_\eta$. Then by N 6 we have

$$\chi_\eta{}^\sigma(A)=\chi_\eta(A^{\sigma^{-1}})=(\eta, \xi^{\sigma^{-1}}\otimes(r/s))=\left(\frac{\eta_n,\ (\xi^{\sigma^{-1}})^{rn/s}\,|\,K_n}{\mathfrak{p}_{K_n}}\right)_n=\left(\frac{\eta_n{}^\sigma,\ \xi^{rn/s}\,|\,K_n}{\mathfrak{p}_{K_n}}\right)_n$$
$$=(\eta^\sigma, A)=\Psi_K(\eta^\sigma)(A).$$

This proves (13), q. e. d.

From Theorem 8 follows finally

THEOREM 9. *Let k be an infinite algebraic extension of Q_p such that k contains all the roots of unity. Then the two class formations over k, namely that of Kummer extensions and that of the natural extension of the local class field theory, are isomorphic to each other. Moreover, the generalized norm-residue* *symbols of these two class formations coinside with each other.*

PROOF. That these two class formations are isomorphic means the existence of an isomorphism $\Psi_k: F(k) \to E(k)$ with the properties (i), (ii) in Theorem 8. Hence we need only to compare two kinds of generalized norm-residue symbols.

Let $\eta \in F(k)$ and $\eta = \mathbf{lim}\, \eta_n$, $\eta_n \in \tilde{k}_n$. Then for any finite cyclic extension $K = k(\beta^{1/m})$, $\beta \in k^*$ the generalized norm-residue symbol $\Phi_k(\eta) = (\eta, k) \in \Gamma(k)$ (defined by class field theory) induces on K/k the automorphism σ

$$(14) \qquad (\beta^{1/m})^{\sigma-1} = \left(\frac{\eta_n, K_n/k_n}{\mathfrak{p}_{k_n}}\right) = \left(\frac{\eta_n, \beta^{n/m} \mid k_n}{\mathfrak{p}_{k_n}}\right)_n$$

by the definition of Hilbert's norm-residue symbol, where we take k_n such that $\beta \in k_n$ and n is divisible by m.

On the other hand, let $\Psi_k(\eta) = \chi_\eta \in E(k)$, and let the generalized norm-residue theorem (defined by Kummer theory) be denoted by $\tau = \Phi_k(\chi_\eta)$. Then by definition (6), (7) of $(\mathrm{III})_K$ we have for any $\beta \otimes (r/s) \in E(k)^\wedge$

$$(15) \qquad \lambda(\tau) = \chi_\eta(\beta \otimes (r/s)) = \left(\frac{\eta_n, \beta^{rn/s} \mid k_n}{\mathfrak{p}_{k_n}}\right)_n,$$

where we take k_n such that $\beta \in k_n$ and n is divisible by s.

Comparing (14) and (15) we see that two kinds of norm-residue symbols are identical, q. e. d.

§ 6. Class formations over a global field which contains all the roots of unity.

Let k_0 be an infinite algebraic extension of the rational field Q. As we have proved (Kawada [**14**, Theorem 7]) we have a class formation theory over k_0 which is a natural extension of class field theory.

Namely, let $k = \bigcup_n k_n$, $k_1 \subset k_2 \subset \cdots$, $[k_n : Q] < \infty$. Let $C(k_n)$ be the idèle class group of k_n and $C_0(k_n)$ be the compact subgroup of $C(k_n)$ consisting of all idèle classes of volume 1. We associate with k the inverse limit group of $\{C_0(k_n)\}$ with respect to the norm-mapping $N_{k_{n+1}/k_n}: C_0(k_{n+1}) \to C_0(k_n)$. We shall call this limit group the *fundamental group* of k and denote it by

$$(1) \qquad F(k) \to \mathrm{inv.}\lim C_0(k_n)\,.$$

$F(k)$ is a compact group. For a finite extension K/k we have a natural injection $\iota_{k,K}: F(k) \to F(K)$, and for a normal extension K/k with the Galois group $\mathfrak{G} = G(K/k)$ $F(K)$ is a $\mathfrak{G}$-group and $\iota_{k,K} F(k) = F(K)^{\mathfrak{G}}$ holds. It was proved that the system $F(k)$ gives a class formation over k_0.

