Journal of the Mathematical Society of Japan Vol. 11, No. 2, April, 1959

Existence of derivations in graded algebras.

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(Received Sept. 12, 1958)

In the present paper we shall discuss on the existence of derivations in the sense of C. Chevalley [**1**] in graded algebras. We shall give a new definition of a homomorphism of graded algebras which is a generalization of the usual one. Such a homomorphism will naturally lead us to a definition of free graded algebras as a generalization of the concept of Z-graded free algebras. The free graded algebras will play a fundamental rôle in our study.

Section 1 shows the existence of derivations of the free graded algebras. Section 2 deals with transferability between the derivation of a graded algebra and that of its homomorphic image. In the last section 3 a criterion for the existence of derivations in any graded algebras is obtained by using new binary operations which are generalizations of the usual partial differential operators.

§**1.** Throughout this paper an algebra means an algebra with a unit element 1, and a homomorphism of algebras means a ring homomorphism which maps unit upon unit. We denote by $\Gamma, \Delta, \cdots$ additive (commutative) groups and by (E, Γ) a Γ-graded algebra over any fixed (commutative or non-commutative) ring A with a unit element.

Let $E=\sum_{\gamma\in\Gamma}^{\oplus}E_\gamma$ and $F=\sum_{\delta\in\Delta}^{\oplus}F_\delta$ be decompositions of (E, Γ) and (F, Δ) into homogeneous modules respectively. Let φ_R be a homomorphism from E onto (into) F as algebras, and φ_G a homomorphism from Γ onto Δ. If $\varphi_R(E_\gamma) \subseteq F_{\varphi_G(\gamma)}$, then $\varphi=(\varphi_R, \varphi_G)$ is called a homomorphism from (E, Γ) onto (into) (F, Δ). For convenience, we write $\varphi(x)=\varphi_R(x)$ for $x\in E$, and $\varphi(\gamma)=\varphi_G(\gamma)$ for $\gamma\in\Gamma$; the kernel of φ means the kernel of φ_R.

Let (E, Γ) be a Γ-graded algebra over A. Let Δ be a homomorphic image or a factor group of Γ. Then it is easy to see that E is also a Δ-graded algebra. An element in the homogeneous module E_δ of E is called Δ-homogeneous of degree δ. A submodule M of E is said to be Δ-homogeneous if $M=\sum_{\delta\in\Delta}^{\oplus}(M\cap E_\delta)$. If a submodule M or an ideal $\mathfrak{A}$ of E is generated by Δ-homogeneous elements, then it is Δ-homogeneous by Theorem 1.3 in [**1**].

THEOREM 1. *Let (E, Γ) be a graded algebra over A. If Δ is a factor group of Γ, then there exists, for any Δ-homogeneous two-sided ideal $\mathfrak{A}$ of E, a homomorphism from (E, Γ) onto $(E/\mathfrak{A}, \Delta)$. Conversely, if $\varphi=(\varphi_R, \varphi_G)$ is a homomor-*

phism from (E, Γ) onto (F, Δ), then there exists a Δ-homogeneous ideal $\mathfrak{A}$ of E such that $(E/\mathfrak{A}, \Delta)$ and (F, Δ) are isomorphic.

PROOF. The former part of this theorem is easily obtained. We now prove the latter part. It is clear that E can be considered as a Δ-graded algebra, and that $\varphi' = (\varphi_R, I_G)$ is a homomorphism from (E, Δ) onto (F, Δ). Let $a = \sum_{\delta \in \Delta} a_\delta$ be an element of the kernel $\mathfrak{A}$ of φ_R. Then $0 = \varphi_R(a) = \sum_{\delta \in \Delta} \varphi_R(a_\delta)$. By the uniqueness of this representation, we have $\varphi_R(a_\delta) = 0$, i. e. $a_\delta \in \mathfrak{A}$. Hence $\mathfrak{A}$ is Δ-homogeneous. Therefore (F, Δ) is isomorphic to $(E/\mathfrak{A}, \Delta)$. This completes the proof.

Let (E, Γ) be a graded algebra over A. If Γ' is the subgroup of Γ generated by the set $\{\gamma; E_\gamma \neq 0\}$, then it is clear that Γ' is uniquely determined by (E, Γ). Such a subgroup Γ' is called the irredundant subgroup of Γ with respect to E, and denoted by $\Gamma(E)$. It is easily seen that E is a $\Gamma(E)$-graded algebra. Hereafter $[E, \Gamma]$ denotes the $\Gamma(E)$-graded algebra E, i. e. $[E, \Gamma] = (E, \Gamma(E))$, and is called an irredundantly graded algebra.

