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Local theory of rings of operators I.

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The theory of rings of operators founded by F. J. Murray and J. v. Neumann [1], [2], [3], [4] was extended from the case of factors to general rings of operators by J. Dixmier [5], I. Kaplansky [6], I. E. Segal [7], and others. In particular, the notions of finiteness, and types I, II etc. of general operator algebras and of the trace of elements of these algebras were defined and investigated by these authors. The aim of this paper is to reestablish and generalize some results of these authors from a unified standpoint by introducing the notion of "local properties" of systems of elements of operator algebras.

We shall explain in §1 what we mean by "local" and "global" properties of systems of elements of a B^*-algebra, and study mutual relations between them.

In §2 we refer to some general theorems as preliminaries to §§3, 4. These are mostly known results, but we give also proofs for completeness' sake. Especially the results on "natural supporters" as named by Ti. Yen [8] after the idea of Dixmier [5], are given here for arbitrary AW^*-algebras, whereas Dixmier [5] introduced them in case of finite W^*-algebras and Ti. Yen considered them only in case of finite AW^*-algebras.

In §3, we shall develope a "local theory" of AW^*-algebras. We shall first reestablish an important theorem of Kaplansky [6] on the equivalence between projections in AW^*-algebras as Proposition 3.5, and obtain finally a "decomposition theorem" as Proposition 3.10. The method of "localization" will turn out to be very useful in the course of this §.

Finally we shall deal with the trace in §4. This concept was introduced by F. J. Murray - J. von Neumann [1], [2] in finite factors, and investigated further by J. Dixmier [5] in case of finite W^*-algebras, by Ti. Yen [8] and M. Goldman [9] in case of finite AW^*-algebras. We shall obtain a necessary and sufficient condition for the existence of "local trace" in finite AW^*-algebras (Proposition 4.1.) and some sufficient conditions for the existence of trace in these algebras (Theorems 4.1, 4.2, 4.3).

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§ 1. Global properties and local properties.

Let R be a B^*-algebra with a unit 1, that is to say, a Banach algebra over the complex number field with a unit 1 and an involution * satisfying following conditions:

(1.1) $x^{**}=x$,

(1.2) $(\alpha x)^*=\bar{\alpha}x^*$ ($\bar{\alpha}$=the conjugate complex number of α),

(1.3) $(x+y)^*=x^*+y^*$,

(1.4) $(xy)^*=y^*x^*$,

(1.5) $\|x^*x\|=\|x\|^2$.

We denote by R_0 the center of R and by Ω the spectrum of R_0, which is a compact Hausdorff space by the usual topology $\sigma(R_0, \Omega)$. It is well known that R_0 is isometric and isomorphic to $C(\Omega)$, the B^*-algebra of continuous functions on Ω by a theorem of I. Gelfand and M. H. Stone. We identify R_0 with $C(\Omega)$ by the canonical isomorphism from R_0 onto $C(\Omega)$. We write $\lambda(a_0)$ instead of $a_0(\lambda)$ for $a_0\in R_0$ and $\lambda\in\Omega$. An element e of R is called a *projection* if we have $e=e^*=e^2$. We denote by E_0 the set of projections of R_0, which forms a Boolean lattice if we define the semi-order $e_{01}\leqq e_{02}$, ($e_{01}, e_{02}\in E_0$) by $\lambda(e_{01})\leqq\lambda(e_{02})$ for any $\lambda\in\Omega$. As to E_0, we shall assume that the following condition is satisfied:

(1.6) R_0 is generated by E_0.

A point of Ω is called a *spectre* of R. For any spectre λ of R, we denote by $E_0(\lambda)$ the set of projections e_0's of E_0 with $\lambda(e_0)=1$. Then, $E_0(\lambda)$ forms the set of characteristic functions of a basis of neighbourhoods at λ.

Let (L) be a property concerned with a system $\mathfrak{a}=(a_\iota; \iota\in I)$ of some elements of R, where I is a set of indices depending on (L). We denote by $E_0(\mathfrak{a}, (L))$ (or briefly by $E_0(\mathfrak{a})$) the set of projections e_0's of E_0 such that $e_0\mathfrak{a}=(e_0a_\iota; \iota\in I)$ has the property (L). A property (L) is called *global*, if, for any system $\mathfrak{a}$ of elements of R, $E_0(\mathfrak{a})$ forms an ideal of E_0, that is to say, a non-empty subset of E_0 containing $e_{01}\cup e_{02}, e_{01}\cap e_{02}'$ with e_{01} and e_{02} for any e_{02}' of E_0. A property (L_λ) is called *local* with respect to a spectre λ of R, if, for any system $\mathfrak{a}$ of elements of R, it holds that $E_0(\mathfrak{a}, (L_\lambda))=E_0$ or $E_0(\lambda)^c$ (the complement of $E_0(\lambda)$ in E_0). For any global property (L) and for any spectre λ of R, we denote by $(L)_\lambda$ the local property with respect to λ, which $\mathfrak{a}$ has if and only if $E_0(\lambda)\cap E_0(\mathfrak{a})\neq\phi$ (we shall denote with ϕ the empty set). We call $(L)_\lambda$ the local property corresponding to (L) with respect to λ, and denote by $\eta(L)$ the system $((L)_\lambda;\ \lambda\in\Omega)$ of local properties corresponding to (L).

The following series of propositions play an essential role in our investigations.

Proposition 1.1. *$\mathfrak{a}$ has a global property (L) if and only if $\mathfrak{a}$ has $(L)_\lambda$ for any spectre λ of R.*

Proof. Necessity. If $\mathfrak{a}$ has (L), we have $E_0(\mathfrak{a})=E_0$, and hence $E_0(\lambda)\cap E_0(\mathfrak{a})\neq\phi$. This means that $\mathfrak{a}$ has $(L)_\lambda$ for any spectre λ of R. Sufficiency. For any spectre λ of R, there exists a projection $e_0(\lambda)$ of $E_0(\lambda)$ with $e_0(\lambda)\in E_0(\mathfrak{a})$. Since Ω is compact, we have $1=e_0(\lambda_1)\cup e_0(\lambda_2)\cup\cdots\cup e_0(\lambda_n)$ for some spectres $\lambda_1, \lambda_2,\cdots, \lambda_n$ of R with $e_0(\lambda_i)\in E_0(\mathfrak{a})$ $(1\leqq i\leqq n)$. Since $E_0(\mathfrak{a})$ is an ideal of E_0, we get $1\in E_0(\mathfrak{a})$. This means that $\mathfrak{a}$ has (L). q. e. d.

For any system $((L_\lambda);\ \lambda\in\Omega)$ of local properties, we denote by $\eta'((L_\lambda);\ \lambda\in\Omega)$ the logical product of local properties of $((L_\lambda);\ \lambda\in\Omega)$, that is to say, the property, which $\mathfrak{a}$ has if and only if $\mathfrak{a}$ has (L_λ) for any spectre λ of R. Then, we have

Proposition 1.2. *$\eta'((L_\lambda);\ \lambda\in\Omega)$ is a global property.*

Proof. Let Ω' be the set of spectres μ's of R, for which $\mathfrak{a}$ has not (L_μ) and let (L) be $\eta'((L_\lambda);\ \lambda\in\Omega)$. Then, we have $E_0(\mathfrak{a},(L))=\cap(E_0(\mu)^c;\ \mu\in\Omega')$, where the intersection means E_0 if Ω' is an empty set. Hence, $E_0(\mathfrak{a},(L))$ is an ideal of E_0. q. e. d.

A system $((L_\lambda);\ \lambda\in\Omega)$ of local properties is called *closed* if Ω' $(=(\mu;\ \mathfrak{a}$ has not $(L_\mu)))$ is a closed subspace of Ω for any system $\mathfrak{a}$ of elements of R. Then, we have

Proposition 1.3. *For a system $((L_\lambda);\ \lambda\in\Omega)$ of local properties, it holds $(L_\lambda)=(\eta'((L_\lambda);\ \lambda\in\Omega))_\lambda$ for any spectre λ of R if and only if the system $((L_\lambda);\ \lambda\in\Omega)$ of local properties is closed.*

Proof. Necessity. For a spectre λ of R with $\lambda\in\Omega'$, it holds that $\mathfrak{a}$ has $(\eta'((L_\lambda);\ \lambda\in\Omega))_\lambda$ by hypothesis, that is, $E_0(\lambda)\cap(\cap(E_0(\mu)^c;\ \mu\in\Omega'))\neq\phi$. Hence, we may find a characteristic function $e_0(\lambda)$ of a neighbourhood of λ separating λ and Ω'. This means that Ω' is closed. Sufficiency. If $\mathfrak{a}$ has not (L_λ), then we have $\lambda\in\Omega'$. Hence, we have $E_0(\lambda)\cap(\cap(E_0(\mu)^c;\ \mu\in\Omega'))=\phi$. This means that $\mathfrak{a}$ has not $(\eta'((L_\lambda);\ \lambda\in\Omega))_\lambda$. Conversely, if $\mathfrak{a}$ has (L_λ), then we have $\lambda\in\Omega'$. Since Ω' is closed, we may find a characteristic function $e_0(\lambda)$ $(\in E_0(\lambda))$ of a neighbourhood of λ separating Ω'. This means that it holds $E_0(\lambda)\cap(\cap(E_0(\eta)^c;\ \mu\in\Omega'))\neq\phi$. Thus, $\mathfrak{a}$ has $(\eta'(L_\lambda);\ \lambda\in\Omega))_\lambda$. q. e. d.

We write 0 instead of $\mathfrak{a}=(a_\iota;\ \iota\in I)$ if each a_ι of $\mathfrak{a}$ is 0. Then, the property "$\mathfrak{a}=0$" is a global property. We denote it by (ϕ). For a local property (L_λ) with respect to a spectre λ of R, we denote by $(L_\lambda)^c$ the property, which $\mathfrak{a}$ has if and only if $\mathfrak{a}$ has $(\phi)_\lambda$ or has not (L_λ). Then, we have

Proposition 1.4. *The property $(L_\lambda)^c$ is a local property with respect to λ.*

Proof. If $\mathfrak{a}$ has (L_λ), we have $E_0(\mathfrak{a}, (L_\lambda)^c) = E_0(\mathfrak{a}, (\phi)_\lambda) = E_0$ or $E_0(\lambda)^c$. On the other hand, if $\mathfrak{a}$ has not (L_λ), $e_0(\lambda)\mathfrak{a}$ has not (L_λ) for any $e_0(\lambda)$ of $E_0(\lambda)$. Hence, we have $E_0(\mathfrak{a}, (L_\lambda)^c) \geqq E_0(\lambda)$. Since $\mathfrak{a}$ has not (L_λ), $\mathfrak{a}$ has not $(\phi)_\lambda$, that is, $E_0(\mathfrak{a}, (\phi)_\lambda) = E_0(\lambda)^c$. Thus, we have $E_0(\mathfrak{a}, (L_\lambda)^c) \geqq E_0(\lambda) \cup E_0(\lambda)^c = E_0$. q. e. d.

This local property $(L_\lambda)^c$ is called the *negation of* (L_λ). Similarly, for a global property (L), we denote by $(L)^c$ the logical product of the system $(((L)_\lambda)^c; \lambda \in \Omega)$ of local properties. This global property $(L)^c$ is called the *negation of* (L).

A global property (L) is called *normal*, if, for any system $\mathfrak{a}$ of elements of R, $E_0(\mathfrak{a}, (L))$ is a principal ideal of E_0, that is to say, an ideal such as e_0E_0 of E_0 for some projection e_0 of E_0. We denote e_0 by $e_0(\mathfrak{a}, (L))$. In the following investigation, we shall add the following assumption

(1.7) (ϕ) is normal.

Under the assumption (1.7), we have the following

Proposition 1.5. *It holds* $((L)_\lambda)^c = ((L)^c)_\lambda$ *for any spectre* λ *of* R *if and only if* (L) *is normal.*

Proof. Necessity. Since we have $((L)_\lambda)^c = ((L)^c)_\lambda$ for any spectre λ of R, the system $(((L)_\lambda)^c; \lambda \in \Omega)$ of local properties is closed by Prop. 1.3. On the other hand, $(\mu; \mathfrak{a}$ has not $((L)_\mu)^c) = (\mu; \mathfrak{a}$ has $(L)_\mu) \cap (\mu; \mathfrak{a}$ has $(\phi)_\mu)^c$ is open by (1.7) and by the definition of $(L)_\mu$. Hence, we may find a projection e_0 of E_0, which is the characteristic function of $(\mu; \mathfrak{a}$ has not $((L)_\mu)^c)$. Thus, we have $E_0(\mathfrak{a}, (L)) = (e_0 \oplus e_0(\mathfrak{a}, (\phi)))E_0$. (We used here and shall use hereafter the notation $e_1 \oplus e_2$ instead of $e_1 + e_2$ if $e_1e_2 = 0$.) Sufficiency. Since $(\mu; \mathfrak{a}$ has not $((L)_\mu)^c) = (\mu; \mathfrak{a}$ has $(L)_\mu) \cap (\mu; \mathfrak{a}$ has $(\phi)_\mu)^c$ is closed, we have $((L)_\lambda)^c = ((L)^c)_\lambda$ for any spectre λ of R by Prop. 1.3. q. e. d.