Let (α_n, k_n) be the generalized norm-residue symbol of k_n which gives the continuous homomorphism of $C_0(k_n)$ onto the Galois group $\Gamma(k_n)$ of the maximal abelian extension Ω_n^a of k_n. The kernel of this mapping $\alpha_n \to (\alpha_n, k_n)$ is the connected component $D_0(k_n)$ of $C_0(k_n)$ (see Weil [**26**] and Artin [**2**]). Let $\alpha = \lim \alpha_n$, $\alpha \in F(k)$, $\alpha_n \in C_0(k_n)$. For an arbitrary abelian extension K/k, $K = k(\theta)$ we have as before $K = \bigcup_n K_n$, $k = \bigcup_n k_n$, $K_n = k_n(\theta)$. Then the symbol (α, k) induces the automorphism $(\alpha, K/k)$ which is given by $(\alpha, K/k) = (\alpha_n, K_n/k_n)$ (for $n \geqq n_0$). From this follows that the kernel $D(k)$ of the mapping $\alpha \to (\alpha, k)$ $(\alpha \in F(k))$ is given by

(2) $$D(k) = \text{inv.}\lim D_0(k_n)\,.$$

Let us put then

(3) $$F^*(k) = F(k)/D(k)\,.$$

We have also

(4) $$F^*(k) = \text{inv.}\lim C_0(k_n)/D_0(k_n)\,.$$

If k is totally imaginary then k_n are also total imaginary (for $n \geqq n_0$). Then for any normal extension K/k, $K = k(\theta)$, $K = \bigcup_n K_n$, $k = \bigcup_n k_n$, $K_n = k_n(\theta)$ the Galois group $\mathfrak{G} = G(K/k)$ has trivial cohomologies for the coefficient groups $D_0(K_n)$ $(n \geqq n_1)$ (see Weil [**26**] and Artin [**2**]). Hence it is the same for $D(k)$. Now we can easily verify the following theorem.

THEOREM 10. *Assume that k_0 is total imaginary. If we associate with each finite extension k of k_0 the compact group $F^*(k)$ we have also a class formation. Moreover, $F^*(k)$ is topologically isomorphic to the Galois group $\Gamma(k)$ of the maximal abelian extension Ω^a/k with the compact Krull topology.*

Now we shall consider the case where the ground field k_0 contains all the roots of unity. In this case we shall compare the two class formations. One is the above defined $F^*(k)$ and the other is $E(k) = (k^* \otimes (Q/Z))^\wedge$ of the Kummer extensions. The purpose of this § is to give explicitly the isomorphism of these two class formations. For this purpose we need some more properties of Hilbert's norm-residue symbol. These properties were prepared in a paper of Satake [**24**] (in Japanese). For the sake of completeness we shall reproduce them here.

Let now k be a finite number field containing a n-th primitive root of unity. We denote by W_n the group of n-th roots of unity as in § 5. The idèle group $J_k = J$ is the restricted direct product of the locally compact groups $k_\mathfrak{p}^*$ with respect to the compact open subgroups $U_\mathfrak{p}$ (unit group of $k_\mathfrak{p}^*$) (except infinite prime divisors $\mathfrak{p}_\infty$). Then the quotient locally compact group J/J^n is the restricted direct product of $k_\mathfrak{p}^*/k_\mathfrak{p}^{*n}$ with resepect to the compact open subgroups $U_\mathfrak{p} k_\mathfrak{p}^{*n}/k_\mathfrak{p}^{*n}$ (except $\mathfrak{p} = \mathfrak{p}_\infty$):

(5) $$J/J^n = \prod{}' k_\mathfrak{p}^*/k_\mathfrak{p}^{*n}\,.$$

k^* can be considered as a subgroup of J by identifying each element $a \in k^*$ with the principal idèle (a).