Let F be a free algebra with a free system of generators $\{x_\lambda; \lambda \in \Lambda\}$ over a ring A, and Δ the additive group freely generated by Λ. Let δ be any element of Δ, and $\delta = \lambda_1 + \cdots + \lambda_n$ an expression of δ in the normal form in Δ. F_δ denotes the submodule of F which is spanned by the set $\{y_{\pi(\delta)} = x_{\pi(\lambda_1)} \cdots x_{\pi(\lambda_n)}; \pi$ runs over the symmetric group $S_n\}$. Then F forms an irredundantly Δ-graded algebra. This algebra $[F, \Delta]$ is called a free graded algebra with a free system of generators $\{x_\lambda; \lambda \in \Lambda\}$.

THEOREM 2. *Let $[E, \Gamma]$ be any irredundantly graded algebra over A. Then there exists a free graded algebra $[F, \Delta]$ such that a homomorphism from $[F, \Delta]$ onto $[E, \Gamma]$ exists.*

PROOF. Let $\{a_\lambda; \lambda \in \Lambda\}$ be a system of homogeneous generators of $[E, \Gamma]$. Take symbols $\{x_\lambda; \lambda \in \Lambda\}$. Then there exists a free graded algebra $[F, \Delta]$ with a free system of generators $\{x_\lambda; \lambda \in \Lambda\}$. The mapping $x_\lambda \to a_\lambda$ $(\lambda \in \Lambda)$ can be clearly extended to a homomorphism φ_R from F onto E as algebras over A. If x is a homogeneous element of degree δ in $[F, \Delta]$, then we have $x = \sum_{\pi \in S_n} \alpha_{\pi(\lambda_1), \ldots, \pi(\lambda_n)} x_{\pi(\lambda_1)} \cdots x_{\pi(\lambda_n)}$ with $\delta = \lambda_1 + \cdots + \lambda_n$. Let a be the φ_R-image of x. Then $a = \sum_{\pi \in S_n} \alpha_{\pi(\lambda_1), \ldots, \pi(\lambda_n)} a_{\pi(\lambda_1)} \cdots a_{\pi(\lambda_n)}$. If a_λ is a homogeneous element of degree $\gamma(\lambda)$, then a is a homogeneous element of degree $\gamma = \gamma(\lambda_1) + \cdots + \gamma(\lambda_n)$. Since Δ is a free additive group, the mapping $\varphi_G: \delta \to \varphi_G(\delta) = \gamma$ is clearly a homomorphism from Δ onto $\Gamma(E)$. Hence $\varphi = (\varphi_R, \varphi_G)$ is a homomorphism from $[F, \Delta]$ onto $[E, \Gamma]$. This completes the proof.

Let (E, Γ) be a graded algebra over A. Let Γ have a subgroup Γ' of index 2. Then the main involution J of E with respect to Γ/Γ' can be defined as in [**1**]. If $\Theta = \Gamma(E) \cap \Gamma' \neq \Gamma(E)$, then it is easily verified that $(\Gamma(E) : \Theta) = 2$. Hence we can form the main involution J_1 of E with respect to $\Gamma(E)/\Theta$.

Then we have $J(x)=J_1(x)$, and also the symbolical power $J^\nu(x)=J_1{}^\nu(x)$ for every ν in $\Gamma(E)$. On the other hand if $\Theta=\Gamma(E)$, then we may define $J_2(x)=x$ for every x in E. Then we have $J(x)=J_2(x)$, and $J^\nu(x)=J_2{}^\nu(x)$ for every ν in $\Gamma(E)$. In the following, we assume that J and J^ν are defined on $[E,\Gamma]$ as above, whether $\Gamma(E)$ has a subgroup of index 2 or not.

Let (E,Γ), (F,Δ) be two graded algebras over A, φ a homomorphism from (E,Γ) into (F,Δ), and let ν be a fixed element of Γ. A linear mapping D from E into F is called of degree $\varphi(\nu)$, if $D(x)\in F_{\varphi(\mu+\nu)}$ for $x\in E_\mu$. A linear mapping D of degree $\varphi(\nu)$ is called a φ-derivation of degree $\varphi(\nu)$ from (E,Γ) into (F,Δ), if it satisfies

$$D(xy)=D(x)\varphi(y)+\varphi(J^\nu(x))D(y)$$

for every x, y in E.