The negation $(L)^c$ is normal with (L) if and only if we have (1.7). In fact, we have $E_0(\mathfrak{a}, (L)^c) = E_0(\mathfrak{a}, (L))' \oplus E_0(\mathfrak{a}, (\phi))$ (the direct sum of $E_0(\mathfrak{a}, (L))'$ and $E_0(\mathfrak{a}, (\phi))$ as Boolean lattices). Here, we denote by $E_0(a, (L))'$ the set of projections e_0's of E_0 such that $e_0E_0(a, (L)) = (0)$. Moreover, under (1.7), we have $(L)^{cc} = (L)$ if (L) is normal. In fact, it holds that $((L)^{cc})_\lambda = ((L)_\lambda)^{cc} = (L)_\lambda$ by Prop. 1.5. Hence, we have $(L)^{cc} = (L)$ by Prop. 1.1.

For example, the property "$a = b$" is a global property which concerns two elements a, b of R. The local property "$a =_\lambda b$" corresponding to "$a = b$" with respect to a spectre λ of R is defined by the existence of a projection $e_0(\lambda)$ of $E_0(\lambda)$ with "$e_0(\lambda)a = e_0(\lambda)b$". On the other hand, we may define "$a = b$" by $\|a - b\| \leqq \varepsilon$ for any positive number ε, whose local property with respect to λ is defined by the existence of a projection $e_0(\lambda)$ of $E_0(\lambda)$ with $\|e_0(\lambda)(a - b)\| \leqq \varepsilon$ for any positive number ε. Thus, two local properties corresponding to the same global property with respect to the same spectre of R is not always equivalent to each other according to the mode of the expression of the definition of the global property. In this paper, we shall use the former as the definition of " $a=b$ ".

§ 2. Preliminary results.

Let R be an AW^*-algebra, that is to say, a B^*-algebra satisfying the following conditions:

(2.1) Any orthogonal system of projections of R has a supremum in the set of projections of R with respect to the semi-order $e_1 \leqq e_2$ (e_1, e_2 being projections of R) defined by $e_1 e_2 = e_1$, where two elements a, b of R are called *orthogonal* to each other if it holds $a^*b = ba^* = 0$,

(2.2) Any maximal commutative subalgebra of R is generated by projections in it, where two elements a, b of R is called *commutative* with each other if it holds that $ab = ba$ and $a^*b = ba^*$.

This algebra was introduced by I. Kaplansky [6]. For the sake of completeness, we shall sketch the proofs of results obtained by C. E. Rickart [10], I. Kaplansky [6], and Ti. Yen [8]. We denote by E the set of projections of R and by U the set of *partial isometries* of R, that is to say, the set of elements u's of R such that u^*u is a projection of R. We notice that uu^* is a projection of R if u is a partial isometry of R.

Lemma 2.1. *In a B^*-algebra,* (2.1) *is equivalent to*

(2.3) *Any chain of E, that is, any linearly ordered subset of E, has a supremum in E.*

Proof. (2.3) implies (2.1). In fact, let E_1 be an orthogonal system of projections of a B^*-algebra. We shall show that E_1 has a supremum in the set E of projections of the B^*-algebra. We denote by $\mathfrak{F}$ the family of subsets E_2's of E_1 having a supremum in E. Obviously, $\mathfrak{F}$ is non-empty. Moreover, from (2.3) it follows that $\mathfrak{F}$ is an inductively ordered set with respect to the inclusion semi-order. Hence, by Zorn's lemma, there exists a maximal subset E_3 of E_1 in $\mathfrak{F}$ with respect to this semi-order. It is easy to see that E_3 coincides with E_1.

(2.1) implies (2.3). First, we shall prove that, if a projection e' is commutative with each element of an orthogonal system E_1 of projections, then e' is commutative also with its supremum e_1 under the assumption (2.1). We denote by e_2 the supremum of $(e'e_\iota ; e_\iota \in E_1)$ and by e_3 that of $((1-e')e_\iota ; e_\iota \in E_1)$. Then, we have $e_2 \leqq e'$, $e_3 \leqq 1-e'$, and $e_1 = e_2 \oplus e_3$. Hence, $e'e_1 = e_2$ and thus $e'e_1 = e_1 e'$. Next, we shall prove that any commutative system E_1 of projections has a supremum in E under the assumption (2.1). We denote by E_2 the set of projections e's such that $e \leqq e_\iota$ for some $e_\iota \in E_1$ and that e is commutative with each projection of a commutative system E_3 of projections containing E_1. (There is no need .to introduce E_3 here, but we do so in order to prove the corollary below by a method without large charge.) Moreover, we denote by E_4 a maximal orthogonal system in E_2, whose supremum we denote by e. Then, we have $e_\iota \leqq e$ for any $e_\iota \in E_1$. In fact $e_\iota(1-e)$ is orthogonal to eash projection of E_4 and commutative with each projection of E_3 and hence $e_\iota(1-e)=0$. On the other hand, if $e_\iota \leqq e'$ for any $e_\iota \in E_1$, then we have $e_\kappa' \leqq e'$for any $e_\kappa' \in E_4$ and hence $e \leqq e'$. Thus, we have $e=\sup(e_\iota ; e_\iota \in E_1)$. q. e. d.

Corollary. *If a projection e_4 of R is commutative with each projection of a commutative system of E, then it is commutative also with the supremum of the system.*

Proof. Under the same notation as in the proof of Lemma 2.1, we take the commutative system consisting of e_4 and E_1 as E_3. Then, we get the assertion by the proof of Lemma 2.1. q. e. d.

Proposition 2.1. *R has a unit (denoted by 1).*

Proof. There exists a maximal orthogonal system E_1 of E, whose supremum we denote by e. Then, we have $ae=a$ for any element a of R. For, otherwise, we could find an element a of R with $a-ae\neq 0$. We denote $(a-ae)^*(a-ae)$ by h. Then, we have $h\neq 0$ and $eh=he=0$. Let A be a maximal commutative subalgebra of R containing h and e. Then, for any natural number n, there exists an orthogonal system $(e_\iota^{(n)} ; 1\leqq \nu \leqq k_n)$ of projections of A and real numbers $(\alpha_\nu^{(n)} ; 1\leqq \nu \leqq k_n)$ such that $\|h-\sum_{\nu=1}^{k_n}\alpha_\nu^{(n)}e_\nu^{(n)}\| \leqq 1/n$. If $ee_\nu^{(n)}\neq 0$ for any n and any ν, we have $\|\sum_{\nu=1}^{k_n}\alpha_\nu^{(n)}ee_\nu^{(n)}\| \leqq 1/n$, from which follows $|\alpha_\nu^{(n)}| \leqq 1/n$, $(1\leqq \nu \leqq k_n)$. Hence, we have $\|h\| \leqq 2/n$, which leads to a contradiction. Thus, we have $ee_\nu^{(n)}=0$ for some n and some ν. But this contradicts the maximality of E_1. Hence, we obtain $ae=a$ for any element a of R. On the other hand, we have $a=(a^*)^*=(a^*e)^*=ea$. Thus, R has a unit e. q. e. d.

An element h of R with $h=h^*$ is called *hermitian*. We denote by N the set of hermitian elements of R. By virtue of M. Fukamiya [11], J. L. Kelley-R. L. Vaught [12], and I. Kaplansky [13] (cf. J. A. Schatz' review [14]), it is known that the set of hermitian elements of a B^*-algebra with a unit 1 forms a semi-ordered set with respect to a semi-order $h\geqq 0$ defined by the following mutially equivalent conditions:

(2.4) Any spectrum of h is positive,

(2.5) $h=k^2$, for some hermitian element k of R,

(2.6) $\|\alpha-h\| \leqq \alpha$ for sufficiently large positive number α,

and with respect to this semi-order it holds that

(2.7) $a^*a\geqq 0$ for any element a of R.

We can easily see that the semi-order of E stated in (2.1) coincides with that of N reduced to E.

Let A be a maximal commutative subalgebra of R and N_A be the set of hermitian elements of A. We say that an element h of N has *a resolution of the unit in* N_A, if there exists a system $(e_\alpha; -\infty<\alpha<\infty)$ of projections of N_A satisfying (1) $e_c=1, e_{-c}=0$ for a sufficiently large positive number c, (2) $e_\alpha \leqq e_\beta$ for $\alpha\leqq\beta$,(3) $\lim_{\beta\downarrow\alpha} e_\beta=e_\alpha$, and

(2.8) $h=\int_{-\infty}^{\infty}\alpha de_\alpha$.

Lemma 2.2. *If each of h, k of N_A has a resolution of the unit in N_A, then*
(2.9) *$h\leqq k$ holds if and only if $e_\alpha(h)\geqq e_\alpha(k)$ for any real number α.*

Proof. Necessity. We denote by $(e_\alpha(h); -\infty<\alpha<\infty)$ a resolution of the unit of an hermitian element h of R if h has a resolution of the unit. We may assume without loss of generality that we have $0\leqq h\leqq k$. We denote $1-e$ by e^c for any projection e of N. Then, we have $\beta e_\beta(h)^c\leqq k$, from which follows $\beta e_\beta(h)\geqq\beta-k$ and hence $\beta e_\beta(h)e_\alpha(k)\geqq(\beta-\alpha)e_\alpha(k)$ for $\beta>\alpha\geqq 0$. Thus, we have $e_\beta(h)\geqq e_\alpha(k)$. Making $\beta\downarrow\alpha$, we have $e_\alpha(h)\geqq e_\alpha(k)$. Sufficiency. First we shall prove that, for any element h of N_A not being $h\geqq 0$, there exists a projection e of N with $eh\geqq 0$ and $eh\neq 0$. In fact, under the same notation as in the proof of Prop. 2.1, we obtain $1/n+\sum_{\nu=1}^{k_n}\alpha_\nu^{(n)}e_\nu^{(n)}\geqq h\geqq -1/n+\sum_{\nu=1}^{k_n}\alpha_\nu^{(n)}e_\nu^{(n)}$, and here we have not always $1/n\geqq\alpha_\nu^{(n)}$, for, otherewise, we would have $h\leqq 0$. Thus, for some n and $\nu, \alpha_\nu^{(n)}-1/n>0$, so $e_\nu^{(n)}h\geqq(\alpha_\nu^{(n)}-1/n)e_\nu^{(n)}>0$ and $e_\nu^{(n)}h\neq 0$. Next, we assume that $e_\alpha(h)\geqq e_\alpha(k)$ for any real number α and that $h\leqq k$ does not hold. Then, we may find a projection e of N_A with $eh\geqq ek$ and $eh\neq ek$. Since each of eh, ek has also a resolution of the unit, we have $ee_\alpha(h)\leqq ee_\alpha(k)$. On the other hand, we see $ee_\alpha(h)\geqq ee_\alpha(k)$ from $e_\alpha(h)\geqq e_\alpha(k)$. Thus, we get $ee_\alpha(h)=ee_\alpha(k)$ for any real number α, that is, $eh=ek$. This leads to a contradiction. Therefore, if $e_\alpha(h)\geqq e_\alpha(k)$ for any α, then we have $h\leqq k$. q. e. d.

