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On regular rings.

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Introduction

The term "*regular ring*" will be understood in this paper in the sense defined by Auslander-Buchsbaum [**2**]. It will mean namely a Noetherian ring R, such that the quotient ring $R_{\mathfrak{p}}$ of R with respect to any prime ideal $\mathfrak{p}$ of R is a regular local ring. A regular intergral domain will be simply called a regular domain. For example every Dedekind domain[1)] is a one-dimensional regular domain. As is well known, the concept of regular local rings was introduced as a generalization of formal power-series rings with finite numbers of variables over fields, whereas a regular ring may be considered as a generalization of a polynomial ring with a finite number of variables over a field. In [**2**], as well as in Serre [**10**], an important characterization of regular local rings is given. (But in [**2**] most proofs are left out). The following theorem, given also in [**2**], [**10**] with homological methods and referred to as Theorem A in the following, is important for us.

THEOREM A. *If R is a regular local ring, then the quotient ring $R_{\mathfrak{p}}$ of R with respect to any prime ideal $\mathfrak{p}$ of R is also a regular local ring.*

According to this theorem, the definition of the regular ring can be restated as follows: *A Notherian ring R is called a regular ring if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a regular local ring.*

In this paper, we shall start from our latter definition of the regular ring, and shall prove properties of regular rings using only purely ideal-theoretical methods. Among the results proved by homological methods, we presuppose only Theorem A above mentioned. Most of the results previously obtained, will be proved by simpler methods in the generalized form. We shall use, in this paper, the notations and terminology of Northcott [**8**]. Moreover, "ideal" will always mean a proper ideal, "ring" a commutative ring with unity e.

In § 1, we shall prove that every regular ring can be expressed as a direct sum of a finite number of regular domains.

1) Dedekind domain is an integral domain which satisfies Noether-Sono's condition.

In § 2 we shall show that the polynomial ring and formal power-series ring with a finite number of variables over a regular ring are also regular.

In § 3 we shall introduce a notion of quasi-ZPE rings as a generalization of ZPE rings in the sense of Krull [4] and prove in particular the following results. Every regular ring R is a quasi-ZPE ring if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is unramified or one- or two-dimensional. If R is a quasi-ZPE regular ring, the polynomial ring $R'=R[X_1,\cdots,X_n]$ with a finite number of variables $X_1, X_2,\cdots, X_n$ over R is also a quasi-ZPE regular ring.

In § 4 we shall examine on semi-local regular ring. Main results are the following: Every maximal ideal $\mathfrak{m}$ of a semi-local regular ring R is generated by d elements where $d=\text{rank }\mathfrak{m}$. Every semi-local quasi-ZPE regular domain is a ZPE ring.

In § 5, we shall give some further results.

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§ 1. Reductions to regular domains.

First we give the following

THEOREM 1. *Every regular ring is integrally closed.*

This follows from the following general proposition.

PROPOSITION 1. *A ring R is integrally closed if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is integrally closed.*

Furthermore, if R is integrally closed and if S is an arbitrary multiplicatively closed set of R which does not contain 0, *the quotient ring R_S of R with respect to S is integrally closed.*

PROOF. The second part can be proved easily (for example, see [1] or [11]). Hence we have only to show the if part. Let K be the total quotient ring of R and let $\alpha=b/a$ be any element of K which is integral over R, where $a, b\in R$. Denote by $\bar{a}, \bar{b}$ the residues of a, b in $R_{\mathfrak{m}}$ respectively and set $\bar{\alpha}=\bar{b}/\bar{a}$. Since $R_{\mathfrak{m}}$ is integrally closed, $\bar{\alpha}$ is contained in $R_{\mathfrak{m}}$. Therefore we can put $\bar{b}/\bar{a}=\bar{r}/\bar{s}$, $\bar{s}\notin\mathfrak{m}/\mathfrak{a}$, $\bar{r},\bar{s}\in R/\mathfrak{a}$, where $\mathfrak{a}=\{a\,;as'=0, s\notin\mathfrak{m}, a\in R\}$. Denote by s, r the representatives of $\bar{s}, \bar{r}$ in R, then by the definition of $\mathfrak{a}$, we have $s'sa=sr\in R$, $s's\notin\mathfrak{m}$, for a suitable element $s'\notin\mathfrak{m}$. Now, if we set $\mathfrak{c}=\{c\,;c\alpha\in R, c\in R\}$, then $\mathfrak{c}$ is not contained in any maximal ideal of R. Therefore $\mathfrak{c}=R$. Consequently $\alpha\in R$. Thus R is integrally closed.

Secondly we shall prove the following theorem which is mentioned without proof in [2].

THEOREM 2. *A ring is regular if and only if it is expressible as a direct* *sum of a finite number of regular domains.*

This theorem can easily be derived from the following general propositions.