THEOREM 3. *Let $[F,\Delta]$ be a free graded algebra with a free system of generators $\{x_\lambda;\lambda\in\Lambda\}$. Let (E,Γ) be an arbitrary graded algebra over A, and φ a homomorphism from $[F,\Delta]$ into (E,Γ). Assume that for each element λ of Λ, a Γ-homogeneous element $y_\lambda\in E$ of degree $\varphi(\lambda+\nu)$ is preassigned arbitrarily, where ν is a fixed element in Δ. Then there exists one and only one φ-derivation D of degree $\varphi(\nu)$ from $[F,\Delta]$ into (E,Γ), which is an extension of the mapping $x_\lambda\to y_\lambda$.*

PROOF. First we put $D(1)=0$ and $D(x_\lambda)=y_\lambda$. Next we set

$$D(x_{\lambda_1}\cdots x_{\lambda_n})=\textstyle\sum_i \varphi(J^\nu(x_{\lambda_1}\cdots x_{\lambda_{i-1}}))D(x_{\lambda_i})\varphi(x_{\lambda_{i+1}}\cdots x_{\lambda_n}),$$

and define

$$D(x)=\textstyle\sum_{(\lambda)}\alpha_{(\lambda)}D(x_{\lambda_1}\cdots x_{\lambda_n}), \qquad (\lambda)=(\lambda_1,\cdots,\lambda_n)$$

for any element $x=\sum_{(\lambda)}\alpha_{(\lambda)}x_{\lambda_1}\cdots x_{\lambda_n}$ in F. Then D forms a linear mapping of degree $\varphi(\nu)$ from $[F,\Delta]$ into (E,Γ). Now we shall prove that D satisfies the condition:

$$D(xy)=D(x)\varphi(y)+\varphi(J^\nu(x))D(y).$$

If $y=\sum_{(\mu)}\beta_{(\mu)}x_{\mu_1}\cdots x_{\mu_m}$, $(\mu)=(\mu_1,\cdots,\mu_m)$, then

$$xy=\textstyle\sum_{(\lambda)(\mu)}\alpha_{(\lambda)}\beta_{(\mu)}x_{\lambda_1}\cdots x_{\lambda_n}x_{\mu_1}\cdots x_{\mu_m}.$$

Then we have

$$\begin{aligned}
D(xy)&=\textstyle\sum_{(\lambda)(\mu)}\alpha_{(\lambda)}\beta_{(\mu)}D(x_{\lambda_1}\cdots x_{\lambda_n}x_{\mu_1}\cdots x_{\mu_m})\\
&=\textstyle\sum_{(\lambda)(\mu)}\alpha_{(\lambda)}\beta_{(\mu)}[\sum_i\varphi(J^\nu(x_{\lambda_1}\cdots x_{\lambda_{i-1}}))D(x_{\lambda_i})\varphi(x_{\lambda_{i+1}}\cdots x_{\lambda_n})\varphi(x_{\mu_1}\cdots x_{\mu_m})\\
&\qquad+\textstyle\sum_j\varphi(J^\nu(x_{\lambda_1}\cdots x_{\lambda_n}))\varphi(J^\nu(x_{\mu_1}\cdots x_{\mu_{j-1}}))D(x_{\mu_j})\varphi(x_{\mu_{j+1}}\cdots x_{\mu_m})]\\
&=\textstyle\sum_{(\lambda)(\mu)}\alpha_{(\lambda)}\beta_{(\mu)}[D(x_{\lambda_1}\cdots x_{\lambda_n})\varphi(x_{\mu_1}\cdots x_{\mu_m})+\varphi(J^\nu(x_{\lambda_1}\cdots x_{\lambda_n}))D(x_{\mu_1}\cdots x_{\mu_m})]\\
&=[\textstyle\sum_{(\lambda)}\alpha_{(\lambda)}D(x_{\lambda_1}\cdots x_{\lambda_n})]\varphi(\sum_{(\mu)}\beta_{(\mu)}x_{\mu_1}\cdots x_{\mu_m})\\
&\qquad+\varphi(J^\nu(\textstyle\sum_{(\lambda)}\alpha_{(\lambda)}x_{\lambda_1}\cdots x_{\lambda_n}))[\sum_{(\mu)}\beta_{(\mu)}D(x_{\mu_1}\cdots x_{\mu_m})]\\
&=D(x)\varphi(y)+\varphi(J^\nu(x))D(y).
\end{aligned}$$

Namely the mapping $x_\lambda \to y_\lambda$ is extended to a φ-derivation D of degree $\varphi(\nu)$ from $[F, \Delta]$ into (E, Γ). If a φ-derivation D' of degree $\varphi(\nu)$ from $[F, \Delta]$ into (E, Γ) is any extension of the mapping $x_\lambda \to y_\lambda$, then it is easily verified that $D-D'$ is a zero derivation. Hence D is uniquely determined by the mapping $x_\lambda \to D(x_\lambda) = y_\lambda$. This completes the proof.