(I) $k^* \cap J^n = k^{*n}$.

PROOF. Let $(a) \in k^* \cap J^n$. Then $k_\mathfrak{p}(a^{1/n}) = k_\mathfrak{p}$ hold for all $\mathfrak{p}$ which implies $k(a^{1/n}) = k$, i. e. $a \in k^{*n}$, q. e. d.

Let $V(\mathfrak{a})$ be the volume of an idèle $\mathfrak{a} \in J$. Let J_0 be the subgroup of J consisting of all idèles with volume 1. Since $J \cong J_0 \times R$ (direct) we have

$$J/J^n \cong J_0/J_0{}^n. \tag{6}$$

Now we shall define the inner product $(\mathfrak{a}, \mathfrak{b} \,|\, k)_n$ for idèles $\mathfrak{a} = \{\mathfrak{a}_\mathfrak{p}\}$ and $\mathfrak{b} = \{\mathfrak{b}_\mathfrak{p}\} \in J$ by

$$(\mathfrak{a}, \mathfrak{b} \,|\, k)_n = \prod_{\text{all finite } \mathfrak{p}} \left(\frac{\mathfrak{a}_\mathfrak{p}, \mathfrak{b}_\mathfrak{p} \,|\, k_\mathfrak{p}}{\mathfrak{p}} \right)_n. \tag{7}$$

Here $\left(\frac{\mathfrak{a}_\mathfrak{p}, \mathfrak{b}_\mathfrak{p}}{\mathfrak{p}} \right)_n = 1$ if (i) n is not divisible by $\mathfrak{p}$ and (ii) $\mathfrak{a}_\mathfrak{p} \in U_\mathfrak{p}$ and $\mathfrak{b}_\mathfrak{p} \in U_\mathfrak{p}$. Hence $(\mathfrak{a}, \mathfrak{b} \,|\, k)_n$ is certainly well defined. This symbol has the following properties which will follow easily from N 1–N 8 in § 5 and the product formula of Hilbert's norm-residue symbol. We shall use also the notation $(\mathfrak{a}, \mathfrak{b})_n$ instead of $(\mathfrak{a}, \mathfrak{b} \,|\, k)_n$.

N*1. $(\mathfrak{a}, \mathfrak{b})_n \in W_n$.

N*2. $(\mathfrak{a}, \mathfrak{b})_n = (\mathfrak{b}, \mathfrak{a})_n{}^{-1}$.

N*3. $(\mathfrak{a}_1\mathfrak{a}_2, \mathfrak{b})_n = (\mathfrak{a}_1, \mathfrak{b})_n(\mathfrak{a}_2, \mathfrak{b})_n$,
$(\mathfrak{a}, \mathfrak{b}_1\mathfrak{b}_2)_n = (\mathfrak{a}, \mathfrak{b}_1)_n(\mathfrak{a}, \mathfrak{b}_2)_n$.

N*4. (Product formula) $(\mathfrak{a}, \mathfrak{b})_n = 1$ if $\mathfrak{a} \in k^*$ and $\mathfrak{b} \in k^*$.

N*5. If $k \supset W_n$ and m divides n then
$(\mathfrak{a}, \mathfrak{b})_m = (\mathfrak{a}, \mathfrak{b})_n{}^{n/m}$.

N*6. Let τ be an automorphism of k then
$(\mathfrak{a}^\tau, \mathfrak{b}^\tau \,|\, k^\tau)_n = (\mathfrak{a}, \mathfrak{b} \,|\, k)_n{}^\tau$ for $\mathfrak{a}, \mathfrak{b} \in J_k$.

N*7. Let $K \supset k$ and $k \supset W_n$. Then
$(\mathfrak{a}, \mathfrak{b} \,|\, K)_n = (\mathfrak{a}, \mathfrak{b} \,|\, k)_n$ for $\mathfrak{a}, \mathfrak{b} \in J_k$.