PROPOSITION 2. *For any ring R with only a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ of rank 0 in R the following properties are equivalent:*

1) *R is expressible as a direct sum of a finite number of integral domains.*

2) *We have $(\mathfrak{p}_i, \mathfrak{p}_j) = R$ for any $i \neq j$ and $\bigcap_{i=1}^{s} \mathfrak{p}_j = (0)$.*

3) *The quotient ring $R_\mathfrak{m}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is an integral domain.*

PROOF. 1)→2) By assumption we have

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_s \quad \text{(direct sum)}$$

where every R_i is an integral domain. Then we have clearly $\mathfrak{p}_i = R_1 \oplus \cdots \oplus \overset{i}{(0)} \oplus \cdots \oplus R_s$ by changing suitably the indexes of R_i. Hence R has the property 2). 2)→1). From 2) it follows directly

$$(\mathfrak{p}_i, \mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_s) = R \text{ for any } i.$$

Now denote by e_i an element of R such that $1 = a_i + e_i$, where $a_i \in \mathfrak{p}_i$, $e_i \in \mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_s$. If we set $R_i = \mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_s$, then R_i is clearly a ring with a unit element e_i. From this we obtain easily

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_s \quad \text{(direct sum)}.$$

2)→3). According to our assumption every maximal ideal $\mathfrak{m}$ of R contains only one $\mathfrak{p}_{i_0}$ of $\mathfrak{p}_i$'s. If we set $\mathfrak{a} = \{a; as = 0, s \notin \mathfrak{m}, a, s \in R\}$, then $\mathfrak{a}$ coincides with $\mathfrak{p}_{i_0}$. So $R_\mathfrak{m}$ is an integral domain.

3)→2). Now set $\mathfrak{a}_\mathfrak{m} = \{a; as = 0, s \notin \mathfrak{m}, a, s \in R\}$ for any maximal ideal $\mathfrak{m}$ of R and $\mathfrak{a} = \bigcap_{\mathfrak{m}} \mathfrak{a}_\mathfrak{m}$, where $\mathfrak{m}$ runs over all maximal ideals of R. Denote by a any element of $\mathfrak{a}$ and set $\mathfrak{c} = \{c; ca = 0, c \in R\}$. Then $\mathfrak{c}$ is not contained in any maximal ideal of R. Therefore $\mathfrak{c} = R$. Hence $\mathfrak{a} = (0)$. Since any $\mathfrak{a}_\mathfrak{m}$ is a prime ideal of R coinciding with one of $\mathfrak{p}_i$'s, we have $\bigcap_{i=1}^{s} \mathfrak{p}_i = \bigcap_{\mathfrak{m}} \mathfrak{a}_\mathfrak{m} = \mathfrak{a} = (0)$. It is obvious that $(\mathfrak{p}_i, \mathfrak{p}_j) = R$, for any $i \neq j$. q. e. d.

PROPOSITION 3.[2] *For each Noetherian ring R the following statements are*

2) This is a generalization of the result in [5]. In a similar way we have obviously the following: "Let R be a Noetherian ring and $\bar{R}$ be the integral closure of R. Then if R has no non-zero nilpotent element, $\bar{R}$ satisfies the following conditions: 1) The quotient ring $\bar{R}_{\bar{\mathfrak{p}}}$ of $\bar{R}$ with respect to every prime ideal $\bar{\mathfrak{p}}$ of rank 1 in $\bar{R}$ is a discrete valuation ring. 2) Every principal ideal of $\bar{R}$ whose generator is a non-zero divisor of $\bar{R}$ is expressible as an intersection of a finite number of primary ideals of rank 1 in $\bar{R}$. If R has non-zero nilpotent elements, $\bar{R}$ does not satisfy 1) but satisfies 2) and so $\bar{R}$ is not a Noetherian ring."

equivalent:

1) *R is integrally closed.*

2) *R is expressible as a direct sum of a finite number of integrally closed integral domains.*

3) *The quotient ring $R_{\mathfrak{m}}$ of R with respect to every maximal ideal $\mathfrak{m}$ of R is an integrally closed integral domain.*

4) *The quotient ring $R_{\mathfrak{p}}$ of R with respect to every prime ideal $\mathfrak{p}$ of rank 1 in R is a discrete valuation ring and every principal ideal of R whose generator is a non zero divisor of R is expressible as an intersection of a finite number of primary ideals of rank 1 in R.*

PROOF. The implication 3)→2)→1) is obvious by Proposition 1 and 2. Hence we have only to prove the implications 1)→3) and 2)⇄4). 1)→3). It follows from Proposition 1 that $R_{\mathfrak{m}}$ is integrally closed. Therefore it is sufficient to show that if R is an integrally closed local ring, R is always an integral domain. Indeed, since R is integrally closed, every ideal $\mathfrak{a}$ of R which contains at least one of non zero divisors of R contains $\bigcap_{i=1}^{s}\mathfrak{p}_i$. If we denote by $\mathfrak{m}$ a maximal ideal of R, we have $\mathfrak{m}^k\supset\bigcap_{i=1}^{s}\mathfrak{p}_i$, for any positive integer k. So $\bigcap_{k=1}^{\infty}\mathfrak{m}^k\supset\bigcap_{i=1}^{s}\mathfrak{p}_i$. Now, by Krull's theorem, $\bigcap_{k=1}^{\infty}\mathfrak{m}^k=(0)$. Hence $\bigcap_{i=1}^{s}\mathfrak{p}_i=(0)$. Let K be the total quotient ring of R. Then we have $R_s=K$, where $S=\bigcap_{i=1}^{s}(R-\mathfrak{p}_i)$. Since $\bigcap_{i=1}^{s}\mathfrak{p}_i=(0)$, we have, by Proposition 2,

$$K=K_1\oplus K_2\oplus\cdots\oplus K_s\,,\ \text{(direct sum)} \qquad \cdots\cdots\ \mathrm{I)}$$

where every K_i is a field isomorphic to $K/\mathfrak{p}_iK$. If we denote by e_i a unit element of K_i for each i, then R can be decomposed as a subring of K as follows:

$$R=Re_1+Re_2+\cdots+Re_s\,. \qquad \cdots\cdots\ \mathrm{II)}$$

Since $e_i^2=e_i$, e_i is integral over R and so $e_i\in R$. This shows

$$R=Re_1\oplus Re_2\oplus\cdots\oplus Re_s\ \text{(direct sum)}\,.$$

Since R is a local ring, the number of the components in the above direct sum should be equal to 1. Consequently R is an integral domain.