§ **2.** Let (E_1, Γ_1), (E_2, Γ_2), (E_1', Γ_1') and (E_2', Γ_2') be four graded algebras over A, and φ a homomorphism from $[E_1, \Gamma_1]$ into (E_2, Γ_2). Let σ_1 be a homomorphism from $[E_1, \Gamma_1]$ onto $[E_1', \Gamma_1']$, and σ_2 a homomorphism from (E_2, Γ_2) onto (E_2', Γ_2'). Let $\mathfrak{A}_i$ be the kernel of σ_i $(i = 1, 2)$ as algebras, and let $\varphi(\mathfrak{A}_1) \subseteqq \mathfrak{A}_2$. Moreover let N_i be the kernel of σ_i as groups, and let $\varphi(N_1) \subseteqq N_2$. Then $\psi = (\psi_R, \psi_G)$ defined by the pair of mappings ψ_R: $\sigma_1(x) \to \sigma_2(\varphi(x))$ and ψ_G: $\sigma_1(\gamma) \to \sigma_2(\varphi(\gamma))$ gives a homomorphism from $[E_1', \Gamma_1']$ into (E_2', Γ_2'). Let Θ be a subgroup of $\Gamma_1(E_1)$ which contains N_1. Then the subgroup $\sigma_1(\Theta)$ of $\Gamma_1'(E_1')$ has the same index as that of Θ in $\Gamma_1(E_1)$. Conversely, if Θ' is a subgroup of $\Gamma_1'(E_1')$, then the subgroup $\sigma_1^{-1}(\Theta')$ of $\Gamma_1(E_1)$ has the same index as that of Θ' in $\Gamma_1'(E_1')$. And we get

$$\sigma_1^{-1}(\sigma_1(\Theta)) = \Theta, \qquad \sigma_1(\sigma_1^{-1}(\Theta')) = \Theta' .$$

Let J_Θ be the main involution of $[E_1, \Gamma_1]$ defined by the subgroup Θ of $\Gamma_1(E_1)$. The main involution $J_{\sigma_1(\Theta)}$ of $[E_1', \Gamma_1']$ defined by $\sigma_1(\Theta)$ is said to be deduced from J_Θ. Conversely, let $J_{\Theta'}$ be a main involution defined by the subgroup Θ' of $\Gamma_1'(E_1')$. Then we get

$$J_{\sigma_1^{-1}(\sigma_1(\Theta))} = J_\Theta, \qquad J_{\sigma_1(\sigma_1^{-1}(\Theta'))} = J_{\Theta'} .$$

If D is a φ-derivation of degree $\varphi(\nu)$ from $[E_1, \Gamma_1]$ into (E_2, Γ_2) with respect to the main involution J_Θ, and if the mapping

$$\bar{D}: \sigma_1(x) \to \bar{D}(\sigma_1(x)) = \sigma_2(D(x))$$

forms a ψ-derivation of degree $\psi(\sigma_1(\nu))$ from $[E_1', \Gamma_1']$ into (E_2', Γ_2') with respect to the involution $J_{\sigma_1(\Theta)}$ deduced from J_Θ, then we say that $\bar{D}$ is deduced from D.

Under the above assumptions, we prove the following Theorems 4 and 5.

THEOREM 4. *If there exists a φ-derivation D of degree $\varphi(\nu)$ from $[E_1, \Gamma_1]$ into (E_2, Γ_2), then in order that the mapping*

$$\bar{D}: \sigma_1(x) \to \bar{D}(\sigma_1(x)) = \sigma_2(D(x))$$

is a deduced ψ-derivation of degree $\psi(\sigma_1(\nu))$ from $[E_1', \Gamma_1']$ into (E_2', Γ_2'), it is necessary and sufficient that $D(\mathfrak{A}_1) \subseteqq \mathfrak{A}_2$.

PROOF. Suppose that $D(\mathfrak{A}_1) \subseteqq \mathfrak{A}_2$, First we prove that $\bar{D}$ is a linear mapping. Let $\sigma_1(x) = \sigma_1(y)$ for some elements x, y in E_1, then $\sigma_1(x-y) = 0$, $x-y = a \in \mathfrak{A}_1$, and $\sigma_2(D(x)) - \sigma_2(D(y)) = \sigma_2(D(x) - D(y)) = \sigma_2 D(x-y) = \sigma_2 D(a)$. Since $D(\mathfrak{A}_1) \subseteqq \mathfrak{A}_2$, we get $\sigma_2 D(a) = 0$. Hence $\sigma_2(D(x)) = \sigma_2(D(y))$, i. e. $\bar{D}$ is a well defined mapping. The linearity of $\bar{D}$ can be easily obtained from the linearity of D and σ_i.