N*8. Let Ω/k be any finite extension and $k \supset W_n$. Then
$(\mathfrak{a}, \mathfrak{b} \,|\, \Omega)_n = (N_{\Omega/k}\mathfrak{a}, \mathfrak{b} \,|\, k)_n$ for $\mathfrak{a} \in J_\Omega$, $\mathfrak{b} \in J_k$.

From N*1 and N*3 follows that $(\mathfrak{a}, \mathfrak{b})_n = 1$ if $\mathfrak{a} \in J^n$ or $\mathfrak{b} \in J^n$. Hence $(\mathfrak{a}, \mathfrak{b})_n$ gives a pairing of J/J^n and J/J^n.

(II)* J/J^n is self-dual with respect to the inner product $(\mathfrak{a}, \mathfrak{b})_n$.

We use the following general lemma. Let G and X be locally compact abelian groups which are orthogonally paired by the inner product (g, χ) $(g \in G, \chi \in X)$ into R/Z. Moreover, if there exist compact open subgroups G_1 of G and X_1 of X such that they are mutually annihilators of the other. Then we have $G \cong X^\wedge$ and $X \cong G^\wedge$, where ^ means the character group in the sense of Pontrjagin.

Now we shall prove (II). By Lemma 4 we can easily see that J/J^n and

J/J^n are orthogonally paired by the inner product $(\mathfrak{a}, \mathfrak{b})_n$. Next we shall prove that there exist compact open subgroups H_1 and H_2 of J/J^n which are mutually annihilators of the other. Let E be a finite set of prime divisors. Then we denote $J^E = \{\mathfrak{a} : \mathfrak{a} \in J, \mathfrak{a}_\mathfrak{p} \in U_\mathfrak{p}$ for $\mathfrak{p} \notin E\}$ and $J^{E,n} = \{\mathfrak{a} ; \mathfrak{a} \in J, \mathfrak{a}_\mathfrak{p} \in k_\mathfrak{p}^n$ for $\mathfrak{p} \in E$ and $\mathfrak{a}_\mathfrak{p} \in U_\mathfrak{p}$ for $\mathfrak{p} \notin E\}$. If we take E such that (i) E contains all infinite prime divisors, (ii) E contains all finite prime divisors which divides n. Then (II)* follows from the following Lemma (III)*:

(III)* $J^E \cdot J^n/J^n$ and $J^{E,n} \cdot J^n/J^n$ are compact open subgroups of J/J^n which are mutually annihilators of the other.

PROOF. That these are compact open subgroups follows immediately from (5). That they are mutually annihilators of the other follows from Lemma 4 and from the fact that $(U_\mathfrak{p}, U_\mathfrak{p} | k_\mathfrak{p})_n = 1$ for $\mathfrak{p} \notin E$, q.e.d.

(IV)* By the above self-duality of J/J^n the discrete subgroup k^*J^n/J^n is the annihilator of itself.

PROOF. Let E be a finite set of prime divisors which satisfy the above conditions (i), (ii) and (iii) $J = J^E k^*$. Then we shall first prove that

$$J^n J^{E,n} k^*/J^n \quad \text{and} \quad J^n k^E/J^n \tag{8}$$

are mutually annihilators of the other, where we denote $k^E = k^* \cap J^E$. By the self-duality of J/J^n it suffices to see that

$$(J^n J^{E,n} k^*, J^n k^E | k)_n = 1 \tag{9}$$

and

$$[J : J^n J^{E,n} k^*] = [J^n k^E : J^n] < \infty . \tag{10}$$

Here (9) is easy to see. We shall prove (10). Since $J = J^E k^*, J^n \subset (J^E)^n k^* \subset J^{E,n} k^*$ we have

$$[J : J^n J^{E,n} k^*] = [J^E k^* : J^{E,n} k^*] = [J^E : J^{E,n}]/[J^E \cap k^* : J^{E,n} \cap k^*] .$$