2)→4). This can be deduced easily from the result in integrally closed integral domains.

4)→2). First we show that $\bigcap_{i=1}^{s}\mathfrak{p}_i=(0)$. Now set $\mathfrak{a}_{\mathfrak{p}}=\{a\,;\,as=0,\ s\notin\mathfrak{p},\ a,s\in R\}$ for any prime ideal $\mathfrak{p}$ of rank 1 in R. Then, since every $R_{\mathfrak{p}}$ is a discrete valuation ring, $\mathfrak{a}_{\mathfrak{p}}$ coincides with one of $\mathfrak{p}_i$'s. Set $\mathfrak{a}=\bigcap_{\mathfrak{p}}\mathfrak{a}_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over all prime ideals of rank 1 in R, and denote by a any element of $\mathfrak{a}$. Further set $\mathfrak{c}=\{c\,;\,ca=0,\,c\in R\}$. Then $\mathfrak{c}$ is not contained in any prime ideals of rank 1 in R. Therefore $\mathfrak{c}$ contains at least one of non-zero divisors of R. Hence $\mathfrak{a}=(0)$. Since $\bigcap_{i=1}^{s}\mathfrak{p}_i=\bigcap_{\mathfrak{p}}\mathfrak{a}_{\mathfrak{p}}$, we have $\bigcap_{i=1}^{s}\mathfrak{p}_i=(0)$. Then we have the expression I), II). We have only to prove that II) is a direct sum. Denote by $\mathfrak{a}_i$ any element of Re_i for every i and set $\alpha=\bar{a}_1\oplus\bar{a}_2\oplus\cdots\oplus\bar{a}_s$. Then α is an element of K. Therefore we can put $\alpha=d/c,\,d,\,c\in R$. Obviously, for every $i,\,\bar{a}_i=de_i/ce_i$. Hence $de_i\in(ce_i)$ in Re_i. Now, by assumption, (c) can be decomposed as follows:

$$(c)=\mathfrak{q}_1'\cap\mathfrak{q}_2'\cap\cdots\cap\mathfrak{q}_t',$$

where every $\mathfrak{q}_j'$ is a primary ideal belonging to a prime ideal $\mathfrak{p}_j'$ of rank 1 in R. Denote by $\mathfrak{p}_{i_1}',\mathfrak{p}_{i_2}',\cdots,\mathfrak{p}_{i_{t_i}}'$ all of $\mathfrak{p}_{j'}'$ containing $\mathfrak{p}_i$ for every i. Then we have $(ce_i)=\mathfrak{q}_{i_1}'e_i\cap\mathfrak{q}_{i_2}'e_i\cap\cdots\cap\mathfrak{q}_{i_{t_i}}'e_i$ in Re_i. Since $de_i\in(ce_i)$, $d\in\mathfrak{q}_{i_1}'\cap\mathfrak{q}_{i_2}'\cap\cdots\cap\mathfrak{q}_{i_{t_i}}'$ for every i. Consequently $d\in\mathfrak{q}_1'\cap\mathfrak{q}_2'\cap\cdots\cap\mathfrak{q}_t'=(c)$. Thus $\alpha\in R$. This completes our proof.

§ 2. Polynomial rings and formal power-series rings.

Nagata has already proved in [7] that the polynomial ring $R'=R[X_1,\cdots,X_n]$ with variables $X_1,X_2,\cdots,X_n$ over a Dedekind domain R is a regular domain. This result is contained in the following general theorem, which we shall prove now by applying Theorem A in the introduction.

THEOREM 3.[3] *Let R be a regular ring. Then the polynomial ring $R'=R[X_1,X_2,\cdots,X_n]$ with variables $X_1,X_2,\cdots,X_n$ over R is also a regular ring.*

PROOF. Without loss of generality we may assume $n=1$ and $R'=R[X]$. Let $\mathfrak{m}'$ be an arbitrary maximal ideal of R'. It is sufficient to prove that the quotient ring $R'_{\mathfrak{m}'}$ of R' with respect to $\mathfrak{m}'$ is a regular local ring. Set $\mathfrak{p}=\mathfrak{m}'\cap R$, then $\mathfrak{p}$ is a prime ideal of R. By theorem A the quotient ring $R_{\mathfrak{p}}$ of R wtih respect to $\mathfrak{p}$ is a regular local ring. Since we have $R'_{\mathfrak{m}'}=(R_{\mathfrak{p}}[X])_{\mathfrak{m}'R_{\mathfrak{p}}[X]}$, we can suppose that R is a regular local ring and that $\mathfrak{m}=\mathfrak{m}'\cap R$ is a maximal ideal of R. Set $K=R/\mathfrak{m}$, then K is a field and we have $K[X]=R[X]/\mathfrak{m}R[X]$. Further, $\bar{\mathfrak{m}}'=\mathfrak{m}'/\mathfrak{m}R[X]$ is a prime ideal of $K[X]$. Since $K[X]$ is a unique factorization ring, $\bar{\mathfrak{m}}'$ is a principal ideal which is generated by an element $\bar{f}$ of $K[X]$. When we replace all coefficients of $\bar{f}$ by the representatives of them in R, we obtain a polynomial f of $R[X]$, and then we have $\mathfrak{m}'=(\mathfrak{m},f(X))$. Accordingly if rank $\mathfrak{m}=d$, we have

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3) This theorem has been proved recently in a more general form in [**3**] by homological methods. Also, after this paper had been prepared, the writer found that in case R is a regular local ring this had been proved in [**9**] by a similar method as used in this paper.

rank $\mathfrak{m}' = d+1$. Since R is a regular local ring, $\mathfrak{m}$ is generated by d elements. So $\mathfrak{m}'$ is generated by $d+1$ elements. Consequently $R'_{\mathfrak{m}'}$ is a regular local ring. This proves our assertion.