Next we shall show the relation $\sigma_1 J_\Theta^\nu(x) = J_{\sigma_1(\Theta)}^{\sigma_1(\nu)}(\sigma_1(x))$. This relation is clear in the case of $\Theta = \Gamma_1(E_1)$. Hence we shall prove the relation in case $(\Gamma_1(E_1) : \Theta) = 2$. Let $x = x_+ + x_-$ be the homogeneous decomposition for the semi-graduation concerning Θ, $\sigma_1(x) = \sigma_1(x)_+ + \sigma_1(x)_-$ the homogeneous decomposition for the semi-graduation concerning $\sigma_1(\Theta)$. Since it is easily verified that $\sigma_1(x_+) = \sigma_1(x)_+$ and $\sigma_1(x_-) = \sigma_1(x)_-$, we have

$$\begin{aligned}\sigma_1 J_\Theta(x) &= \sigma_1 J_\Theta(x_+ + x_-) = \sigma_1(x_+ - x_-) = \sigma_1(x_+) - \sigma_1(x_-)\\ &= \sigma_1(x)_+ - \sigma_1(x)_- = J_{\sigma_1(\Theta)}(\sigma_1(x)_+ + \sigma_1(x)_-) = J_{\sigma_1(\Theta)}(\sigma_1(x)).\end{aligned}$$

Hence we get $\sigma_1 J_\Theta^\nu(x) = J_{\sigma_1(\Theta)}^{\sigma_1(\nu)}(\sigma_1(x))$.

We obtain therefore

$$\begin{aligned}\bar{D}(\sigma_1(x)\sigma_1(y)) &= \bar{D}(\sigma_1(xy)) = \sigma_2(D(xy))\\ &= \sigma_2[D(x)\varphi(y) + \varphi(J_\Theta^\nu(x))D(y)]\\ &= \sigma_2 D(x)\cdot\sigma_2\varphi(y) + \sigma_2\varphi(J_\Theta^\nu(x))\cdot\sigma_2 D(y)\\ &= \bar{D}(\sigma_1(x))\cdot\psi\sigma_1(y) + \psi\sigma_1(J_\Theta^\nu(x))\cdot\bar{D}(\sigma_1(y))\\ &= \bar{D}(\sigma_1(x))\psi(\sigma_1(y)) + \psi[J_{\sigma_1(\Theta)}^{\sigma_1(\nu)}(\sigma_1(x))]\bar{D}(\sigma_1(y)).\end{aligned}$$

This completes the proof of the sufficient part.

The necessary part can be easily obtained. Because, if $D(\mathfrak{A}_1) \not\subseteq \mathfrak{A}_2$, then $\bar{D}$ is not well defined.

COROLLARY 4.1. *Let (E^*, Γ^*), (E, Γ) and (F, Δ) be three graded algebras. Let σ be a homomorphism from $[E^*, \Gamma^*]$ onto $[E, \Gamma]$, ψ a homomorphism from $[E, \Gamma]$ into (F, Δ) and let $\varphi = \psi\sigma$, i.e. $\varphi_R = \psi_R\sigma_R$ and $\varphi_G = \psi_G\sigma_G$. Suppose that there exists a φ-derivation D of degree $\varphi(\nu)$ from $[E^*, \Gamma^*]$ into (F, Δ). Then in order that the mapping $\bar{D} : \sigma(x) \to \bar{D}(\sigma(x)) = D(x)$ is a deduced ψ-derivation of degree $\psi(\sigma(\nu))$ from $[E, \Gamma]$ into (F, Δ), it is necessary and sufficient that the kernel of σ is contained in the kernel of D.*

THEOREM 5. *Assume that $[E_1, \Gamma_1]$ is a free graded algebra with a free system of generators $\{x_\lambda; \lambda \in \Lambda\}$, and that $(\sigma_2)_G$ is an isomorphism from Γ_2 onto Γ_2'. If there exists a ψ-derivation $\bar{D}$ of degree $\psi(\sigma_1(\nu))$ from $[E_1', \Gamma_1']$ into (E_2', Γ_2') with respect to the main involution $J_{\Theta'}$, then there exists a φ-derivation D of degree $\varphi(\nu)$ from $[E_1, \Gamma_1]$ into (E_2, Γ_2) which deduces $\bar{D}$, i.e. $\bar{D}(\sigma_1(x)) = \sigma_2(D(x))$ for every x in E_1.*