Lemma 2.3. *Any upper-bounded system $(h_\iota; \iota\in I)$ of N_A, whose member has a resolution of the unit in N_A, has a supremum with a resolution of the unit in N_A.*

Proof. We denote by $(e_\alpha(h_\iota); -\infty<\alpha<\infty)$ a resolution of the unit of h_ι. Then, $(e_\alpha(h_\iota); \iota\in I)$ forms a commutative system of projections of N_A. We put $e_\alpha'=\inf(e_\alpha(h_\iota); \iota\in I)$. By the Corollary of Lemma 2.1, e_α' is a projection of N_A. We put $\inf_{\beta\downarrow\alpha} e_\beta'=e_\alpha$. Then, $(e_\alpha; -\infty<\alpha<\infty)$ forms a resolution of the unit of an element h $(h=\int_{-\infty}^{\infty}\alpha de_\alpha)$. It can be proved that h is a supremum of $(h_\iota; \iota\in I)$. In fact, we have $h_\iota\leqq h$ from $e_\alpha(h_\iota)\geqq e_\alpha(h)$ for any real number α. On the other hand, if $h_\iota\leqq k$ for any $\iota\in I$, there exist, for any natural number n, projections $(e_\nu^{(n)}; 1\leqq\nu\leqq k_n)$ and real numbers $(\alpha_\nu^{(n)};$

$1\leqq\nu\leqq k_n$) such that $\|k-k^{(n)}+1/n\|<1/n$ with $k^{(n)}=1/n+\sum_{\nu=1}^{k_n}\alpha_\nu^{(n)}e_\nu^{(n)}$, which has a resolution of the unit (denoted by $(e_\alpha(k^{(n)})\,;\ -\infty<\alpha<\infty)$). Since $h_\iota\leqq k^{(n)}$, we have from Lemma 2.2 $e_\alpha(h_\iota)\geqq e_\alpha(k^{(n)})$ and hence $e_\alpha(h)\geqq e_\alpha(k^{(n)})$. Thus, it follows $h\leqq k_n$. Making $n\to\infty$, we get $h\leqq k$. q. e. d.

We say that an element h of N has *a resolution of the unit* if there exists a system $(e_\alpha(h)\,;\ -\infty<\alpha<\infty)$ of projections of N satisfying (1)-(3) and (2.8) as before. Then, we have the so-called spectral theorem as follows.

PROPOSITION 2.2. *Any element of N has a unique resolution of the unit.*

PROOF. Existence. Let k be an element of N and let A be a maximal commutative subalgebra of R containing k. Then, under the same notation as in the proof of Lemma 2.3, it holds that $k=\lim_{n\to\infty} k^{(n)}$ $(=\inf(\sup(k^{(\nu)}\,;\ \nu\geqq n)\,;\ 1\leqq n<\infty))$ and $\sup(k^{(n)}\,;\ n\leqq\nu\leqq m)$ has a resolution of the unit in N_A. Thus, by Lemma 2.3, k has a resolution of the unit (in N_A). Uniqueness. We denote by $(k)'$ the set of elements of R commutative with k and by $(k)''$ the set of elements of R commutative with each element of $(k)'$. Then, the uniqueness and $e_\alpha(k)\in(k)''$ are proved by a similar argument as in the proof of the spectral theorem of hermitian operators on a Hilbert space. q. e. d.

The following proposition is an immediate consequence of Lemma 2.3 and Prop. 2.2.

PROPOSITION 2.3. *In any maximal commutative subalgebra A of R, every upper-bounded system of hermitian elements has a supremum in A.*

For any element a of R, we denote $e_0(a^*a)^c$ $(=1-e_0(a^*a))$ by $e_*(a)$ and $e_*(a^*)$ by $e(a)$, where $(e_\alpha(a^*a)\,;\ -\infty<\alpha<\infty)$ is the resolution of the unit of a^*a. After C. E. Rickart [10] and I. Kaplansky [6], we call $e_*(a)$ *the initial projection* of a and $e(a)$ *the final projection* of a. We say that a projection e *fixes* an element *a from right side* if $ae=a$.

LEMMA 2.4. *$e_*(a)$ is the minimal projection fixing a from right side.*

PROOF. Since $e_*(a)^c(a^*ae_*(a)^c)=0$, we have $ae_*(a)^c=0$. Hence, we have $ae_*(a)=a$. On the other hand, if $ae=a$ for a projection e of R, we get $ae^c=0$ and hence $a^*ae^c=0$. By easy computation, it follows that $e_\alpha(a^*a)^ce^c=0$ for $\alpha>0$. Making $\alpha\downarrow 0$, we get $e_*(a)e^c=0$ and so $e_*(a)\geqq e$. q. e. d.

PROPOSITION 2.4. *For any system F of elements of R, there exists the minimal projection fixing each element of F from right side.*

PROOF. We denote by E_1 a maximal orthogonal system of projections e's of R with $ae=0$ for any $a\in F$ and by e_1 its supremum, which satisfies $ae_1=0$ for any $a\in F$. If $ae=0$ for any $a\in F$, then $e(e-e_1e)$ (the final projection of $e-e_1e$) is orthogonal to e_1 and $ae(e-e_1e)=0$ for any $a\in F$. Hence, $e(e-e_1e)=0$, that is, $e\leqq e_1$. Thus, e_1^c is the projection in question. q. e. d.

We denote by $e_*(F)$ the minimal projection fixing each element of F from right side and by $e(F)$ that from left side. It is obvious that $e(F)=e_*(F^*)$, where $F^*=(a^*; a\in F)$. As a corollary of Prop. 2.4, we have (cf. I. Kaplansky [6])

PROPOSITION 2.5. *The set E of projections of R forms a complete lattice.*

As to $e(a)$ and $e(F)$, we have the following

LEMMA 2.5. *It holds that*

(2.9) $e(ab)=e(ae(b))$,

(2.10) $e(a(\cup(e_\iota; \iota\in I)))=\cup(e(ae_\iota); \iota\in I)$,

(2.11) $e(F)=\cup(e(a); a\in F)$,

(2.12) $e(e_1{}^c e_2)=e_1\cup e_2-e_1$,

where $(e_\iota; \iota\in I)$ is a system of projections of R and F is a system of elements of R.

PROOF. (2.9) follows from the fact that $eab=0$ is equivalent to $eae(b)=0$ for any $a, b\in R$ and any $e\in E$ (the set of projections of R). We shall prove (2.10). From $e(a(\cup(e_\iota; \iota\in I)))a(\cup(e_\iota; \iota\in I))=a(\cup(e_\iota; \iota\in I))$ it follows that $e(a(\cup(e_\iota; \iota\in I)))ae_\iota=ae_\iota$, that is, $e(a(\cup(e_\iota; \iota\in I)))\geqq e(ae_\iota)$. Hence, we have $e(a(\cup(e_\iota; \iota\in I)))\geqq\cup(e(ae_\iota); \iota\in I)$. On the other hand, denoting $\cup(e(ae_\iota); \iota\in I)$ briefly by e, we see $eae_\iota=ae_\iota$ and hence $(ea-a)e_\iota=0$ and so $e_*(ea-a)e_\iota=0$. Therefore, by Prop. 2.5, it holds that $e_*(ea-a)(\cup(e_\iota; \iota\in I))=0$ and so $(ea-a)(\cup(e_\iota; \iota\in I))=0$. Thus, we get $e(a(\cup(e_\iota; \iota\in I)))\leqq e$. (2.11) is an immediate consequence of Prop. 2.5. We shall see (2.12). It is easy to see that $e(e_1{}^c e_2)\leqq e_1\cup e_2-e_1$. Conversely, putting $e'=e(e_1{}^c e_2)$, we have $e'^c e_1{}^c e_2=0$, that is, $e_1{}^c-e'\leqq e_2{}^c$. Hence, we have $e_1{}^c-e'\leqq e_1{}^c\cap e_2{}^c$, that is, $e_1\cup e_2-e_1=e_1{}^c-e_1{}^c\cap e_2{}^c\leqq e'$. Thus, we get (2.12). q. e. d.

After Ti. Yen [8], a subalgebra R_1 of R is called *an AW^*-subalgebra of R* if it is a B^*-subalgebra of R with structure of an AW^*-algebra. We shall consider two AW^*-subalgebras R_i $(i=1, 2)$ of R, whose units are denoted by I_i $(i=1, 2)$, and a system A of elements of R satisfying $b^*a\in R_1$, $ab^*\in R_2$, and $R_2aR_1\subseteqq A$ for any $a, b\in A$. We denote by R_{0i}, E_i, and E_{0i} the center of R_i, the set of projections of R_i, and the set of projections of R_{0i} respectively.

To e_1 of E_1, we associate the minimal projection $e_1{}^\natural$ of E_1 in E_1 fixing each element of e_1R_1 from right side. This notation $e_1{}^\natural$ was introduced by J. Dixmier [5] in another expression $e_1{}^\natural=\cup(s_1{}^*e_1s_1; s_1$ unitary of $R_1)$ in finite W^*-algebras and was called by Ti. Yen [8] *the natural supporter* of e_1. (An element s_1 in R_1 is called *unitary* if it holds that $s_1{}^*s_1=s_1s_1{}^*=I_1$.) Similarly, to e_1 of E_1, we associate the minimal projection $e_1{}^\#$ of E_2 in E_2 fixing each element of Ae_1 from left side. In the same way, we can associate with any $e_2\in E_2$ an $e_2{}^\natural\in E_2$ and an $e_2{}^\#\in E_1$ by considering R_2e_2 and e_2A respectively.

Then, we have the following

PROPOSITION 2.6. *As to operation $\sharp$ the following statements hold.*

(2.13) $e_1^\sharp \in E_{02}$,

(2.14) $e_1^{\natural\sharp} = e_1^\sharp$,

(2.15) $(\cup(e_\iota; \iota \in I))^\sharp = \cup(e_\iota^\sharp; \iota \in I)$,

(2.16) $(e_1^\natural \cap e_1')^\sharp = e_1^\sharp \cap e_1'^\sharp$,

(2.17) $e_1^{\sharp\sharp} = e_1^\natural$ *if and only if* $e_1^\natural \leqq I_2^\sharp$,

(2.18) $e_2 A e_1 = 0$ *if and only if* $e_2^\sharp e_1^\natural = 0$.

PROOF. The proof of (2.13). For any element b of R with $bb^* \in R_2$, we denote also by $e(b)$ the minimal projection in E_2 fixing b from left side. Moreover, we denote Ae_1 by F. Then, for any element a of F, we have $aa^* \in R_2$ and $R_2 a \subseteqq F$. By (2.11), we have $e_1^\sharp = \cup(e(a); a \in F)$. Putting $c = be_1^\sharp - e_1^\sharp b e_1^\sharp$ for any (but fixed) element b of R_2, we have $ca = e_1^{\sharp c} ba = 0$ for any $a \in F$ and hence $e_*(c)a = 0$ for any $a \in F$, that is, $e_1^\sharp \leqq e_*(c)^c$. Thus, we get $e_*(c)e_1^\sharp = 0$. On the other hand, $ce_1^{\sharp c} = 0$ and so $e_*(c)e_1^\sharp = 0$. Hence, there holds $e_*(c) = 0$ and hence $c = 0$, that is $be_1^\sharp = e_1^\sharp b e_1^\sharp$. Since arbitrary elements of R_2 are generated by hermitian elements of R_2, this completes the proof of (2.13).

In the proof below, we shall use the fact that $e_1^\natural \in E_{01}$. This fact is nothing but (2.13)′ below and so the proof is omitted.

The proof of (2.15). Let $(e_{1\iota}; \iota \in I)$ be a system of projections of R_1. Then, from (2.10) and (2.13) it follows that $(\cup(e_{1\iota}; \iota \in I))^\sharp = \cup(e(a(\cup(e_{1\iota}; \iota \in I))); a \in A) = \cup(e(ae_{1\iota}); a \in A, \iota \in I) = \cup(e_{1\iota}^\sharp; \iota \in I)$.

The proof of (2.14). Since $e_1 \leqq e_1^\natural$, we have $e_1^\sharp \leqq e_1^{\natural\sharp}$. On the other hand, we have $e_1^{\natural\sharp} = \cup(e(a(\cup(e_*(e_1 b); b \in R_1)); a \in A) = \cup(e(a(\cup(e(b^* e_1); b \in R_1)); a \in A)) = \cup(e(a(\cup(e(be_1); b \in R_1)); a \in A)) = \cup(e(abe_1); b \in R_1, a \in A) \leqq e_1^\sharp$. Thus, we get (2.14).

The proof of (2.16). First we prove $e_1^{\sharp\sharp} \leqq e_1^\natural$. In fact, we have $e_1^{\sharp\sharp} = \cup(e(ae_1)^\sharp; a \in A) = \cup(e_*(e(ae_1)b); a, b \in A) = \cup(e(b^* a e_1); a, b \in A) \leqq e_1^\natural$. Next, we prove $ae_1^\natural e_1' = e_1^\sharp a e_1'$. In fact, from (2.14) and from the fact above, we have $ae_1^\natural e_1' = e_1^{\natural\sharp} a e_1^\natural e_1' = e^\sharp a e_1^\natural e_1' = e_1^\sharp a e_1^{\sharp\sharp} e_1^\natural e_1' = e_1^\sharp a e_1^{\sharp\sharp} e_1' = e_1^\sharp a e_1'$. Hence, we have $(e_1^\natural e_1')^\sharp = \cup(e(ae_1^\natural e_1'); a \in A) = \cup(e(e_1^\sharp a e_1'); a \in A) = e(e_1^\sharp(\cup(e(ae_1'); a \in A)) = e(e_1^\sharp e_1'^\sharp) = e_1^\sharp e_1'^\sharp$.