Next we shall prove a similar theorem concerning a formal power-series ring. In this case we need not Theorem A, but it is impossible to prove this by the same method as used in Theorem 3. We shall prove the following proposition which is essential for a formal power-series ring.

PROPOSITION 4. *Let $R^* = R[[X_1, X_2, \cdots, X_n]]$ be a formal power-series ring with variables $X_1, X_2, \cdots, X_n$ over a ring R and $M = \{\mathfrak{m}\}$ be the set of all maximal ideals of R. Then $M^* = \{(\mathfrak{m}, X_1, X_2, \cdots, X_n)R^*, \mathfrak{m} \in M\}$ is the set of all maximal ideals of R^*.*

PROOF. Without loss of generality we may assume $n=1$. It is obvious that every $\mathfrak{m}^* = (\mathfrak{m}, X)R^*$ is a maximal ideal of R^*. Let $\mathfrak{m}^*$ be an arbitrary maximal ideal of R^* and let $\mathfrak{a}$ be an ideal generated by all constant terms of power-series contained in $\mathfrak{m}^*$. Then $\mathfrak{a}$ does not coincide with R itself, because, if $\mathfrak{a} = R$, $\bar{\mathfrak{m}}^*$ contains an element f such that $f = 1 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots$, which is a contradiction. Now, we choose a maximal ideal $\mathfrak{m}$ of R which contains $\mathfrak{a}$. Then we have

$$\mathfrak{m}^* = (\mathfrak{m}, X)R^* \supset (\mathfrak{a}, X)R^* \supset \bar{\mathfrak{m}}^* .$$

Consequently we have $\mathfrak{m}^* = \bar{\mathfrak{m}}^*$. The proof is complete.

THEOREM 4. *Let R be a regular ring. Then the formal power-series ring $R^* = R[[X_1, X_2, \cdots, X_n]]$ with variables $X_1, X_2, \cdots, X_n$ over R is also a regular ring.*

PROOF. We may again assume $n=1$. It is sufficient to prove that the quotient ring $R^*_{\mathfrak{m}^*}$ of R^* with respect to any maximal ideal $\mathfrak{m}^*$ of R^* is a regular local ring. By Proposition 4 we have $\mathfrak{m}^* = (\mathfrak{m}, X)R^*$, where $\mathfrak{m}$ is a maximal ideal of R. Therefore $R^*_{\mathfrak{m}^*}$ contains $R_{\mathfrak{m}}$. Hence $\mathfrak{m}^*R^*_{\mathfrak{m}^*} = (\mathfrak{m}, X)R^*_{\mathfrak{m}^*} = (\mathfrak{m}R_{\mathfrak{m}}, X)R^*_{\mathfrak{m}^*}$. Since $R_{\mathfrak{m}}$ is a regular local ring, $\mathfrak{m}R_{\mathfrak{m}}$ is generated by d elements, where rank $\mathfrak{m} = d$. Therefore $\mathfrak{m}^*R^*_{\mathfrak{m}^*}$ is generated by $d+1$ elements. On the other hand we have rank $\mathfrak{m}^* = d+1$. Consequently $R^*_{\mathfrak{m}^*}$ is a regular local ring.

COROLLARY. *Suppose that R is a Noetherian integrally closed ring. Then, both R' and R^* as defined in Theorem 3 and 4 respectively, are also Noetherian integrally closed ring.*

PROOF. We may suppose that R is a integral domain, if we apply Proposition 3 to this. As is well known, we have $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of rank 1 in R, and every $R_{\mathfrak{p}}$ is a discrete valuation ring. Then both $R_{\mathfrak{p}}[X_1, X_2, \cdots, X_n]$ and $R_{\mathfrak{p}}[[X_1, X_2, \cdots, X_n]]$ are regular rings by Theorem 3 and 4, and therefore they are integrally closed by Theorem 1. On the other hand we have $R' = R[X_1, X_2, \cdots, X_n] = \bigcap_{\mathfrak{p}} (R_{\mathfrak{p}}[X_1, X_2, \cdots, X_n])$ and

$R^* = R[[X_1, X_2, \cdots, X_n]] = \bigcap_{\mathfrak{p}} (R_{\mathfrak{p}}[[X_1, X_2, \cdots, X_n]])$. Consequently, both R' and R^* are integrally closed. This proves our assertion.

§ 3. Inversibilities of prime ideals of rank 1 in regular rings.

A unique factorization ring is called a ZPE ring according to Krull.

Now we shall define a quasi-ZPE ring as follows:

DEFINITION. *A Noetherian ring R is called a quasi*-ZPE *ring if every prime ideal of rank 1 in R is inversible.*

It is well known that unramified or one- or two-dimensional regular local rings are ZPE ring. (See for example [**1**] or [**11**].) Analogously we shall obtain the following

THEOREM 5.[4] *If R is a regular ring whose quotient ring with respect to any maximal ideal of it is unramified or one- or two-dimensional, then R is a quasi*-ZPE *ring.*

In order to prove this theorem, we need the following three propositions.