PROOF. For each generator x_λ of $[E_1, \Gamma_1]$, we consider the set $\sigma_2^{-1}\bar{D}\sigma_1(x_\lambda)$. Since

$$\bar{D}\sigma_1(x_\lambda) \in (E_2')_{\psi(\sigma_1(\lambda+\nu))} = (E_2')_{\sigma_2(\varphi(\lambda+\nu))}$$

and since σ_2 gives a module-homomorphism from $(E_2)_{\varphi(\lambda+\nu)}$ onto $(E_2')_{\sigma_2\varphi(\lambda+\nu)}$, it is easy to see that there exists an element a_λ contained in $\sigma_2^{-1}\bar{D}\sigma_1(x_\lambda) \cap$

$(E_2)_{\varphi(\lambda+\nu)}$. Hence by Theorem 3, the mapping $x_\lambda \to a_\lambda$ can be extended to a φ-derivation D of degree $\varphi(\nu)$ from $[E_1, \Gamma_1]$ into (E_2, Γ_2) which concerns the main involution $J_{\sigma^{-1}(\Theta')}$. Now we shall prove that the derivation D deduces the given derivation $\bar{D}$. $\bar{D}\sigma_1$ is a $\psi\sigma_1$-derivation of degree $\psi\sigma_1(\nu)$ from $[E_1, \Gamma_1]$ into (E_2', Γ_2'), and $\sigma_2 D$ is a $\sigma_2\varphi$-derivation of degree $\sigma_2\varphi(\nu)$ from $[E_1, \Gamma_1]$ into (E_2', Γ_2'). Since $\psi\sigma_1 = \sigma_2\varphi$, and since $\bar{D}\sigma_1(x_\lambda) = \sigma_2 D(x_\lambda)$ for every generator x_λ, it is easy to see that $\bar{D}\sigma_1$ and $\sigma_2 D$ are the same derivation, i. e. $\bar{D}\sigma_1(x) = \sigma_2 D(x)$ for every element x in E_1. If $\sigma_1(x) = 0$, then $\bar{D}\sigma_1(x) = 0$, i. e. $\sigma_2 D(x) = 0$. Hence the kernel $\mathfrak{A}_1$ of σ_1 is contained in the kernel of $\sigma_2 D$, i. e. $D(\mathfrak{A}_1) \subseteqq \mathfrak{A}_2$. Then by Theorem 4 the mapping

$$\tilde{D} : \sigma_1(x) \to \tilde{D}(\sigma_1(x)) = \sigma_2 D(x)$$

is a deduced ψ-derivation of degree $\psi(\sigma_1(\nu))$ from $[E_1', \Gamma_1']$ into (E_2', Γ_2'), which concerns the main involution $J_{\Theta'}$. Now we have

$$\tilde{D}(\sigma_1(x)) = \sigma_2(D(x)) = \bar{D}(\sigma_1(x))$$

for every element x of E_1. Hence we get $\tilde{D}(\sigma_1(x)) = \bar{D}(\sigma_1(x))$ for every element $\sigma_1(x)$ in E_1'. This completes the proof.

§ 3. Let (E, Γ) be any graded algebra over A, and $[F, \Delta]$ a free graded algebra with a free system of generators $\{x_\lambda ; \lambda \in \Lambda\}$ over A. Let φ be a homomorphism from $[F, \Delta]$ into (E, Γ). Let f be any element of F, and let

$$(*) \qquad f = \sum\nolimits_{\lambda_1, \cdots, \lambda_n} \alpha_{\lambda_1, \cdots, \lambda_n} x_{\lambda_1} \cdots x_{\lambda_n}$$

be an expression of normal form in the free algebra F. Let Λ_f denote the set of all λ each of which appears in $(*)$. Let a be any element of E. Now we define a binary operation $\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle$ from $E \times F$ into E as follows:

$$a\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle f = \begin{cases} \sum_{\lambda_1, \cdots, \lambda_n} \alpha_{\lambda_1, \cdots, \lambda_n} \sum_{\lambda_i = \lambda} \varphi(J^\nu(x_{\lambda_1} \cdots x_{\lambda_{i-1}})) \cdot a \cdot \varphi(x_{\lambda_{i+1}} \cdots x_{\lambda_n}) & \text{if } \lambda \in \Lambda_f, \\ 0 & \text{if } \lambda \notin \Lambda_f. \end{cases}$$

If D is a φ-derivation of degree $\varphi(\nu)$ from $[F, \Delta]$ into (E, Γ), then, by using the above operation $\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle$, we have

$$D(f) = \sum\nolimits_{\lambda \in \Lambda} D(x_\lambda) \left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle f.$$