The proof of (2.17). First we prove $e_1^{\sharp\sharp\sharp} \leqq e_1^\sharp$. In fact, we have $e_1^{\sharp\sharp\sharp} \leqq e_1^{\natural\sharp} = e_1^\sharp$. Conversely, it holds that $e_1^{\sharp\sharp\sharp} a e_1 = (e_1^\sharp((e_1^{\sharp\sharp\sharp} a)e_1))e_1^{\sharp\sharp} = e_1^{\sharp\sharp\sharp}(e_1^\sharp(ae_1))e_1^{\sharp\sharp} = ae_1$. Hence, we have $e_1^\sharp \leqq e_1^{\sharp\sharp\sharp}$. Thus, we have $e_1^{\sharp\sharp\sharp} = e_1^\sharp$. If $e_1^\natural \leqq I_2^\sharp$, then it holds that $e_1^\natural = e_1^{\sharp\sharp}$, because, as we have $I_2^\sharp = I_2^{\sharp\sharp\sharp} = e_1^{\sharp\sharp} \oplus (I_2^\sharp e_1^{\natural c})^{\sharp\sharp}$, we must have equality sign in both of $e_1^\natural \geqq e_1^{\sharp\sharp}$, and $I_2^\sharp e_1^{\natural c} \geqq (I_2^\sharp e_1^{\natural c})^{\sharp\sharp}$. Conversely, if $e_1^\natural = e_1^{\sharp\sharp}$, then it holds that $e_1^\natural = (e_1^\sharp)^\sharp \leqq I_2^\sharp$.

The proof of (2.18). It holds that $e_2 A e_1 = 0$ if and only if $e_1^\sharp \leqq e_2^c$. Hence, we have $e_1^\sharp R_2 e_2 = 0$. Thus, we obtain $e_1^\sharp e_2 = 0$ if and only if $e_2 A e_1 = 0$. q. e. d.

In what follows, it may happen to use $\sharp$ simply without specifying A, R_1, R_2. In that case, we are supposing that $R=R_1=R_2=A$, and, by virtue of the * operation, $\natural$ and $\sharp$ are identified. As an immediate consequence of Prop. 2.6, we have

PROPOSITION 2.7. *In an AW^*-algebra R, whose center is denoted by R_0, it holds that*

(2.13)′ $e^{\natural}\in R_0$,

(2.14)′ $e^{\natural\natural}=e^{\natural}$,

(2.15)′ $(\bigcup(e_\iota;\iota\in I))^{\natural}=\bigcup(e_\iota^{\natural};\iota\in I)$,

(2.16)′ $(e^{\natural}\cap e')^{\natural}=e^{\natural}\cap e'^{\natural}$,

(2.18)′ $e_2Re_1=0$ *if and only if* $e_1^{\natural}e_2^{\natural}=0$.

For an element a of R, we denote $e(a)^{\natural}$ by $e_0(a)$. It is easy to see that $e_0(a)$ is the minimal projection of R_0 fixing a.

Since the mapping $e_1^{\natural}\rightarrow e_1^{\sharp}$ is an isomorphism from $I_2^{\sharp}E_{01}$ onto $I_1^{\sharp}E_{02}$ as structure of Boolean lattices, it is extended to an algebraic isomorphism from $I_2^{\sharp}R_{01}$ onto $I_1^{\sharp}R_{02}$, which we denote also by $\sharp$. (Here, we say that φ is an algebraic isomorphsm from an AW^*-algebra R_1 onto an AW^*-algebra R_2 if φ is a one-to-one mapping from R_1 onto R_2 with following conditions: (1) $\varphi(a_1+a_1')=\varphi(a_1)+\varphi(a_1')$, (2) $\varphi(a_1a_1')=\varphi(a_1)\varphi(a_1')$, and (3) $\varphi(a_1^*)=\varphi(a_1)^*$ (for any $a_1, a_1'\in R_1$).)

PROPOSITION 2.8. *With the same notation as before, let φ_0 be an algebraic isomorphism from R_{01} onto R_{02} and suppose that A satisfies the property $\varphi_0(c_{01})a=ac_{01}$ for any $a\in A$ and any $c_{01}\in R_{01}$. Then, we have $\varphi_0(c_{01})^{\sharp}=c_{01}$ for any $c_{01}\in I_2^{\sharp}R_{01}$. Moreover, if $\varphi_0(I_2^{\sharp})=I_1^{\sharp}$, it holds that $\varphi_0(c_{01})=c_{01}^{\sharp}$ for any $c_{01}\in R_{01}$.*

PROOF. Since $\varphi_0(e_{01})ae_{01}=ae_{01}$ for any $a\in A$ and $e_{01}\in E_{01}$, it holds that $e_{01}^{\sharp}\leqq\varphi_0(e_{01})$ and so $e_{01}^{\sharp\sharp}\leqq\varphi_0(e_{01})^{\sharp}$. Moreover, from $\varphi_0(e_{01})a=\varphi_0(e_{01})ae_{01}$ it follows that $\varphi_0(e_{01})^{\sharp}\leqq e_{01}$. Thus, we have $\varphi_c(e_{01})^{\sharp}=e_{01}$ if $e_{01}\in I_2^{\sharp}E_{01}$ and hence $\varphi_0(c_{01})^{\sharp}$ c_{01} if $c_{01}\in I_2^{\sharp}R_{01}$. q. e. d.

§ 3. Local relative dimension.

Again, let R be an AW^*-algebra. Hereafter, throughout this paper, a local property with respect to a spectre of R is called briefly a local property, if the spectre is considered as given once for all. In this case we use also the term "locally" in the corresponding sense. A global property is simply called a property.

In order to apply results in § 1 to AW^*-algebras, we must prove the following

PROPOSITION 3.1. *R satisfies* (1.6) *and* (1.7).

PROOF. The proof of (1.6). For an hermitian element h of R_0, we denote by $(e_\alpha(h); -\infty<\alpha<\infty)$ the resolution of the unit of h. Then, from the definition itself, h is the limit of linear combinations of $e_\alpha(h)$, and, as we saw in the proof of Prop. 2.2, $e_\alpha(h)\in E_0$. We thus see that h is in the subalgebra generated by E_0. As regards to a general element of R_0, being able to be written as a linear combination of two hermitian elements of R_0, it is also in that named subalgebra. This shows (1.6).

The proof of (1.7). Let $\mathfrak{a}=(a_\iota; \iota\in I)$ be a system of elements of R. Then, by (2.11), we get $E_0(\mathfrak{a},(\phi))=(\cup((e_0(a_\iota); \iota\in I))^c E_0$. Hence, (ϕ) is normal. q. e. d.

First, we introduce some concepts.

Definition 3.1. 1) *Two projections $e_i (i=1,2)$ are called equivalent to each other and denoted by $e_1\sim e_2$ if there exists a partial isometry u of R with $e_*(u)=e_1$ and $e(u)=e_2$. Here, we say that an element u of R is a partial isometry if u^*u is a projection of R. In this case, uu^* is also a projection of R.*

2) *A projection e of R is called finite if $e=e_1$ follows from $e\sim e_1$ and $e\geqq e_1$.*

3) *A projection e of R is called infinite if it is not finite.*

4) *A projection e of R is called irreducible if $e_1=e_2=0$ follows from $e\geqq e_1\oplus e_2$ and $e_1\sim e_2$.*

These properties are normal properties except for infiniteness (as this is easily seen from Lemma 3.1 below for 1) and is clear for 2), 4)) and are properties concerning with one projection of R except for equivalence, which concerns with two projections of R. We call negation of finiteness *the normal infiniteness.*

We denote by U the set of partial isometries of R. We write $u_1\leqq_1 u_2$ for $u_1, u_2\in U$ if u_2-u_1 is a partial isometry orthogonal to u_1. Then, it is easy to see that U forms a semi-ordered set.

The first aim of this § is to prove a theorem of I. Kaplansky [6] (cf. Prop. 3.5). By the local consideration, his proof will be slightly shortened. We denote by (L_0) the property concerning with an orthogonal system $\mathfrak{a}=(u_\iota; \iota\in I)$ of elements of U, which $\mathfrak{a}$ has if and only if $\mathfrak{a}$ has a supremum in U.

Lemma 3.1. *(L_0) is normal.*

Proof. If each system $\mathfrak{a}_\iota$ of partial isometries has a supremum u_ι in U and if $(u_\iota; \iota\in I)$ has a supremum u in U, then u is the supremum of $(v; v\in\mathfrak{a}_\iota$ for some $\iota\in I)$ in U. Hence, we need only to prove that an orthogonal system $\mathfrak{a}=(u_\iota; \iota\in I)$ of elements of U has a supremum in U, if $(e_0(u_\iota); \iota\in I)$ is an orthogonal system of projections of R_0. We denote by A a maximal commutative subalgebra of R containing $(u_\iota^*+u_\iota; \iota\in I)$ and denote by $(e_{\alpha,\iota}; -\infty<\alpha<\infty)$ the resolution of the unit associated to $u_\iota^*+u_\iota$. Putting $e_\alpha=\sup(e_{\alpha,\iota}; \iota\in I)$, the system $(e_\alpha; -\infty<\alpha<\infty)$ forms the resolution of the unit associated to some hermitian element of A. We denote it by $[u^*+u]$. Then, we have $e_0(u_\iota)[u^*+u]=u_\iota^*+u_\iota$. Similarly, we may find an element $[u^*-u]$ of N with $e_0(u_\iota)[u^*-u]=u_\iota^*-u_\iota$. We put $u=\frac{1}{2}([u^*+u]-[u^*-u])(\oplus(e_0(u_\iota);\iota\in I))$. Then, it is easy to see that u is the supremum of $(u_\iota;\iota\in I)$ in U. q. e. d.

The following lemma is due to [6]. But the present proof need less calculation.

Lemma 3.2. *Let $(u_\iota;\iota\in I)$ be an orthogonal system of elements of U, whose initial projection $e_*((u_\iota;\iota\in I))$ and final one $e((u_\iota;\iota\in I))$ are orthogonal to each other. Then, $(u_\iota;\iota\in I)$ has a supremum in U.*

Proof. We denote $e_*((u_\iota;\iota\in I))$ by e_* and $e((u_\iota;\iota\in I))$ by e. Since $u_\iota^*+u_\iota$ is hermitian and unitary in $(e_*(u_\iota)\oplus e(u_\iota))R(e_*(u_\iota)\oplus e(u_\iota))$, we have $u_\iota^*+u_\iota=e_\iota'-e_\iota''$, where e_ι', e_ι'' are mutually orthogonal projections of $(e_*(u_\iota)\oplus e(u_\iota))R(e_*(u_\iota)\oplus e(u_\iota))$. Putting $e'=\oplus(e_\iota';\iota\in I)$, $e''=\oplus(e_\iota'';\iota\in I)$, and $u=e(e'-e'')$, we have $e(u_\iota)u=e(u_\iota)(e_*(u_\iota)\oplus e(u_\iota))(e'-e'')=e(u_\iota)(e_\iota'-e_\iota'')=e(u_\iota)(u_\iota^*+u_\iota)=e(u_\iota)u_\iota=u_\iota$ and similarly we have $ue_*(u_\iota)=u_\iota$. Further we get $uu^*=e(e'-e'')^2e=e(e_*\oplus e)=e$ and $u^*u=(e'-e'')e(e'-e'')=\oplus(e_*(u_\iota);\iota\in I)=e_*$. q. e. d.

Proposition 3.1. *For two projections e_1, e_2 of R, there exists a non-zero element u of U with $e_*(u)\leqq e_1$ and $e(u)\leqq e_2$ if and only if it holds $e_1^\natural e_2^\natural\neq 0$.*

Proof. Necessity. Since $e_2Re_1\neq 0$, we get $e_1^\natural e_2^\natural\neq 0$ by (2.18)′. Sufficiency. By (2.18), we may find a non-zero element a of R with $e_*(a)\leqq e_1$ and $e(a)\leqq e_2$. Let $(e_\alpha;0\leqq\alpha<\infty)$ be the resolution of the unit of a^*a. We denote by k_α the inverse of $a^*ae_\alpha^c$ in $e_\alpha^cRe_\alpha^c$ for a sufficiently small positive number α. Then, $u=ak_\alpha^{\frac{1}{2}}$ is a non-zero element of U with $e_*(u)\leqq e_*(a)$ and $e(u)\leqq e(a)$. Thus, we get the assertion. q. e. d.