PROPOSITION 5. *Let R be an integral domain with one maximal ideal $\mathfrak{m}$. Then every inversible ideal of R is principal.*

PROOF. Let $\mathfrak{a}$ be an arbitrary inversible ideal of R. Then $\mathfrak{a}\mathfrak{m}$ is strictly contained in $\mathfrak{a}$. Indeed, if $\mathfrak{a} = \mathfrak{a}\mathfrak{m}$, $R = \mathfrak{a}^{-1}\mathfrak{a} = \mathfrak{a}^{-1}\mathfrak{a}\mathfrak{m} = \mathfrak{m}$. This is obviously a contradiction. Then we can choose an element a which is not contained in $\mathfrak{a}\mathfrak{m}$ but contained in $\mathfrak{a}$. Set $\mathfrak{b} = \mathfrak{a}^{-1}a$, the $\mathfrak{b}$ becomes an ideal of R. However, since $a \notin \mathfrak{a}\mathfrak{m}$, $\mathfrak{b} = \mathfrak{a}^{-1}\mathfrak{a} \not\subset \mathfrak{m}$. So $\mathfrak{b}$ must coincide with R. Thus $\mathfrak{a} = (a)$.

COROLLARY. *Every quasi*-ZPE *ring is an integrally closed ring.*

PROPOSITION 6. *Let R be an integral domain and let $\mathfrak{a}$ be an ideal generated by a finite number of elements. Then $\mathfrak{a}$ is inversible if and only if in the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{a}$, $\mathfrak{a}R_{\mathfrak{m}}$ is principal.*

PROOF. The only if part is obvious by Proposition 5, for $\mathfrak{a}$ is inversible in the quotient ring $R_{\mathfrak{p}}$ of R with respect to any prime ideal $\mathfrak{p}$ of R. To prove the if part, it is sufficient to show that $\mathfrak{a}^{-1}\mathfrak{a}$ is not contained in any maximal ideal $\mathfrak{m}$ of R, for we have $\mathfrak{a}^{-1}\mathfrak{a} = R$ if no maximal ideal of R contains $\mathfrak{a}^{-1}\mathfrak{a}$. Let $\mathfrak{m}$ be an arbitrary maximal ideal of R. We may suppose $\mathfrak{a} \subset \mathfrak{m}$. By our assumption we can choose a suitable element a of $\mathfrak{a}$ which generates $\mathfrak{a}R_{\mathfrak{m}}$. If we put $\alpha = a^{-1}$, we have $\alpha\mathfrak{a} \subset R_{\mathfrak{m}}$. Hence we can write $\alpha a_i = r_i/s_i$, $s_i \notin \mathfrak{m}$, $r_i, s_i \in R$ for every i. Set $s = \prod_{i=1}^{t} s_i$, then $s\alpha\mathfrak{a} \in R$ and so $s\alpha$ is contained

4) In case R is a one- or two-dimensional, this is given without proof in [**2**].
The problem if every regular local ring is a ZPE ring is open. If the answer is affirmative, we can assert according to Proposition 7 that every regular ring is a quasi-ZPE ring (cf. p. 170).

in $\mathfrak{a}^{-1}$. Since $\alpha a = 1$, we have $s\alpha a = s$. The left hand side of this formula is contained in $\mathfrak{a}^{-1}\mathfrak{a}$, but the right hand side is not contained in $\mathfrak{m}$. This shows that $\mathfrak{a}^{-1}\mathfrak{a} = R$. This proves our assertion.

Under the assumption that R is Noetherian we have the following general proposition from which Theorem 5 can be derived directly.

PROPOSITION 7. *Let R be a Noetherian ring. Then R is a quasi-*ZPE *ring if and only if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a* ZPE *ring.*

PROOF. By Proposition 3 in § 1 we may assume that R is an integral domain. It is well known that a Noetherian integral domain is a ZPE ring if and only if every prime ideal of rank 1 in it is principal. Hence our assertion is an obvious consequence of Proposition 6.

As is well known, the polynomial ring $R' = R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over a ZPE ring R is also a ZPE ring. Now, we can generalized this fact to a quasi-ZPE ring.

PROPOSITION 8. *If R is a quasi-*ZPE *ring, then the polynomial ring $R' = R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over R is also a quasi* ZPE *ring.*

PROOF. We may assume that R is an integral domain. By Proposition 7 we have only to prove that the quotient ring $R'_{\mathfrak{m}'}$ of R' with respect to any maximal ideal $\mathfrak{m}'$ of R' is a ZPE ring. Set $\mathfrak{p} = \mathfrak{m}' \cap R$, then $\mathfrak{p}$ is a prime ideal of R. Denote by $\mathfrak{m}$ a maximal ideal containing $\mathfrak{p}$, then by our assumption $R_{\mathfrak{m}}$ is a ZPE ring, and therefore the ring $R_{\mathfrak{p}} = (R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}}$ is also a ZPE ring. Hence the polynomial ring $R_{\mathfrak{p}}[X_1, X_2, \cdots, X_n]$ over $R_{\mathfrak{p}}$ is a ZPE ring. Also we have $R'_{\mathfrak{m}'} = (R_{\mathfrak{p}}[X_1, X_2, \cdots, X_n])_{\mathfrak{m}R_{\mathfrak{p}}[X_1, \cdots, X_n]}$. This shows that $R'_{\mathfrak{m}'}$ is a ZPE ring, and this proves our proposition.