It is convenient to consider $a\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle$ as an operator for F. We now define an addition $\boxplus$ of the operators as follows:

$$\left(a\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle \boxplus b\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle \right) f = a\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle f + b\left\langle \frac{\partial}{\partial\varphi(x_\lambda)} \right\rangle f.$$

Then we have

$$D(f)=\sum_{\lambda\in\Lambda}D(x_\lambda)\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle f=\Big(\sum_{\lambda\in\Lambda}^{\boxplus}D(x_\lambda)\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle\Big)(f)\,.$$

If E is commutative, then defining

$$\frac{\partial}{\partial\varphi(x_\lambda)}f=1\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle f,$$

we have

$$a\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle f=(a\cdot 1)\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle f=a\Big(1\left\langle\frac{\partial}{\partial\varphi(x_\lambda)}\right\rangle f\Big)=a\Big(\frac{\partial}{\partial\varphi(x_\lambda)}f\Big)\,.$$

REMARK. Let $[F, \Delta]$ be a free graded algebra freely generated by $\{x_\lambda;\ \lambda\in\Lambda\}$ over a field k, and let K be the polynomial ring generated by $\{X_\lambda;\ \lambda\in\Lambda\}$ over k. Then K forms a Δ-graded algebra, and the mapping $x_\lambda\to X_\lambda$ can be extended to a homomorphism σ from $[F, \Delta]$ onto $[K, \Delta]$:

$$\sigma: f\to\sigma f=f^\sigma(X_\lambda),\qquad f\in F,\ f^\sigma(X_\lambda)\in K.$$

Let (E, Γ) be any commutative graded algebra over k, ψ a homomorphism from $[K, \Delta]$ into (E, Γ), and $\varphi=\psi\sigma$. If J^ν is identity on F, then the above $\frac{\partial}{\partial\varphi(x_\mu)}f$ coincides with the usual symbol $\frac{\partial f^\sigma}{\partial\psi(X_\mu)}$, which is the result of the substitution of $\psi(X_\lambda)$ $(=\varphi(x_\lambda))$ for X_λ in the partial derivative $\frac{\partial f^\sigma}{\partial X_\mu}$ [**2**; p. 12].

THEOREM 6. *Let $[F, \Delta]$ be a graded algebra over A with a system of homogeneous generators $\{a_\lambda;\ \lambda\in\Lambda\}$, and $[F^*, \Delta^*]$ a free graded algebra over A with a free system of generators $\{x_\lambda;\ \lambda\in\Lambda\}$. Let σ be a homomorphism from $[F^*, \Delta^*]$ onto $[F, \Delta]$ which is an extension of the mapping $x_\lambda\to a_\lambda$, and let $\mathfrak{A}=(f_\rho;\ \rho\in P)$ be the kernel of σ. Let (E, Γ) be a graded algebra over A, and ψ a homomorphism from $[F, \Delta]$ into (E, Γ). Then in order that the mapping*

$$d: a_\lambda\to d(a_\lambda)=b_\lambda\in E_{\psi(\sigma(\lambda)+\sigma(\nu))}\ (\lambda,\ \nu\in\Delta^*)$$

can be extended to a ψ-derivation $\bar{D}$ of degree $\psi(\sigma(\nu))$ from $[F, \Delta]$ into (E, Γ), it is necessary and sufficient that

$$\sum_{\lambda\in\Lambda}b_\lambda\left\langle\frac{\partial}{\partial\psi\sigma(x_\lambda)}\right\rangle f_\rho=0$$

for every $\rho\in P$.

PROOF. By Theorem 3, the mapping $x_\lambda\to b_\lambda$ can be extended to a $\psi\sigma$-derivation D of degree $\psi\sigma(\nu)$ from $[F^*, \Delta^*]$ into (E, Γ). It is verified by Corollary 4.1 and Theorem 5 that the mapping

$$d: a_\lambda=\sigma(x_\lambda)\to D(x_\lambda)=b_\lambda$$

can be extended to a ψ-derivation $\bar{D}$ of degree $\psi(\sigma(\nu))$ from $[F, \Delta]$ into (E, Γ), if and only if the kernel $\mathfrak{A}$ of σ is contained in the kernel of D. Hence it is sufficient to prove that

$$(**) \qquad \mathfrak{A} \subseteqq \text{kernel } (D) \Leftrightarrow \sum_{\lambda\in\Lambda} b_\lambda \left\langle \frac{\partial}{\partial\psi\sigma(x_\lambda)} \right\rangle f_\rho = 0 \quad (\rho\in P).$$

Since $D(f_\rho)=\sum_{\lambda\in\Lambda} b_\lambda \left\langle \frac{\partial}{\partial\psi\sigma(x_\lambda)} \right\rangle f_\rho$, the implication $\Rightarrow$ is evident. To prove the converse implication of (**), it suffices only to show that $D(gf_\rho h)=0$ for any elements g, h of F^*. Since $D(f_\rho)=0$ and $\sigma(f_\rho)=\sigma(J^\nu(f_\rho))=0$, we have

$$\begin{aligned} D(gf_\rho h) &= D(g)\psi\sigma(f_\rho)\psi\sigma(h)+\psi\sigma(J^\nu(g))D(f_\rho)\psi\sigma(h)+\psi\sigma(J^\nu(g))\psi\sigma(J^\nu(f_\rho))D(h) \\ &= 0. \end{aligned}$$

This completes the proof.