Proposition 3.2. *The following statements are mutually equivalent: (1) e is normally infinite, (2) $e=\oplus(e_n;1\leqq n<\infty)$ with $e_1\sim e_n$ $(1\leqq n<\infty)$ for some e_n of E, and, (3) $e\sim e'\sim e'^c$ for some e' of E.*

Proof. (1) implies (2). Since e is normally infinite, we may find a projection e' satisfying that $e'\leqq e$ and $e\neq e'\sim e$. Then, by the well known way, we may find a decomposition $\oplus(e_n';1\leqq n<\infty)\leqq e$ with $e_1'\sim e_n'$ $(1\leqq n<\infty)$. Hence, we may find a maximal orthogonal system $(e_\iota';\iota\in I)$ of projections of R with $e_1'\sim e_\iota'(\iota\in I)$ containing $(e_n';1\leqq n<\infty)$. Furthermore, using Lemma 3.2, we may find a maximal pair (e', e'') of projections of R with $e_1'\geqq e'\sim e''\leqq(\oplus(e_\iota';\iota\in I))^c$. Then, we get $(e_1'-e')^\natural((\oplus(e_\iota';\iota\in I))^c-e'')^\natural=0$ by Prop. 3.1. Since $(e_1'-e')^\natural\neq 0$, there exists at least one spectre λ of R, with respect to which it holds that $e=\oplus(e_\iota'';\iota\in I)$ with $e_1\sim e_\iota''$ $(\iota\in I)$ locally by the well known way. Since the cardinal number of I is greater than $\aleph_0$ we find a decomposition (2) with respect to λ locally. The property that a projection decomposes into the sum of a countable orthogonal system consisting of mutually equivalent projections being normal, one easily sees the validity of (2) from this.

(2) implies (3). We may use the index (n, m) instead of n and a decomposition $e=\bigoplus(e_{n,m}; 1\leqq n, m<\infty)$ instead of (2). We write $e_n=\bigoplus(e_{n,m}; 1\leqq m<\infty)$. Applying Lemma 3.2 to $e_{2n}-1\sim e_{2n}$ $(1\leqq n<\infty)$. we have a partial isometry u of R with $e_*(u)=\bigoplus(e_{2n-1}; 1\leqq n<\infty)$ and $e(u)=\bigoplus(e_{2n}; 1\leqq n<\infty)$. Similarly, we have a partial isometry v of R with $e_*(v)=\bigoplus(e_{2n}; 1\leqq n<\infty)$ and $e(v)=\bigoplus(e_{2n+1}; 1\leqq n<\infty)$. We put $w=u+v$. Then, w is a partial isometry of R with $e_*(w)=e$ and $e(w)=e_1{}^c$. Moreover, it holds $e_1\sim e_1{}^c$ by Lemma 3.2.

(3) implies (1). Let $e_0(\lambda)$ be a projection of $E_0(\lambda)$. Since $e\sim e'^c$, it holds that $(e_0(\lambda)e'^c)^\natural=e_0(\lambda)e'^{c\natural}=e_0(\lambda)e^\natural$. Hence, if $e_0(\lambda)e^\natural\neq 0$, we get $e_0(\lambda)e\neq e_0(\lambda)e'\sim e_0(\lambda)e$, that is, $e_0(\lambda)e$ not being locally finite. This means that e is locally zero or not locally finite. Hence, e is locally normally infinite with respect to any spectre λ of R. Thus, e is normally infinite by the definition of normal infiniteness.

We say that R is *discrete, finite*, and *normally infinite* if 1 is discrete, finite, and normally infinite respectively, where a projection e_0 of R_0 is called *discrete* if there exists an irreducible projection e of E with $e^\natural=e_0$. We call the negation of discreteness the non-discreteness. These properties are obviously normal properties concerning one element of R, namely the unit 1 of R. Hence, we can say about these local properties.

Definition 3.2. *A projection e of R is called simple of order n if there exists a decomposition $e^\natural=\bigoplus(e_\nu; 1\leqq\nu\leqq n)$ with $e\sim e_\nu$ $(1\leqq\nu\leqq n)$.*

This property is also normal.

Lemma 3.3. *There exists a unique decomposition*

(3.1) $1=\bigoplus(e_0(I_n); 1\leqq n<\infty)\bigoplus e_0(II_1)\bigoplus e_0(\infty)$, *where $e_0(I_n)$, $e_0(II_1)$, and $e_0(\infty)$ are the uniquely determined maximal projections of R_0 among the projections e_0's of R_0 with a simple irreducible projection e of order n satisfying $e^\natural=e_0$, with finiteness and non-discreteness, and with normal infiniteness respectively.*

Proof. It is easy to see that there exist the maximal projections $e_0(I_n)$ $(1\leqq n<\infty)$, $e_0(II_1)$, $e_0(\infty)$ in question and that they are orthogonal to each other. We put $e_0=(\bigoplus(e_0(I_n); 1\leqq n<\infty)\bigoplus e_0(II_1)\bigoplus e_0(\infty))^c$. Then, e_0 is finite and discrete. We shall prove that, if $e_0\neq 0, e_0$ must be 0. From this, we can conclude the assertion.

Now, assume that $e_0\neq 0$. Since e_0 is discrete, we may find a non-zero irreducible projection e_1 of R with $e_1{}^\natural=e_0$. Then, there exists a maximal orthogonal system $(e_\iota; \iota\in I)$ of projections of R containing e_1 and satisfying $e_1\sim e_\iota$ $(\iota\in I)$. If the cardinal number of I is greater than $\aleph_0$, we see from Prop. 3.2 that, for any countable subsystem $(e_n; 1\leqq n<\infty)$ of $(e_\iota; \iota\in I)$, $\bigoplus(e_n; 1\leqq n<\infty)$ is normally infinite. Since $(\bigoplus(e_n; 1\leqq n<\infty))^\natural=e_1{}^\natural=e_0$, e_0 is normally infinite. This is a contradiction. Hence, I is a finite set (say

$I=(\nu; 1\leqq\nu\leqq n)$. We put $e^{(1)}=e_0(\oplus(e_\nu; 1\leqq\nu\leqq n))^c$. Then, we may find a maximal pair (e', e'') of projections of R with $e_1\geqq e'\sim e''\leqq e^{(1)}$. By Prop. 3.1, we have $(e_1e'^c)^\natural(e^{(1)}e''^c)^\natural=0$. Since $(e_1e'^c)^\natural\neq 0$, we have $e_1\geqq e'\sim e^{(1)}$ locally. Since e_1 is irreducible, we have $(e'^\natural e_1e'^c)^\natural=e'^\natural(e_1e'^c)^\natural=0$, that is, $e'^\natural e_1e'^c=0$. Hence, we get $e^{(1)\natural}e_1=e'^\natural e_1=e'$. Thus, we obtain $e^{(1)\natural}=\oplus(e^{(1)\natural}e_\nu; 1\leqq\nu\leqq n)\oplus e^{(1)}$ with $e_1\sim e^{(1)}$ locally. This means that $e^{(1)\natural}\leqq_\lambda e_0\cap e_0(I_{n+1})=0$. Hence, we get $e^{(1)}=_\lambda 0$, that is, $e_0=_\lambda\oplus(e_\nu; 1\leqq\nu\leqq n)$ and so $e_0\leqq_\lambda e_0(I_n)$. Since e_0 is locally orthogonal to $e_0(I_n)$, e_0 must be locally zero. From this it follows that e_0 is zero by Prop. 1.1. q. e. d.

LEMMA 3.4. *If R is locally normally infinite, then an orthogonal system $\mathfrak{a}$ of elements of U has (L_0).*

PROOF. We may assume without loss of generality that R is normally infinite. By Prop. 3.2, we may find a projection e of R with $1\sim e\sim e^c$. Hence, there exist a partial isometry u of R with $e_*(u)=1, e(u)=e$ and a partial isometry v of R with $e_*(v)=e, e(v)=e^c$. By Lemma 3.2, $vu\mathfrak{a}u^*=(vua_\iota u^*; \iota\in I)$ has (L_0). By multiplying u^*v^* from the left and u from the right, we see that $\mathfrak{a}$ has (L_0). q. e. d.

We denote by "$a=_\lambda b$" the local notion of "$a=b$" and similarly by "$a\geqq_\lambda b$" that of "$a\geqq b-\varepsilon$ for any positive number ε". Moreover, we denote by $e_1\sim_\lambda e_2$ the local notion of $e_1\sim e_2$.

We say that an element a of R is *locally non-zero* (denoted by $a\neq_\lambda 0$) if it is not locally zero and that a projection e of R is *locally minimal* if e is locally non-zero and if $e=_\lambda e_1$ follows from $e\geqq_\lambda e_1$ (e_1 being locally non-zero). Though these properties are never local properties, we shall use such terminologies for convinience. Denoting by (L^λ) one of them, (L^λ) satisfies that $1\in E_0(a, (L^\lambda))$ is equivalent to $E_0(a, (L^\lambda))=E_0(\lambda)$.

PROPOSITION 3.3. *As to the locally irreducble projections, we have* (1) *any locally non-zero irreducible projection is locally minimal,* (2) *any locally non-zero projection contains a locally non-zero irreducible projection if R is locally discrete, and* (3) *locally non-zero irreducible projections of R are mutually locally equivalent.*

PROOF. (1) Let e be a locally non-zero irreducible projection of R and suppose that $e\geqq_\lambda e'$ holds for a locally non-zero projection e' of R. Then, we have $e=_\lambda e'\oplus ee'^c$, $e'^\natural e'^{c\natural}=0$ and hence $e=_\lambda e'^\natural e=_\lambda e'$. (2) Let e' be a locally non-zero projection of R and e be a locally non-zero irreducible projection of R. Then, there exists a maximal pair (e_1, e_1') of projections of R with $e'\geqq e_1'\sim e_1\leqq e$ by Lemma 3.1. By Prop. 3.1, it holds that $(e'e_1'^c)^\natural(ee_1^c)^\natural=0$. If $\lambda((e'e_1'^c)^\natural)=0$, we have $e'=_\lambda e_1'\sim e_1\leqq e$ and $e_1^\natural=_\lambda e'^\natural\in E_0(\lambda)$. Thus, we have $e'\sim_\lambda e_1=_\lambda e$. On the other hand, if $\lambda((e'e_1'^c)^\natural)=1$, we have $\lambda((ee_1^c)^\natural)=0$ and so $e=_\lambda(e'e_1'^c)^\natural e=(e_1e_1'^c)^\natural e_1^\natural e=(e'e_1'^c)^\natural e_1\sim(e'e_1'^c)^\natural e_1'\leqq e_1$. The fact (3) follows immediately from (1) and (2). q. e. d.

LEMMA 3.5. *If R is locally finite and has a locally simple irreducble projection of order n, then an orthogonal system $\mathfrak{a}$ of partial isometries of R has (L_0).*

PROOF. We may assume without loss of generality that R is finite and has a simple irreducible projection of order n. Let P be the set of projections $e_0e_*(u_\iota)$'s of R for arbitrary $e_0 \in E_0$ and $\iota \in I$. We denote e_0u_ι by $\rho(e_0e_*(u_\iota))$. Then, there exists a maximal orthogonal system P_1 of projections of P satisfying that $(e^\natural; e \in P_1)$ is an orthogonal system of projections of R_0. Similarly, by induction, we may find a maximal orthogonal system P_ν of projections of P satisfying that $(e^\natural; e \in P_\nu)$ is an orthogonal system of projections of R_0 and that P_ν is orthogonal to P_μ $(1 \leqq \mu < \nu)$. Then, $(\rho(e); e \in P_\nu)$ has (L_0) by Lemma 3.1. We denote by $e_0^{(\nu)}$ the supremum of $(e^\natural; e \in P_\nu)$ and by $e^{(\nu)}$ that of P_ν. Then, it holds that $e_0^{(\nu)} \geqq e_0^{(\nu+1)}$.

We shall prove that $e_0^{(n+1)} = 0$. For otherwise, we can take a spectre λ of R with respect to which $e^{(\nu)} =_\lambda e_0^{(n+1)} \neq_\lambda 0$. As $e^{(\nu)}$'s are mutually orthogonal and $e^{(\nu)\natural} = e_0^{(\nu)}$ and R is discrete, we see from Prop. 3.3 that R has no locally simple irreducible projection of order n with respect to λ, and this contradicts with the assumption by virtue of Prop. 1.1. Hence, we have $e_0^{(n+1)} = 0$. Consequently, we may assume without loss of generality that I is a finite set. Then, $\sum (u_\iota; \iota \in I)$ is the supremum of $\mathfrak{a}$. q. e. d.

The following proposition is due to J. Dixmier [5].