Since $k_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}^{1/n}) = k_\mathfrak{p}$ for $\mathfrak{a}_\mathfrak{p} \in U_\mathfrak{p}$ if $\mathfrak{p} \nmid n$ we can prove

$$k^* \cap J^{E,n} = (k^E)^n \tag{11}$$

quite similarly as (I)*. Hence we have

$$[J^E \cap k^* : J^{E,n} \cap k^*] = [k^E : (k^E)^n] = n^{s+1} , \tag{12}$$

where s is the number of prime divisors of E (see Chevalley [**3**, § 3, p. 121, Corollary of Theorem 3]). Also we know that

$$[J^E : J^{E,n}] = n^{2(s+1)} \tag{13}$$

(see Chevalley [**3**, § 8.2, p. 129]). Substituting (12), (13) into the above equality we have

$$[J : J^n J^{E,n} k^*] = n^{s+1} . \tag{14}$$

On the other hand we have by (11)

(15) $$[J^n k^E : J^n] = [k^E : k^E \cap J^n] = [k^E : (k^E)^n] = n^{s+1}.$$

From (14) and (15) follows (10), q. e. d.

Now the proof of (IV)* is immediate. Since $k^* = \cup_E k^E$ and $J^n = \cap_E J^n J^{E,n}$ where E runs over the family of all finite sets of prime divisors, we have

$$(k^* J^n / J^n)^{\perp} = \cap_E (k^E J^n / J^n)^{\perp} = \cap_E (k^* J^n J^{E,n} / J^n) = k^* J^n / J^n.$$

Here $^{\perp}$ means the annihilator with respect to the inner product. q. e. d.

As a corollary of (IV)* we have

(V)* The compact group $J/k^* J^n \cong C(k)/C(k)^n$ and the discrete group $k^* J^n / J^n \cong k^*/k^{*n}$ are mutually the character groups of the other.

Applying the above results of Satake, it is now easy to prove the isomorphism of two class formations.

Let k_0 be an infinite algebraic number field such that *k contains all the roots of unity.* We denote by W the group of all the roots of unity as in § 5. In this case we can see that three conditions K 1, K 2, and K 3 in § 5 hold. Here the condition K 3 follows from a theorem of Hasse [**5**]. Hence we have a class formation theory of Kummer extensions by assosiating with each finite extension k of k_0 the compact group $E(k) = (k^* \otimes (Q/Z))^{\wedge}$. We shall compare $E(k)$ and the group $F^*(k)$ defined in Theorem 10.

(I) Let $k = \cup_n k_n$ $[k_n : Q] < \infty$ and $W_n \subset k_n$. We shall define a function $F^*(k) \times E(k) \to W$ as follows. Let $\eta = \mathbf{lim}\, \eta_n$, $\eta \in F^*(k)$, $\eta_n \in C_0(k_n)/D_0(k_n)$ and $A = a \otimes (r/s) \in E(k)^{\wedge}$, $a \in k^*$. Then we can take k_n such that $a \in k_n$ and n is divisible by s. Then we put

(16) $$(\eta, A) = (\mathfrak{a}_n, a^{rn/s})_n \in W,$$

where $\mathfrak{a}_n$ is an arbitrary representative idèle of the class η_n. If we take another representative $\mathfrak{a}_n'$ then $\mathfrak{a}_n' = \mathfrak{a}_n(a')\mathfrak{b}$ holds where $a' \in k_n$ and $\mathfrak{b} \in D_0(k_n)$. Since $D_0(k_n)$ is infinitely divisible (Weil [**26**] and Artin [**2**]) we have $\mathfrak{b} \in J_{k_n}^n$. Hence by N*4 we have $(\mathfrak{a}_n, a^{rn/s})_n = (\mathfrak{a}_n', a^{rn/s})_n$. That the value of (16) is independent of the choice of n and the representation $A = a \otimes (r/s)$ can be proved just as in local case by N*5, N*7, N*8.