Combining this proposition with Theorem 3 in § 2, we obtain the following

THEOREM 6. *If R is a quasi-*ZPE *regular ring, then the polynominal ring $R' = R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over R is a quasi-*ZPE *regular ring.*

By this theorem we can find quasi-ZPE regular rings which do not satisfy the condition in Theorem 5. (For example, the polynomial ring $R' = R[X_1, X_2, \cdots, X_n]$ with a finite number of variables $X_1, X_2, \cdots, X_n$ over a general Dedekind domain R is always a $n+1$ dimensional quasi ZPE regular ring.)

§ 4. Semi-local regular rings.

It is well known that the completion of a regular local ring is a regular local ring. Similarly we have the following

THEOREM 7. *If R is a semi-local regular ring, then the completion R^* of R is also a semi-local regular ring.*

PROOF. Let $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ be all maximal ideals of R. Then we have

$$R/(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_t)^k \cong R/\mathfrak{m}_1{}^k \oplus R/\mathfrak{m}_2{}^k \oplus \cdots \oplus R/\mathfrak{m}_t{}^k$$

$$R_{\mathfrak{m}_1}/\mathfrak{m}_1{}^k R_{\mathfrak{m}_1} \oplus R_{\mathfrak{m}_2}/\mathfrak{m}_2{}^k R_{\mathfrak{m}_2} \oplus \cdots \oplus R_{\mathfrak{m}_t}/\mathfrak{m}_t{}^k R_{\mathfrak{m}_t} \quad \text{(direct sum)}.$$

for every positive integer k. Now set $R_i = R_{\mathfrak{m}_i}$ and let R_i^* be the completion of R_i. Then have

$$R^* = R_1^* \oplus R_2^* \oplus \cdots \oplus R_t^* \quad \text{(direct sum)}.$$

Since all R_i are regular local rings, all R_i^* are also regular local rings. Consequently R^* is a semi-local regular ring according to Theorem 2.

Furthermore, as is well known, every semi-local Dedekind domain is a principal ideal domain (for example, see [**1**] or [**4**]). This fact can be generalized to semi-local regular rings as follows:

THEOREM 8. *Let R be a semi-local regular ring and let $\mathfrak{m}$ be any maximal ideal of R. Let $d = \operatorname{rank} \mathfrak{m}$. Then $\mathfrak{m}$ is generated by d elements in R.*

This theorem can be obtained easily from the following proposition which holds in general semi-local rings.

PROPOSITION 9. *Let R be a semi-local ring and let $\mathfrak{q}$ be a primary ideal belonging to an arbitrary maximal ideal $\mathfrak{m}$ of R. If $\mathfrak{q}R_{\mathfrak{m}}$ is generated in $R_{\mathfrak{m}}$ by s elements, then $\mathfrak{q}$ is generated in R by s elements.*

PROOF. Denote by $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ all maximal ideals of R. When $t = 1$, our assertion is obvious. Therefore we suppose $t > 1$. Further without loss of generality, we may suppose $\mathfrak{m} = \mathfrak{m}_1$. Since R is Noetherian, we can set $\mathfrak{q} = (u_1, u_2, \cdots, u_r)$ by choosing a finite number of elements $u_1, u_2, \cdots, u_r$ of R. If $r \leqq s$, our proposition is true. Accordingly suppose $r > s$. Now we shall show that all u_i can be chosen not to be contained in $\mathfrak{m}_2, \mathfrak{m}_3 \cdots, \mathfrak{m}_t$. There exists a positive integer k such that u_1 is not contained in $\mathfrak{m}_2, \mathfrak{m}_3, \cdots, \mathfrak{m}_k$ but contained in $\mathfrak{m}_{k+1}, \cdots, \mathfrak{m}_t$, if we change suitably the indexes of $\mathfrak{m}_i$'s. If $k = t$, u_1 is an element as is required. Therefore we suppose $k < t$. Since $(u_2, u_3, \cdots, u_r) \not\subset \mathfrak{m}_{k+1}, \mathfrak{m}_{k+2}, \cdots, \mathfrak{m}_t$, there exists an element a_i for every i $(k+1 \leqq i \leqq t)$ which is not in $\mathfrak{m}_i$ but in $(u_2, \cdots, u_r)$. On the other hand we can take an element b_i for every i $(k+1 \leqq i \leqq t)$ which is not in $\mathfrak{m}_i$ but in $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_i \cap \cdots \cap \mathfrak{m}_t$. Set $d_i = a_i b_i$ and $e = \sum_{i=k+1}^{t} d_i$. Then e is contained in $(u_2, u_3, \cdots, u_r) \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_k$ but not contained in $\mathfrak{m}_{k+1}, \cdots, \mathfrak{m}_t$. Set $u_1' = u_1 + e$. Then we have $\mathfrak{q} = (u_1', u_2, \cdots, u_r)$ and u_1' is not contained in $\mathfrak{m}_2, \mathfrak{m}_3, \cdots, \mathfrak{m}_t$. By repeating this procedure we can take a base of $\mathfrak{q}$ none of which base elements is contained in $\mathfrak{m}_2, \mathfrak{m}_3, \cdots, \mathfrak{m}_t$. Now let $(u_1, u_2, \cdots, u_r)$ be such a base of $\mathfrak{q}$ in R. Since $\mathfrak{q}R_{\mathfrak{m}}$ is generated by s elements, $\mathfrak{q}R_{\mathfrak{m}}$ is generated by suitable s elements $u_{i_1}, u_{i_2}, \cdots, u_{i_s}$ of u_i's. That is

$$\mathfrak{q}R_{\mathfrak{m}} = (u_{i_1}, u_{i_2}, u_{i_3}, \cdots, u_{i_s})R_{\mathfrak{m}} .$$

Set $\mathfrak{a} = (u_{i_1}, u_{i_2}, \cdots, u_{i_s})$ in R. Then $\mathfrak{a}$ is an ideal of R such that $\mathfrak{a} \not\subset m_2, \cdots, m_t$ and $\mathfrak{a}R_{\mathfrak{m}} = \mathfrak{q}R_{\mathfrak{m}}$. So $\mathfrak{a}$ must coincide with $\mathfrak{q}$. This proves our proposition.