PROPOSITION 3.4. *If R is non-discrete, there exists two projections $e^{(1)}$ and $e^{(2)}$ of R satisfying $e = e^{(1)} \oplus e^{(2)}$ and $e^{(1)} \sim e^{(2)}$ for any projection e of R.*

PROOF. There exists a maximal pair $(e^{(1)}, e^{(2)})$ of projections satisfying $e^{(1)} \sim e^{(2)}$ and $e = e^{(1)} \oplus e^{(2)} \oplus e^{(3)}$ for some projection $e^{(3)}$ of R. Hence, $e^{(3)}$ is irreducible and must be 0. q. e. d.

LEMMA 3.6. *If R is locally finite and locally non-discrete, then an orthogonal system $\mathfrak{a}$ of partial isometries of R has (L_0).*

PROOF. We may assume without loss of generality that R is finite and non-discrete. First we shall prove that two simple projections e_i $(i=1,2)$ of order 2 with $e_i^\natural = 1$ are equivalent to each other. By Prop. 3.1, we may find a pair (e_1', e_2') of non-zero projections of R with $e_1 \geqq e_1' \sim e_2' \leqq e_2$. Since $e_1' \neq 0$, there exists a spectre λ of R, with respect to which it holds that $e_1' \geqq e_1''$ for some locally simple projection e_1'' of order 2^r $(r > 1)$ (cf. Lemma 4.2.). Since $e_1' \sim e_2'$, we may find a projection e_2'' of R with $e_1'' \sim e_2'' \leqq e_2'$. We shall prove that e_2'' is also locally simple of order 2^r. There exists a maximal pair (e', e'') of projections of R satisfying $e_2''^c \geqq e' \sim e'' \leqq e_2''$ by lemma 3.2. If $e_2''^c e'^c$ is locally non-zero, we can repeat this process. Thus, we get a decomposition $1 =_\lambda \oplus (e_2''^{(\nu)}; 1 \leqq \nu \leqq n+1)$, where $e_2'' = e_2''^{(1)} \sim_\lambda e_2''^{(\nu)}$ $(1 \leqq \nu \leqq n)$ and $e_2''^{(n+1)} \sim_\lambda e''' \leqq e_2''$ for some $e''' \in E$. Hence, multiplying some projection $e_0(\lambda)$ of $E_0(\lambda)$, we obtain a decomposition $e_0(\lambda) = \oplus (e_0(\lambda) e_2''^{(\nu)};$

$1 \leqq \nu \leqq n+1$), where $e_0(\lambda)e_2'' = e_0(\lambda)e_2''^{(1)} \sim e_0(\lambda)e_0''^{(\nu)}$ ($1 \leqq \nu \leqq n$) and $e_0(\lambda)e_2''^{(n+1)} \sim e_0^{(\nu)}e''' \leqq e_0(\lambda)e_2''$. Since e_0Re_0 is finite, we have, by the well known method, $n=2^r$ and $e_2''^{(n+1)} =_\lambda 0$. (On the same time, by the same method, we can prove that above local decomposition is possible.) Thus, e_2'' is locally simple of order 2^r. Here, we notice that "$e_2'' \leqq 1$" is trivial, but this fact is an essential part of above proof. Therefore, remembering that $e_i'' \leqq e_i$ ($i=1,2$), we may find a decomposition $e_i = \bigoplus(e_i''^{(\nu)}; 1 \leqq \nu \leqq 2^{r-1})$, where $e_i'' = e_i''^{(1)} \sim e_i''^{(\nu)}$ ($1 \leqq \nu \leqq 2^{r-1}$). Since $e_i''^{(\nu)}$ ($1 \leqq \nu \leqq 2^{r-1}, i=1,2$) are mutually locally equivalent to each other, we can easily conclude that e_1 is locally equivalent to e_2. Since equivalence is normal, e_1 is equivalent to e_2.

Let $\mathfrak{a}$ $(=(u_\iota; \iota \in I))$ be an orthogonal system of elements of U. For each $\iota \in I$, we may find a decomposition $e(u_\iota) = e_\iota^{(1)} \oplus e_\iota^{(2)}$ with $e_\iota^{(1)} \sim e_\iota^{(2)}$. We denote by $\mathfrak{a}^{(1)}$ the orthogonal system $(e_\iota^{(1)}u_\iota; \iota \in I)$ of elements of U. We need only to see that $\mathfrak{a}^{(1)}$ has (L_0). Then, we can find two simple projections e_1, e_2 of order 2 of R with $\bigoplus(e_*(e_\iota^{(1)}u_\iota); \iota \in I) \leqq e_1$ and $\bigoplus(e(e_\iota^{(1)}u_\iota); \iota \in I) \leqq e_2$. Since e_1, e_1^c, e_2 are simple projections of order 2, we may find partial isometries u, v of R such that $e_*(u)=e_1$, $e(u)=e_1^c$, $e_*(v)=e_2$, and $e(v)=e_1$. Applying Lemma 3.2 to $uv\mathfrak{a}^{(1)} = (uve^{(1)}u_\iota; \iota \in I)$, $uv\mathfrak{a}^{(1)}$ has the supremum w in U. Hence, $\mathfrak{a}^{(1)}$ has the spremum. q. e. d.

Under these preparations, we shall prove the following important proposition already discovered and proved by I. Kaplansky [6], which plays an essential role in AW^*-algebras.

PROPOSITION 3.5. *Any orthogonal system of elements of U has a supremum in U.*

PROOF. Since an orthogonal system $\mathfrak{a}$ of elements of U has (L_0) with respect to any λ according to Lemmas 3.3–3.6, we see that $\mathfrak{a}$ has (L_0) by virtue of Prop. 1.1. q. e. d.

COROLLARY. *Any chain of elements of U has a supremum in U.*

PROOF. Let $(u_\iota; \iota \in I)$ be a chain of elements of U. We denote by E_1 a maximal commutative system of projections of R containing $(e(u_\iota); \iota \in I)$ and by U_1 the system $(e_1u_\iota; e_1 \in E_1, \iota \in I)$. Moreover, we denote by U_2 a maximal orthogonal system of elements of U_1. Then, there exists the supremum u of U_2 in U by Prop. 3.5. It is easily seen that u is the supremum of $(u_\iota; \iota \in I)$. q. e. d.

Recently, Ti. Yen [8] has given the proof of the canonical decomposition theorem in AW^*-algebras (cf. Lemma 2.1 and its Corollary, [8]). The following proposition contains the results of Ti. Yen [8].

PROPOSITION 3.6. *For any element a of R, we may find a unique system* $(u_\alpha(a); 0 \leqq \alpha < \infty)$ *of elements of U satisfying* (1) $u_0=0$, $u_\alpha = const. \in U$ *for* $\alpha_0 \leqq \alpha$, (2) $\alpha \leqq \beta$ *implies* $u_\alpha \leqq_1 u_\beta$; *and* (3) $\lim_{\beta \downarrow \alpha} u_\beta = u_\alpha$ *and further*

(3.2) $a=\int_{-\infty}^{\infty}\alpha du_{\alpha}$,

and, from these it follows that

(3.3) $e_{*}(a)=\int_{-\infty}^{\infty}de_{*}(u_{\alpha})$,

(3.4) $e(a)=\int_{-\infty}^{\infty}de(u_{\alpha})$,

(3.5) $a=uh$, *where* $h=(a^{*}a)^{\frac{1}{2}}$, *and* $u=\int_{0}^{\infty}du_{\alpha}$ *(the canonical decomposition of* a*), and that*

(3.6) $a^{*}a=u^{*}aa^{*}u$.

Proof. The proof of (3.2). Let $(e_{\alpha};\ 0\leqq\alpha\leqq<\infty)$ be the resolution of the unit of $(a^{*}a)^{\frac{1}{2}}$. We denote by k_{α} the inverse of $a^{*}ae_{\alpha}{}^{c}$ in $e_{\alpha}{}^{c}Re_{\alpha}{}^{c}$ for $0<\alpha<\|a\|$. We put $u_{\alpha}{}^{c}=ak_{\alpha}^{\frac{1}{2}}$ and $h_{\alpha}=(a^{*}a)^{\frac{1}{2}}e_{\alpha}{}^{c}$. Then, we have $ae_{\alpha}{}^{c}=u_{\alpha}{}^{c}h_{\alpha}$. Since $(u_{\alpha}{}^{c};\ 0<\alpha<\|a\|)$ is a chain of elements of U, we may find its supremum u in U by the Corollary of Prop. 3.5 and hence we have $ae_{\varepsilon}{}^{c}=uh_{\varepsilon}=u\int_{\varepsilon}^{\infty}\alpha de_{\alpha}=\int_{\varepsilon}^{\infty}\alpha due_{\alpha}$. We put $u_{\alpha}=ue_{\alpha}$ for $\alpha>0$ and $u_{0}=\lim_{\alpha\downarrow 0}u_{\alpha}=0$. It is easy to see that $(u_{\alpha};\ 0\leqq\alpha<\infty)$ satisfies (1)-(3). Making $\varepsilon\downarrow 0$, we have (3.2) from $ae_{\varepsilon}{}^{c}=\int_{\varepsilon}^{\infty}\alpha du_{\alpha}$.

The proof of uniqueness. Let $(u_{\alpha}{}';\ 0\leqq\alpha<\infty)$ be another system of elements of U satisfying (1)-(3) and (3.2). From $a^{*}a=\int_{0}^{\infty}\alpha^{2}du_{\alpha}{}^{*}u_{\alpha}=\int_{0}^{\infty}\alpha^{2}du_{\alpha}{}'^{*}u_{\alpha}{}'$, it is easy to compute that $u_{\alpha}{}^{*}u_{\alpha}=u_{\alpha}{}'^{*}u_{\alpha}{}'$ for $0\leqq\alpha<\infty$. Hence, we have $u(u_{\alpha}{}^{*}u_{\alpha})^{c}=a\int_{\alpha}^{\infty}\alpha^{-1}du_{\alpha}{}^{*}u_{\alpha}=a\int_{\alpha}^{\infty}\alpha^{-1}du_{\alpha}{}'^{*}u_{\alpha}{}'=u'(u_{\alpha}{}'^{*}u_{\alpha}{}')^{c}$, where we put $u'=\int_{0}^{\infty}du_{\alpha}{}'$. Making $\alpha\downarrow 0$, we see that $u=u'$ and so $u_{\alpha}{}'=u'-u'(u_{\alpha}{}'^{*}u_{\alpha}{}')^{c}=u-u(u_{\alpha}{}^{*}u_{\alpha})^{c}=u_{\alpha}$.

The proof of (3.3) and (3.4). To see (3.3), we have $e_{*}(a)=e_{*}(a^{*}a)=e_{*}(\int_{0}^{\infty}\alpha^{2}de_{*}(u_{\alpha}))=\int_{0}^{\infty}de_{*}(u_{\alpha})$. Thus, we have (3.3). Similarly, we get (3.4).

The proof of (3.5) and (3.6). To see (3.5), we have $uh=\int_{0}^{\infty}du_{\alpha}\int_{0}^{\infty}\alpha de_{*}(u_{\alpha})=\int_{0}^{\infty}\alpha du_{\alpha}=a$. Moreover, we have $u^{*}aa^{*}u=\int_{0}^{\infty}\alpha^{2}du^{*}u_{\alpha}u_{\alpha}{}^{*}u=\int_{0}^{\infty}\alpha^{2}du_{\alpha}{}^{*}u_{\alpha}=a^{*}a$, because $u^{*}u_{\alpha}u_{\alpha}{}^{*}u=u^{*}ue_{\alpha}u^{*}u=e_{\alpha}e_{0}{}^{c}=u_{\alpha}{}^{*}u_{\alpha}$. q. e. d.

Corollary 1. *It holds that* $e_{1}\sim e_{2}$ *if and only if there exists an element* a *of* R *with* $e_{1}=e_{*}(a)$ *and* $e_{2}=e(a)$.

Proof. We need only to see the sufficiency. By (3.3) and by (3.4), we have $e_{*}(a)=e_{*}(u)$ and $e(a)=e(u)$. q. e. d.

Corollary 2. *For any equi-bounded orthogonal system* $(a_{\iota};\ \iota\in I)$ *of elements of* R*, there exists an element* a *of* R *satisfying* $e(a_{\iota})a=ae_{*}(a_{\iota})=a_{\iota}$ *for any* $\iota\in I$.

Proof. We denote by $(u_{\alpha}(a_{\iota});\ 0\leqq\alpha<\infty)$ the system of elements of U for a_{ι} satisfying (1)-(3) and (3.2). Since $(a_{\iota};\ \iota\in I)$ is an orthogonal system of elements of $R, (u_\alpha(a_\iota); \iota\in I)$ is that of elements of U. We denote its supremum in U by u_α. Then, $(u_\alpha; 0\leqq\alpha<\infty)$ satisfies (1)-(3). We put $a=\int_0^\infty \alpha du_\alpha$. It is easy to see that it holds that $e(a_\iota)a=ae_*(a_\iota)=a_\iota$ for any $\iota\in I$. q. e. d.