(II) (η, A) is a continuous bilinear function on $F^*(k) \times E(k)^{\wedge}$ with respect to the compact topology of $F^*(k)$ and the discrete topology of $E(k)^{\wedge}$.

Proof. That (η, A) is bilinear follows from N*3. To prove the continuity of (η, A) it suffices to see that for any given $A \in E(k)^{\wedge}$ there exists an open subgroup H of $F^*(k)$ such that

(17) $$(H, A) = 1$$

holds. This follows immediately from the fact that for any given $a_n \in k_n$ the set H consisting of all idèles $\mathfrak{b} \in J_{k_n}$ with the property $(\mathfrak{b}, a_n)_n = 1$ is an open subgroup of J_{k_n} of index $[k_n(a_n^{1/n}) : k_n]$ and that H contains $k_n^* D_0(k_n)$.

(III) (Orthogonality) $(F^*(k), A)=1$ implies $A=0$ and $(\eta, E(k)^\wedge)=1$ implies $\eta=1$.

PROOF. The first statement can be proved just as in local case by applying class field theory. For the proof of the second statement we use the result of Satake. Namely, let $\eta=\mathbf{lim}\,\eta_n$. Then $(\eta, E(k)^\wedge)=1$ implies $(\eta_n, k_n^*)=1$. Hence by (IV) we have $\eta_n \in k_n^* J_{k_n}{}^n D_0(k_n)/k_n^* D_0(k_n)$. If we take any multiple m of n then $\eta_n = N_{k_m/k_n}\eta_m \in k_n^* J_{k_n}{}^m D_0(k_n)/k_n^* D_0(k_n)$. Hence we have

$$\eta_n \in \bigcap_m k_n^* J_{k_n}{}^m D_0(k_n)/k_n^* D_0(k_n) = 1\,.$$

This proves $\eta=1$, q. e. d.

From (II) follows that each $\eta \in F^*(k)$ determines uniquely an element $\chi_\eta \in E(k)$ defined by

$$\chi_\eta(A) = (\eta, A) \qquad (A \in E(k)^\wedge)\,. \tag{17}$$

Since $F^*(k)$ is compact and $E(k)^\wedge$ is discrete we have by (III) the topological isomorphism

$$\Psi_k:\ F^*(k) \cong E(k) = (k^* \otimes (Q/Z))^\wedge \tag{18}$$

by the mapping $\Psi_k: \eta \to \chi_\eta$.

(IV) Let $k \subset K$. Then we have the following commutative diagram:

$$\begin{array}{ccc} F^*(k) & \xrightarrow{\iota_{k,K}} & F^*(K) \\ \Psi_k \downarrow & & \downarrow \Psi_K \\ E(k) & \xrightarrow{\varphi_{k,K}} & E(K) \end{array} \quad . \tag{19}$$

(V) Let K/k be normal with the Galois group $\mathfrak{G}=G(K/k)$. Then Ψ_K: $F^*(K) \cong E(K)$ is an $\mathfrak{G}$-isomorphism:

$$\sigma \circ \Psi_K = \Psi_K \circ \sigma \qquad \sigma \in \mathfrak{G}\,. \tag{20}$$

We can prove (IV), (V) just as in local case by N*4, N*6, N*8. Summing up (I)—(V) we have the following theorem:

THEOREM 11. *Let k_0 be an infinite algebraic number field such that k_0 contains all the roots of unity. There are two kinds of class formations: one is that of defined in Theorem* 10 *and the other is that of Kummer extensions. Then they are isomorphic by the mapping* (18).