COROLLARY 6.1. *If E is commutative in Theorem 6, then in order that the mapping d can be extended to a ψ-derivation D, it is necessary and sufficient that*

$$\sum_{\lambda\in\Lambda} b_\lambda \left(\frac{\partial}{\partial\psi\sigma(x_\lambda)} f_\rho \right) = 0$$

for every $\rho\in P$.

THEOREM 7. *Under the same assumption of Theorem 6, in order that a system of equations*

$$(***) \qquad \sum_{\lambda\in\Lambda} Z_\lambda \left\langle \frac{\partial}{\partial\psi\sigma(x_\lambda)} \right\rangle f_\rho = 0 \qquad (\rho\in P)$$

has a non-trivial solution for Z_λ, it is necessary and sufficient that there exists a system of one-variable equations

$$Z_\mu \left\langle \frac{\partial}{\partial\psi\sigma(x_\mu)} \right\rangle f_\rho = 0 \qquad (\rho\in P)$$

with a non-trivial solution for Z_μ.

PROOF. Let $\{b_\lambda;\ \lambda\in\Lambda\}$ be a non-trivial solution of (***), i.e.

$$\sum_{\lambda\in\Lambda} b_\lambda \left\langle \frac{\partial}{\partial\psi\sigma(x_\lambda)} \right\rangle f_\rho = 0 \qquad (\rho\in P).$$

Then, by Theorem 6, the mapping $\sigma(x_\lambda)\to b_\lambda$ can be extended to a ψ-derivation $\bar{D}$ from $[F, \Delta]$ into (E, Γ), and by Theorem 3, the mapping $x_\lambda\to b_\lambda$ can be extended to a $\psi\sigma$-derivation D from $[F^*, \Delta^*]$ into (E, Γ). Then it is clear that $D(x)=\bar{D}(\sigma(x))$, and hence, by Corollary 4.1, the kernel of σ is contained in the kernel of D. Suppose that $b_\mu\neq 0$, and $c_\lambda=\delta_{\lambda\mu}b_\mu$. It is clear from Theorem 3 that the mapping $x_\lambda\to c_\lambda$ can be extended to a $\psi\sigma$-derivation D' from $[F^*, \Delta^*]$ into (E, Γ). Then the kernel of D is clearly contained in the kernel of D'. Hence the kernel of σ is contained in the kernel of D'. Therefore, by Corollary 4.1, there exists a ψ-derivation $\bar{D}'$: $\sigma(x)\to D'(x)$ which is an extension of the mapping $a_\lambda=\sigma(x_\lambda)\rightarrow D'(x_\lambda)=c_\lambda$. By Theorem 6 we obtain

$$\sum_{\lambda\in\Lambda} c_\lambda\left\langle\frac{\partial}{\partial\psi\sigma(x_\lambda)}\right\rangle f_\rho=0, \text{ i.e. } c_\mu\left\langle\frac{\partial}{\partial\psi\sigma(x_\mu)}\right\rangle f_\rho=0 \qquad (\rho\in P).$$

Conversely, if there exists a system of one-variable equations

$$Z_\mu\left\langle\frac{\partial}{\partial\psi\sigma(x_\mu)}\right\rangle f_\rho=0 \qquad (\rho\in P)$$

with a non-trivial solution c_μ, then putting $b_\lambda=\delta_{\lambda\mu}c_\mu$, we obtain

$$\sum_{\lambda\in\Lambda} b_\lambda\left\langle\frac{\partial}{\partial\psi\sigma(x_\lambda)}\right\rangle f_\rho=c_\mu\left\langle\frac{\partial}{\partial\psi\sigma(x_\mu)}\right\rangle f_\rho=0 \qquad (\rho\in P).$$

This completes the proof.

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References

[1] C. Chevalley, The construction and study of certain important algebras, Tokyo, Publications of the Math. Soc. of Japan, 1955.

[2] A. Weil, Foundations of algebraic geometry, New York, Amer. Math. Soc. Colloq. Publ., 1946.