If we set $\mathfrak{m} = \mathfrak{q}$, then Theorem 8 can be derived from this.

COROLLARY 1. *If R is a semi-local regular ring and if we denote by $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ all maximal ideals of R, then the Jacobson radical $\mathfrak{a} = \bigcap_{i=1}^{t} \mathfrak{m}_t$ is generated by d elements, where $d = \dim R$.*

PROOF. According to Proposition 9 we have $\mathfrak{m}_i = (u_{i_1}, \cdots, u_{id_i})$ where $d_i =$ rank $\mathfrak{m}_i$ and for every fixed i any u_{ij} is not contained in $\mathfrak{m}_1, \cdots, \hat{\mathfrak{m}}_i, \cdots, \mathfrak{m}_t$. If $d > d_i$, we choose an element v_i such that $v_i \in \mathfrak{m}_i, \notin \mathfrak{m}_1, \cdots, \hat{\mathfrak{m}}_i, \cdots, \mathfrak{m}_t$ and set $u_{i,di+1} = u_{i,di+2} = \cdots = u_{i,d} = v_i$. Furthermore set $w = \prod_{i=1}^{d} u_{ij}$ and $\mathfrak{c} = (w_1, w_2, \cdots, w_d)$. Then $\mathfrak{c}R_{\mathfrak{m}_i} = \mathfrak{q}R\mathfrak{m}_i$ for every i. This shows $\mathfrak{a} = \mathfrak{c} = (w_1, w_2, \cdots, w_d)$.

COROLLARY 2. *Let R be a semi-local ring any of which maximal ideals is of the same rank d equal to $\dim R$. Then R is a regular ring if and only if the Jacobson radical of R is generated by d elements.*

PROOF. This follows immediately from Corollary 1.

On the other hand we can also assert the following theorem as another generalization of the fact described before Theorem 8.

THEOREM 9. *Let R be a semi-local regular ring whose quotient ring with respect to any maximal ideal of it is unramified or one- or two-dimensional. Then every prime ideal of rank 1 in R is a principal ideal.*

Since R may be supposed to be an integral domain, we have only to prove that R is a ZPE ring, in case R is an integral domain. To prove this, we begin with the following two propositions.

PROPOSITION 10. *If R is an integral domain which has a finite number of maximal ideals of it, then every inversible ideal of R is principal.*

This is a slight generalization of the proposition 5.

PROOF. Let $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_s$ be all maximal ideals of R and let $\mathfrak{a}$ be any inversible ideal of R. Then every $\mathfrak{a}\mathfrak{m}_i$ is strictly contained in $\mathfrak{a}$. If there exists now an element a in $\mathfrak{a}$ which is not in any one of $\mathfrak{a}\mathfrak{m}_i$, we have easily $\mathfrak{a} = (a)$. Accordingly, it is sufficient to prove the existence of such an element. Obviously there exists an element b_i which is not contained in $\mathfrak{m}_i$, but in $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \hat{\mathfrak{m}}_i \cap \cdots \cap \mathfrak{m}_s$. Then $\mathfrak{a}b_i$ is not contained in $\mathfrak{a}\mathfrak{m}_i$, but in $\mathfrak{a}\mathfrak{m}_1 \cap \cdots \cap \widehat{\mathfrak{a}\mathfrak{m}_i} \cap \cdots \cap \mathfrak{a}\mathfrak{m}_s$, and therefore we can choose a suitable element a_i in $\mathfrak{a}b_i$ which is not contained in $\mathfrak{a}\mathfrak{m}_i$. Set $a = \sum_{i=1}^{s} a_i$, then this is an element which satisfies the above condition.

When we connect this proposition with Proposition 7, the following general proposition can be obtained.

Proposition 11.[5] *Let R be a semi-local integral domain. Then R is a* ZPE *ring if and only if the quotient ring of R with respect to any maximal ideal of R is a* ZPE *ring.*

Theorem 9 is a corollary to this proposition.

§ 5. Some further results.

It is well known that if R is a Dedekind domain every ideal of R is generated by 2 elements (for example see [**12**]). Here we give a generalization of this fact to regular rings.