Finally we shall mention here briefly the case of an algebraic function field k of one variable with absolutely algebraic and algebraically closed constant field Λ of characteristic p. Namely, Λ is the union of all finite fields $GF(q)$, $q=p^n$ $(n=1,2,\cdots)$. Then for a suitable finite field $GF(q_0)$, $q_0=p^{n_0}$ there is an algebraic function field k_{n_0} with constant field $GF(q_0)$ such that k is the union of k_{n_0} and Λ. Hence if we put $k_n = k_0 \cdot GF(p^{nn_0})$ which has

$GF(p^{nn_0})$ as constant field, then we have $k = \bigcup_n k_n$. Therefore, we can derive a class formation over k as the limit of the usual class field theory over k_n.

Let J_{k_n} be the group of all idèles of k_n and let $J_{k_n}{}^0$ be the group of all idèles with degree 0. We denote the group of idèle classes by $C(k_n) = J_{k_n}/k_n{}^*$ and also $C_0(k_n) = J_{k_n}{}^0/k_n{}^*$. Here $C_0(k_n)$ is a compact group and $C(k_n)/C_0(k_n) \cong Z$. As in Weil [**26**] we can compactify the group $C(k_n)$. We denote the compactification by $C^*(k_n)$. Then $C^*(k_n) \supset C_0(k_n)$, $C^*(k_n)/C_0(k_n) \cong \tilde{Z}$, $\tilde{Z} = \prod_p Z_p$. Moreover, we denote $J_{k_n}{}^\phi = \prod_\mathfrak{p} U_\mathfrak{p}$ and $\mathfrak{C}_0(k_n) = J_{k_n}{}^0/k^* \cdot J_{k_n}{}^\phi$ (group of divisor classes of degree 0), which is a finite group.

Now the *fundamental group* $F(k)$ is defined by

$$F(k) = \text{inv.}\lim C^*(k_n) \tag{21}$$

with respect to the norm-mapping $N_{k_{n+1}/k_n}: C^*(k_{n+1}) \to C^*(k_n)$. By local class field theory the image $N_{k_{n+1}/k_n} C^*(k_{n+1})$ is the group of all idèle classes whose degrees are divisible by $[k_{n+1}: k_n]$. From this follows that if we express $\eta = \mathbf{lim}\, \eta_n$, $\eta \in F(k)$, $\eta_n \in C^*(k_n)$ the range of η_n is $C_0(k_n)$. Hence we can write

$$F(k) = \text{inv.}\lim C_0(k_n) \tag{22}$$

where $N_{k_{n+1}/k_n} C_0(k_{n+1}) = C_0(k_n)$. $F(k)$ contains a closed subgroup

$$F^\phi(k) = \text{inv.}\lim k_n{}^* J_{k_n}{}^\phi / k_n{}^* \tag{23}$$

such that we have the factor group:

$$\mathfrak{F}(k) = F(k)/F^\phi(k) = \text{inv.}\lim \mathfrak{C}_0(k_n) \tag{24}$$

where $N_{k_{n+1}/k_n} \mathfrak{C}_0(k_{n+1}) = \mathfrak{C}_0(k_n)$ holds.

By the general theory (Kawada [**14**]) we have a class formation over such a field k if we associate with every finite separable extension K the fundamental group $F(K)$ and $F(K)$ is topologically isomorphic to the Galois group $\Gamma(K)$ of the maximal separable abelian extension $\Omega^a(K)/K$ with the compact Krull topology.

In a former paper (Kawada-Satake [**13**, § 3]) we have proved the isomorphism of two kinds of class formations of separable p-extensions of k_n. Namely, let $\mathfrak{V}(k_n)$ be the group of all Witt's vectors over k_n and let

$$\mathfrak{W}(k_n) = (\mathfrak{V}(k_n)/\mathfrak{p}\mathfrak{V}(k_n)) \otimes (Q^{(p)}/Z) \tag{25}$$

where $Q^{(p)} = \{a/p^n;\ a \in Z,\ n = 0, 1, 2, \cdots\}$ and

$$E(k_n) = \mathfrak{W}(k_n)^\wedge. \tag{26}$$

Then $\mathfrak{W}(k_n)$ and the compact group $Z_p \times (C_0(k_n)/C_0(k_n)^\infty)$ are orthogonally paired by the inner product defined in § 3 of our joint paper. Here we put $C_0(k_n)^\infty = \bigcap_m C_0(k_n)^{p^m}$.