The element a is called *the direct sum* of $(a_\iota; \iota\in I)$.

After F. J. Murray-J. v. Neumann [1], J. Dixmier [5], and I. Kaplansky [6], we shall introduce the following

DEFINITION 3.3. *A mapping d, which carries E onto some semi-ordered set, satisfying the following condition:*

(3.7) $d(e_1)\leqq d(e_2)$ *if and only if* $e_1\sim e_2'\leqq e_2$ *for some* e_2' *of* E,

is called a relative dimension of R.

By virtue of the complete additivity of equivalence, it is easy to verify that there exists one and only one relative dimension in R (except for isomorphism of the semi-ordered set as structure of semi-ordered set). We denote it by d.

DEFINITION 3.4. *A mapping* d_λ, *which carries E onto some semi-ordered set and satisfies the following condition:*

(3.8) $d_\lambda(e_1)\leqq d_\lambda(e_2)$ *if and only if* $e_1\sim_\lambda e_2'\leqq e_2$ *for some* e_2' *of* E,

is called a local relative dimension of R, where $e_1\sim_\lambda e_2'$ *means that* e_1 *is locally equivalent to* e_2'.

It is easy to see that there exists one and only one local relative dimension in R (except for isomorphism of the semi-ordered set as structure of semi-ordered set). We denote it by d_λ.

As to d_λ, we have the following important

PROPOSITION 3.7. *The semi-order of* $d_\lambda(E)$ *introduced by* d_λ *is linealy ordered.*

PROOF. For any two projections e_i $(i=1,2)$ of R, we may find a maximal pair (e_1', e_2') of projections of R with $e_1\geqq e_1'\sim e_2'\leqq e_2$. By Prop. 3.2, we have $(e_1e_1'^c)^\natural(e_2e_2'^c)^\natural=0$. Hence, it holds either $e_1e_1'^c=_\lambda 0$ or $e_2e_2^c=_\lambda 0$. q. e. d.

The property $d_\lambda(e_1)\leqq d_\lambda(e_2)$ concerned with a pair (e_1, e_2) of projections is the local property of the property $d(e_1)\leqq d(e_2)$ concerned with a pair (e_1, e_2) of projections. Hence, we have $d(e_1)\leqq d(e_2)$ if and only if $d_\lambda(e_1)\leqq d_\lambda(e_2)$ with respect to any spectre λ of R by Prop. 1.1.

PROPOSITION 3.8. (1) *A projection e of R is locally finite if and only if* $e_1=_\lambda e$ *follows from* $e_1\leqq_\lambda e$ *and* $e_1\sim_\lambda e$.

(2) *A projection e of R is locally normally infinite if and only if there exists a decomposition* $e=_\lambda e_1\oplus e_2$ *with* $e\sim_\lambda e_1\sim_\lambda e_2$.

(3) *A projection e of R is locally irreducible if and only if it is locally zero or locally minimal.*

PROOF. The proof of (1). We need only to prove the sufficiency. If e is not locally finite, then e is locally normally infinite. Hence, from Prop. 3.2, we may find a decomposition $e=_\lambda e_1\oplus e_2$ with $e\sim_\lambda e_1\sim_\lambda e_2$. Therefore, it holds that $e_1 \leqq_\lambda e, e_1 \sim_\lambda e$, and $e_1 \neq_\lambda e$. The statement (2) is an immediate consequence of (1) and Prop. 3.2.

The proof of (3). We have already proved the necessity (cf. Prop. 3.3). Hence, we need only to prove the sufficiency. Let e be locally minimal. Then, in the same way as J. Dixmier [5] has done, we may find a maximal orthogonal pair (e_1, e_2) of projections of R satisfying $e \geqq e_1 \sim e_2 \leqq e$ and hence $e_3 = e - (e_1 \oplus e_2)$ is irreducible. Since e is locally minimal, we have $e =_\lambda e_3$. For otherwise, it holds that $e_1 =_\lambda e$, which leads to a contradiction. Thus, e is locally irreducible. q. e. d.

Proposition 3.9. *The following statements are equivalent to each other.*

(1) *A projection e of R is irreducble,*

(2) *$e_1 = e_2$ follows from $e \geqq e_1 \sim e_2 \leqq e$,*

(3) *eRe is commutative.*

Proof. It is obvious that (2) implies (1). New we shall prove that (1) implies (3). In fact, for any $e_1 \leqq e$ and any $e_2 \leqq e$, we have $e_1 =_\lambda 0$ or e and $e_2 =_\lambda 0$ or e with respect to λ and hence $e_1 e_2 =_\lambda e_2 e_1$, from which follows $e_1 e_2 = e_2 e_1$ by Prop. 1.1. Next we shall prove that (3) implies (2). In fact, from $e \geqq e_1 \sim e_2 \leqq e$ it follows that there is a partial isometry u of eRe satisfying $u^* u = e_1$ and $uu^* = e_2$ and hence $e_1 = e_2$ by (3). q. e. d.

Under these definitions, we may classify AW^*-algebras into six local types (analoguous to F. J. Murray-J. v. Neumann [1], cf. J. Dixmier [5] and I. Kaplansky [6]):

Definition 3.5. (a) *R is called of locally finite discrete type $(I_n)_\lambda$ (or locally homogenous type of order n) if R is locally finite and there exists a locally irreducible and locally simple projection of order n.*

(b) *R is called of locally finite discrete limiting type $(I_0)_\lambda$ if R is locally finite and there exists a locally irreducible projection, which is never locally simple of order n for any natural number n.*

(c) *R is called of locally infinite discrete type $(I_\infty)_\lambda$ if R is locally discrete and locally normally infinite.*

(d) *R is called of locally finite continuous type $(II_1)_\lambda$ if R is locally non-discrete and locally finite.*

(e) *R is called of locally infinite continuous type $(II_\infty)_\lambda$ if R is locally non-discrete, locally normally infinite, and R cotains a locally finite and locally non-zero projection.*

(f) *R is called of locally purely infinite type $(III_\infty)_\lambda$ if R is locally normally infinite and contains no locally finite and locally non-zero projection.*

A projection e_0 of R_0 is called of local type (*) if it holds that $\lambda(e_0) = 0$ or that $e_0 R e_0$ is of the same local type, where $* = I_n, I_0, I_\infty, II_1, II_\infty, III_\infty$. We define type (*) by globalization of local type $(*)_\lambda$ (that is to say, R is called of type (*)

if R is of local type $(*)_\lambda$ with respect to any spectre λ of R), except for (I_0), which we define by globalization of being $(I_0)_\lambda$ or $(I_n)_\lambda$ for some n. Finally, a projection e_0 of R_0 is called of type $(*)$ if e_0Re_0 is of the same type.

Since these are normal, we have (cf. Lemma 3.3)

Proposition 3.10. *There exists a decomposition*

(3.9) $1=\oplus(e_0(*)\,;\ *=I_0, I_\infty, II_1, II_\infty, III_\infty)$,

(3.10) $e_0(I_0)=\oplus(e_0(I_n)\,;\ 1\leqq n<\infty)$.

where $e_0(*)$ is uniquely determined as the maximal projection of E_0 of type $(*)$.

§ 4. Local trace.

Let R be a finite AW^*-algebra. First we shall introduce the following

Definition 4.1. *A functional t_λ of R satisfying the following statements:*

(4.1) $t_\lambda(1)=1$,

(4.2) $t_\lambda(\alpha a)=\alpha t_\lambda(a)$ *for $a\in R$ and for any complex number α,*

(4.3) $t_\lambda(a+b)=t_\lambda(a)+t_\lambda(b)$ *for $a, b\in R$,*

(4.4) $t_\lambda(a^*)=\overline{t_\lambda(a)}$ *for $a\in R$ ($\overline{t_\lambda(a)}$=the complex conjugate of $t_\lambda(a)$),*

(4.5) $t_\lambda(a^*a)\geqq 0$ *for $a\in R$,*

(4.6) $t_\lambda(e_0(\lambda)a)=t_\lambda(a)$ *for $a\in R$ and $e_0(\lambda)\in E_0(\lambda)$,*

(4.7) $t_\lambda(ab)=t_\lambda(ba)$ *for $a, b\in R$,*

is called a local trace of R (with respect to λ).

By virtue of Prop. 3.2, a local trace of R may exist only if R is locally finite. Concerning this trace, in this §, we shall prove the following

Proposition 4.1. *There exists a local trace of R if and only if there exists a locally non-zero and locally simple projection e of R with $\bar{t}_\lambda(e)<1$. (cf. Def. 4.5 as to $\bar{t}_\lambda$.)*

We denote by N the set of hermitian elements of R, by N_0 the set of hermitian elements of R_0, and by Σ the set of (real-coefficient) linear operators σ's of R defined by

(4.8) $\sigma(a)=\sum_{\nu=1}^{n}\alpha_\nu s_\nu^* a s_\nu$ for any a of N,

where $s_1, s_2, \cdots, s_n$ are unitary elements of R and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are positive numbers with $\sum_{\nu=1}^{n}\alpha_\nu=1$.

The following proposition is due to J. Dixmier ([5], théorème 7). His proof was based on an AW^*-algebra. Recently, M. Goldman [9] has given a proof of this theorem in algebras satisfying weaker conditions than AW^*-algebras by a similar method of J. Dixmier [5]. Here, we shall give an alternative proof of this theorem in an AW^*-algebra.

Proposition 4.2. *For any element a of N and for any number $\varepsilon>0$ there exist an element a_0 of N_0 and a linear operator $\sigma\in\Sigma$ satisfying*

(4.9) $\|\sigma(a)-a_0\|\leqq\varepsilon$. *(Here, R is not necessarily finite.)*

Proof. Let R be a (not necessarily finite) AW^*-algebra. First we shall prove (4.9) locally for $a=e\in E$. If, for any natural number n, there exists an orthogonal system $(e_\nu;1\leqq\nu\leqq n)$ of projection of R satisfying $e=e_1\sim_\lambda e_\nu$ $(1\leqq\nu\leqq n)$ and $\oplus(e_\nu;1\leqq\nu\leqq n)\leqq 1$, then (4.9) holds good locally for $a_0=0$, as we can verify it by taking such an n that $n^{-1}<\varepsilon$ and making $s_\nu=u_\nu^*+u_\nu+(e_\nu\oplus e)^c(1\leqq\nu\leqq n)$, where u_ν is a partial isometry of R with $e_*(u_\nu)=e_\nu$ and $e(u_\nu)=e$. In the other case, namely, if the above-mentioned condition is not satisfied by e, there exists, for any natural number m, a commutative system $(e_\nu;1\leqq\nu\leqq n+1)$ of projections of R satisfying (1) $m=_\lambda\sum_{\nu=1}^{n+1}e_\nu$, (2) $e=e_1, e_1^ce_\nu\sim e_\nu^ce_1$ $(1\leqq\nu\leqq n)$, (3) $d_\lambda(e_{n+1})\leqq d_\lambda(e)$, and (4) $m\leqq n$. We shall see it by induction. It is obvious for $m=1$. For an m, suppose that such a system has been constructed. Since the system $(e_\nu;1\leqq\nu\leqq n+1)$ is commutative, we may find an orthogonal system P of projections of R such that P consists of a finite number of projections of R and that each e_ν is expressed as a direct-sum of members of P. We denote by $P(e')$ the set of projections e'''s of P such that $e''\leqq e'$. Then, we can construct a locally equivalent pair $\oplus(e_\nu';1\leqq\nu\leqq r'+1)$, $\oplus(e_\nu'';1\leqq\nu\leqq r''+1)$ satisfying (5) e_ν' $(1\leqq\nu\leqq r')\in P(ee_{n+1})$, e_ν'' $(1\leqq\nu\leqq r'')\in P(e^ce_{n+1})$, either (6) $e_{r'+1}'$ being commutative with each projection of P and $e_{r''+1}''=_\lambda 0$ or (7) $e_{r'+1}'=_\lambda 0$ and $e_{r''+1}''$ being commutative with each projection of P and (8) r' is best possibly maximal. By Prop. 3.7, it is easy to see that $e=\oplus(e_\nu';1\leqq\nu\leqq r'+1)\oplus(\oplus(e';e'\in P(ee_{n+1}^c))$ or $e_{n+1}=\oplus(e_\nu'';1\leqq\nu\leqq r''+1)\oplus(\oplus(e';e'\in P(ee_{n+1}))$. If $e=\oplus(e_\nu';1\leqq\nu\leqq r'+1)\oplus(\oplus(e';e'\in P(ee_{n+1})))$, then we have $d_\lambda(e)\leqq d_\lambda(e_{n+1})$, and so $d_\lambda(e)=d_\lambda(e_{n+1})$ by (3). This means that $e\sim_\lambda e_{n+1}$. Starting from e, we can construct a decomposition $1=\oplus(e_\nu;n+2\leqq\nu\leqq r+1)$ satisfying (9) $e\sim_\lambda e_\nu(n+2\leqq\nu\leqq r)$, (10) each e_ν is expressed as a direct-sum of projections of a common orthogonal system of projections of R commutative with each projection of P, (11) $d_\lambda(e_{r+1})\leqq d_\lambda(e)$. Therefore, we can take $(e_\nu;1\leqq\nu\leqq r+1)$ as a commutative system in question for $m+1$. On the other hand, if $e_{n+1}=\oplus(e_\nu'';1\leqq\nu\leqq r''+1)\oplus(\oplus(e'';e''\in P(e^ce_{n+1}))$, then we have $d_\lambda(e_{n+1}\oplus e_{r'+1}')=d_\lambda(e)$ and, by starting from $e_{n+1}\oplus e_{r'+1}'$, we can construct a decomposition $1=\oplus(e_\nu;n+2\leqq\nu\leqq r+1)$ satisfying (9)–(11) by the same way as above. Hence, we can construct $(e_\nu;1\leqq\nu\leqq r+1)$ as a commutative system in question for $m+1$. It is easy to see that for some $\sigma\in\Sigma$, $\sigma(e)=(1/n)\sum_{\nu=1}^n e_\nu$. Thus, we have $\|\sigma(e)-\alpha\|_\lambda\leqq 1/n$ for some scalar α. (Here, we denote by $\|a\|_\lambda$ the local norm of a, that is, $\inf(\|e_0(\lambda)a\|;e_0(\lambda)\in E_0(\lambda))$). Hence, (4.9) holds locally for e.