Theorem 10. *Let R be a d dimensional regular ring and let $\mathfrak{m}$ be any maximal ideal of R. Then $\mathfrak{m}$ is generated by $d+1$ or less elements.*

Proof. Set $s = \operatorname{rank} \mathfrak{m}$, then obviously $s \leqq d$. Since R is a regular ring, we can choose s elements $u_1, u_2, \cdots, u_s$ from $\mathfrak{m}$ satisfying the following conditions: 1) $\mathfrak{m}R_{\mathfrak{m}} = (u_1, u_2, \cdots, u_s)R_{\mathfrak{m}}$. 2) rank $(u_1, u_2, \cdots, u_s) = s$ in R. If $\mathfrak{m} \neq (u_1, u_2, \cdots, u_s)$, then there exists a primary decomposition of $(u_1, u_2, \cdots, u_s)$ in R as follows:

$$(u_1, u_2, \cdots, u_s) = \mathfrak{m} \cap \mathfrak{q}_1^{(s)} \cap \mathfrak{q}_2^{(s)} \cap \cdots \cap \mathfrak{q}_{t_0}^{(s)},$$

where every $\mathfrak{q}_i^{(s)}$ is a primary ideal belonging to a prime ideal $\mathfrak{q}_i^{(s)}$ of rank s in R. Further we take an element u_{s+1} from $\mathfrak{m}$ which is not contained in any $\mathfrak{p}_i^{(s)}$. Then, if any $\mathfrak{p}_i^{(s)}$ is maximal in R, we obtain immediately $\mathfrak{m} = (u_1, u_2, \cdots, u_{s+1})$. Hence our theorem holds in this case. On the other hand, if $\mathfrak{m} \neq (u_1, u_2, \cdots, u_{s+1})$, we have $d > s$ and at least one of $\mathfrak{p}_i^{(s)}$'s is not maximal in R. Then we also have a primary decomposition of $(u_1, u_2, \cdots, u_s, u_{s+1})$ as follows:

$$(u_1, u_2, \cdots, u_s, u_{s+1}) = \mathfrak{m} \cap \mathfrak{q}_1^{(s+1)} \cap \mathfrak{q}_2^{(s+1)} \cap \cdots \cap \mathfrak{q}_{t_1}^{(s+1)},$$

where every $\mathfrak{q}_i^{(s+1)}$ is a primary ideal belonging to a prime ideal $\mathfrak{p}_i^{(s+1)}$ of rank $s+1$ or more in R. We choose again an element u_{s+2} from $\mathfrak{m}$ which is not contained in any $\mathfrak{p}_i^{(s+1)}$. By repeating this procedure $d-s+1$ times we obtain finally $\mathfrak{m} = (u_1, u_2, \cdots, u_s, \cdots, u_{d+1})$. This proves our assertion.

We say that a ring R' dominates a ring R, if the following conditions are satisfied: 1) $R' \supset R$. 2) $\dim R' = \dim R$. 3) If $\mathfrak{m}$ is any maximal ideal of R, then $\mathfrak{m}R'$ is a maximal ideal of R'.

Now we give the following

Theorem 11. *Let R be a one dimensional regular ring. Then there exists always at least one one dimensional regular ring R' dominating R such that for every prime ideal $\mathfrak{p}$ of rank 1 in R $\mathfrak{p}R'$ is a principal prime ideal in R', that is,*

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5) This proposition has already been obtained by Akizuki-Nagata [1] by a different method.

such that any element of R is expressible as the products of a finite number of prime elements of R' in R'.

PROOF. By Theorem 1, 5 and 9 we may assume R is a non ZPE Dedekind domain with an infinite number of prime ideals. Now, according to Theorem 10, any non principal prime ideal $\mathfrak{p}$ of R is generated by 2 elements. Hence we can set $\mathfrak{p}=(p_1, p_2)$.

Denote by X, Y two variables over R and set $\bar{R}=R[X, Y]$. Further put $\bar{S}=\bigcap_{\mathfrak{p}^*}(\bar{R}-\mathfrak{p}\bar{R})$, where $\mathfrak{p}^*$ runs over all prime ideals of R and all prime ideals of rank 1 in $R[X]$. Then the quotient ring $\bar{R}_{\bar{S}}$ of $\bar{R}$ with respect to $\bar{S}$ is a ring as is required. For, we can show easily that $\mathfrak{p}\bar{R}_{\bar{S}}=(p_1X+p_2Y)\bar{R}_{\bar{S}}$ and $\bar{R}_{\bar{S}}$ is a Dedekind domain.

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Bibliography

[1] Y. Akizuki and M. Nagata, Kindai Daisugaku (modern algebra), (In Japanese) Kyoritsu, 1957.

[2] M. Auslander and D. A. Buchsbaum, Homological dimension in Noetherian rings, Proc. Nat. Acad. U. S. A., **42** (1956), 36-38.

[3] S. Eilenberg, A. Rosenberg and D. Zelinsky, On the dimension of modules and algebras VIII, Nagoya Math. J., **12** (1957), 71-94.

[4] W. Krull, Ideal theorie, Erg. d. Math., 1955.

[5] M. Nagata, On the derived normal rings of Noetherian integral domains, Mem. Coll. Sci, Univ. Kyoto, Ser. A, Math., **29** (1955), 293-303.

[6] M. Nagata, The theory of mutiplicity in general local rings, Proc. Intern. Symp. Tokyo (1955), 191-226.

[7] M. Nagata, A general theory of algebraic geometry over Dedekind domains, Amer. J. of Math., **78** (1956), 78-116.

[8] D. G. Northcott, Ideal theory, Cambridge Tracts, 1953.

[9] D. G. Northcott, A note on polynomial rings, J. London Math., **33** (1958), 36-39.

[10] J. P. Serre, Sur la dimension homologique des anneaux et des modules noetheriens, Proc. Intern. Symp. Tokyo, 1955, 175-190.

[11] I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc., **59** (1946), 54-106.

[12] I. S. Cohen, Commutative rings with restricted minimum condition, Duke Math. J., **17** (1950), 27-42.

Added in proof. In the course of reading the proofs, the auther was informed that M. Auslander succeeded to prove that every regular local ring is a ZPE ring (cf. Footnote on p. 165).