Similarly we define

(27) $$E(k)=\mathfrak{W}(k)^{\wedge},\ \mathfrak{W}(k)=(\mathfrak{V}(k)/\wp\mathfrak{V}(k))\otimes(Q^{(p)}/Z)).$$

Then we have $\mathfrak{W}(k)=\text{dir.}\lim\mathfrak{W}(k_n)$ by the natural injection and dually

$$E(k)=\text{inv.}\lim E(k_n).$$

Let $F(k)^{\infty}=\bigcap_m F(k)^{p^m}$. Then we can define a pairing (η, A) of $\eta\in F(k)/F(k)^{\infty}$ and $A\in\mathfrak{W}(k)$ as follows. Since A belongs to some $\mathfrak{W}(k_n)$ we define

(28) $$(\eta, A)=(\eta_n, A)$$

where $\eta=\mathbf{lim}\,\eta_n$, $\eta_n\in C_0(k_n)/C_0(k_n)^{\infty}$. We can prove similarly to the formula (38) in §2 of Kawada-Satake [**13**], that the value of (28) is independent of the choice of n, i.e. $(N_{k_m/k_n}\eta_m, A)=(\eta_m, A)$ for $m>n$. It is not difficult to prove the following theorem:

THEOREM 12. *Let k_0 be an algebraic function field of one variable with absolutely algebraic and algebraically closed constant field of characteristic p. Let us consider separable p-extensions of k_0. Then two kinds of class formations $\{E(k)\}$ and $\{F(k)/F(k)^{\infty}\}$ are isomorphic.*

Now we shall consider only separable unramified p-extensions of k_0. We have then also a class formation in this case (Kawada-Satake [**13**, §4]). Namely let us associate with k the compact group

(29) $$^*E(k)={}^*\mathfrak{W}(k)^{\wedge},\ {}^*\mathfrak{W}(k)=({}^*\mathfrak{V}(k)/\wp{}^*\mathfrak{V}(k))\otimes(Q^{(p)}/Z))$$

where $^*\mathfrak{V}(k)$ is the group of all unramified Witt's vectors over k. We can consider $^*\mathfrak{W}(k)$ naturally as a subgroup of $\mathfrak{W}(k)$. Then we have

LEMMA 5. *By the duality of $\mathfrak{W}(k_n)$ and $Z_p\times(C_0(k_n)/C_0(k_n)^{\infty})$ the subgroup $^*\mathfrak{W}(k_n)$ of $\mathfrak{W}(k_n)$ and the subgroup $Z_p\times(J_{k_n}{}^{\phi}k_{k_n}{}^*(J_{k_n}{}^0)^{\infty})/k_n{}^*(J_{k_n}{}^0)^{\infty}$ of the latter group are muatually annihilators of the other.*

Hence by taking the limit groups we have

THEOREM 13. *By the orthogonal pairing of the discrete group $\mathfrak{W}(k)$ and the compact group $F(k)/F(k)^{\infty}$ the subgroup $^*\mathfrak{W}(k)$ of $\mathfrak{W}(k)$ and the closed subgroup $F^{\phi}(k)/F(k)^{\infty}$ of $F(k)/F(k)^{\infty}$ are mutually annihilators of the other.*

As a corollary we have the topological isomorphism of $^*E(k)={}^*\mathfrak{W}(k)^{\wedge}$ and $\mathfrak{F}(k)/\mathfrak{F}(k)^{\infty}$.

We know already that both groups $^*E(k)$ and $\mathfrak{F}(k)/\mathfrak{F}(k)^{\infty}$ are topologically isomorphic to the direct sum of ρ copies of Z_p where ρ is the Hasse-Witt's invariant of k (Hasse-Witt [**7**]). Theorem 13 gives us an explicit isomorphism of these groups.

University of Tokyo.

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