Next we shall prove (4.9) locally for $a=\sum_{\mu=1}^r\alpha_\mu e_\mu$ with $e_1\leqq e_2\leqq\cdots\leqq e_r$. It is obvious for $r=1$. Moreover, by the assumption of induction for r, there exists a linear operator $\sigma\in\Sigma$ of e_2Re_2 with $\|\sigma(e_1)-\alpha e_2\|\leqq\mathrm{Min}(\varepsilon/2,\varepsilon/2|\alpha_1|)$. We denote it by $\sigma(a)=\sum_{\nu=1}^n r_\nu s_\nu^* a s_\nu$, where s_ν $(1\leqq 1\leqq n)$ are unitary elements of e_2Re_2 and r_ν $(1 \leqq \nu \leqq n)$ are positive numbers with $\sum_{\nu=1}^{n} r_\nu = 1$. The operator σ may be extended to a linear operator $a \rightarrow \sum_{\nu=1}^{n} \alpha_\nu s_\nu'^* a s_\nu'$ of R, where $s_\nu' = s_\nu + ee^c$. We denote it again by σ. Then, we have $\sigma(e_\mu) = e_\mu (2 \leqq \mu \leqq r)$ and so $\| \sigma(a) - \sum_{\mu=2}^{r} \alpha_\mu' e_\mu \|_\lambda \leqq \varepsilon/2$, where $\alpha_2' = \alpha_2 - \alpha\alpha_1$ and $\alpha_\mu' = \alpha_\mu (3 \leqq \mu \leqq r)$. By the assumption of induction we have $\| \sigma_1(\sum_{\mu=2}^{r} \alpha_\mu' e_\mu) - \beta \|_\lambda \leqq \varepsilon/2$ for some $\sigma_1 \in \Sigma$ and some scalar β. Thus, we have $\| \sigma_1 \sigma(a) - \beta \|_\lambda \leqq \varepsilon$. Hence, we have (4.9) locally for such an element a of N as before. Since any element a of N is uniformly approximated by those elements as before, we get (4.9) locally.

Finally, we shall prove (4.9) globally. Since we have already proved (4.9) locally, for any spectre λ of R, we may find $\sigma_\lambda \in \Sigma$ such that $\| \sigma_\lambda(a) - \alpha_\lambda \|_\lambda \leqq \varepsilon/2$ for some scalar α_λ. Hence, we have $\| e_0(\lambda)(\sigma_\lambda(a) - \alpha_\lambda) \| \leqq \varepsilon$ for some $e_0(\lambda) \in E_0(\lambda)$. As we know, Ω is compact and so we get $1 = \cup(e_0(\lambda_i); 1 \leqq i \leqq m)$ for some $\lambda_i \in \Omega$ and some $e_0(\lambda_i) \in E_0(\lambda_i)$. We put $e_0^{(1)} = e_0(\lambda_1)$ and $e_0^{(i)} = e_0(\lambda_i)(\cup(e_0(\lambda_j); 1 \leqq j \leqq i-1))^c (1 \leqq i \leqq n)$. Then, we have $\| e_0^{(i)}(\sigma_{\lambda i}(a) - \alpha_{\lambda i}) \| \leqq \varepsilon$. Suppose that $\sigma_{\lambda i}$ is defined by $\sigma_{\lambda i}(a) = \sum_{\nu i=1}^{n(i)} \alpha_{\nu i}^{(i)} s_{\nu i}(i)^* a s_{\nu i}(i)$, where $e_0(\lambda_i) s_{\nu i}(i)$ is a unitary element of $e_0(\lambda_i)R$. Then, we may find a unitary element $s(\nu_1, \nu_2, \cdots, \nu_m)$ of R satisfying that $e_0^{(i)} s(\nu_1, \nu_2, \cdots, \nu_m) e_0^{(i)} s_{\nu i}(i)$ $(1 \leqq i \leqq m)$ for any $\nu_1, \nu_2, \cdots, \nu_m$ with $1 \leqq \nu_i \leqq n_i$ $(1 \leqq i \leqq m)$. We denote by σ the linear operator of R defined by $\sigma(a) = \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} \cdots \sum_{\nu n=1}^{n m} \alpha_{\nu_1}(1) \alpha_{\nu_2}(2) \cdots \alpha_{\nu m}(m) s(\nu_1, \nu_2, \cdots, \nu_m)^* a s(\nu_1, \nu_2, \cdots, \nu_m)$ and by a_0 the element $\sum(\alpha_{\lambda i} e_0^{(i)}; 1 \leqq i \leqq m)$ of R_0. Thus, we have $\| e_0^{(i)}(\sigma(a) - a_0) \| = \| e_0(\lambda_i)(\sigma_{\lambda i}(a) - \alpha_{\lambda i}) \| \leqq \varepsilon$ and so $\| \sigma(a) - a_0 \| \leqq \varepsilon$, q. e. d.

Lemma 4.1. *There exists a local trace of R, if R is of local type $(I_n)_\lambda$.*

Proof. It is easy to see that R is locally isomorphic to the full matric algebra of degree n over the complex number field, whose local isomorphism is denoted by $\varphi(a) = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij}$, where $(\alpha_{ij}; 1 \leqq i, j \leqq n)$ is a system of complex numbers and $(e_{ij}; 1 \leqq i, j \leqq n)$ is a system of matrix units. Thus, we have $t_\lambda(a) = \sum_{i=1}^{n} \alpha_{ii}$ as a local trace of R. q. e. d.

For a while, we denote by R a finite and non-discrete AW^*-algebra. But we notice that the results in this § are all valid, even if we drop the non-discreteness assumption and assume simply that R is finite.

Since R has the unit 1, we may assume without loss of generality that R contains the complex number field and so the unit 1 may consider as coinciding with the number 1.

Definition 4.2. (1) *A projection e of R is called elementary if it is simple of order 2^n. We put $2^{-n} e^{\natural} = D(e)$ and $2^{-n} \lambda(e^{\natural}) = D(e)$.*

(2) *A projection e of R is called locally singular if $d_\lambda(e) \leqq d_\lambda(e_1)$ for any locally non-zero and locally elementary projection e_1 of R.*

The projection 0 is considered as simple of order n for any n. In this way, the elementarity of a projection of R is a normal (global) property concerning a projection of R. Hence, we say about being locally elementary.

The local singularity of a projection of R is a local property concerning a projection of R. Hence, the property concerning a projection of R that the projection is not locally singular, is never a local property. But, for convenience, we call this property the local non-singularity. What property is the global form of local singularity? The following lemma is an answer to this question. Here is a property, which does not coincide with the local form of its global form (cf. § 1).

LEMMA 4.2. *If e is a locally singular projection of R with respect to any spectre of R, then $e=0$.*

PROOF. Using Prop. 3.4 step by step, we find a decomposition $1=\oplus(e_n; 1\leqq n<\infty)$ with $D(e_n)=2^{-n}(1\leqq n<\infty)$, Since e is locally singular with respect to any spectre of R, we may find a projection e_n' with $e\sim e_n'\leqq e_n$ for each natural number n. If e is a non-zero projection, then $\oplus(e_n'; 1\leqq n<\infty)$ is also a non-zero projection. According to Prop. 3.2, $\oplus(e_n'; 1\leqq n<\infty)$ is normally infinite, and we arrive at a contradiction. q. e. d.

As an immediate consequence of Lemma 4.2, we have the following

COROLLARY. *Every projection of R is expressed as a direct-sum of elementary projections of R.*

The following lemma is due to J. Dixmier [5] and the present proof is essentially the same as his.

LEMMA 4.3. *For a decomposition $1=\oplus(e_\iota; \iota\in I)$ we have*

(4.10) $1=\sum(D(e_\iota); \iota\in I)$

where each e_ι is an elementary projection of R.

PROOF. We denote by I_n the set of indices ι's of I with $D(e_\iota)=2^{-n}e_\iota^\natural$ and by P_n the set of projections e_0e_ι's of R for $\iota\in I_n$ and $e_0\in E_0$. Then, there exists a maximal orthogonal system $P_n^{(1)}$ of projections of P_n with $e_1^\natural e_2^\natural=0$ for $e_1\neq e_2$ $(e_1, e_2\in P_n^{(1)})$. Furthermore, by induction, we may find a maximal orthogonal system $P_n^{(k)}$ $(k\geqq 2)$ of projections of P_n with $e_1^\natural e_2^\natural=0$ for $e_1\neq e_2$ $(e_1, e_2\in P_n^{(k)})$ and orthogonal to $P_n^{(\nu)}$ $(1\leqq\nu\leqq k-1)$. We put $\oplus(e; e\in P_n^{(k)})=e_n^{(k)}$. Then, we have $D(e_n^{(k)})=2^{-n}e_n^{(k)\natural}$, because the simplicity is normal. Moreover, it holds that $e_n^{(1)\natural}\geqq e_n^{(2)\natural}\geqq\cdots$. If $\cap(e_n^{(k)\natural}; 1\leqq k\leqq 2^n+1)$ (say$=e_0$) is not 0, we must have $\oplus(e_n^{(k)}; 1\leqq k\leqq 2^n+1)\leqq 1$ and $D_\lambda(e_n^{(k)})=2^{-n}$ for a spectre λ of R with $\lambda(e_0)=1$. This leads to a contradiction. Hence, we may find a natural number $k(n)$ with $e_n^{(k)}\neq 0$ $(1\leqq k\leqq k(n))$ and $e_n^{(k)}=0$ $(k(n)<k)$. This implies that $\oplus(e_\iota; \iota\in I_n)=\oplus(e_n^{(k)}; 1\leqq k\leqq k(n))$. Since it holds that $e_\iota=\oplus(e_\iota e_n^{(k)}; 1\leqq k\leqq k(n))$ for $\iota\in I_n$, we see that $D(e_\iota)=\sum(D(e_\iota e_n^{(k)}); 1\leqq k\leqq k(n))$ (finite sum). On the other hand, we have $D(e_n^{(k)})=\sum(D(e_\iota e_n^{(k)}); \iota\in I_n)$ (direct sum). Thus, the equality $\sum(D(e_\iota); \iota\in I)=\sum(D(e_n^{(k)}); 1\leqq k\leqq k(n), 1\leqq n<\infty)$ is obtained.

By the above arguement, we may assume without loss of generality that I is a countable set (say$(n; 1\leqq n<\infty)$). If it does not hold that $1=$

$\sum(D(e_n); 1\leqq n<\infty)$, we may find a non-zero elementary projection e of R such that $1\geqq D(e)+\sum(D(e_n); 1\leqq n<\infty)$. For, putting $e_0=\sum(D(e_n); 1\leqq n<\infty)$, we have $\lambda(1-e_0)>0$ for fome $\lambda\in\Omega$, that is, $e_0(\lambda)((1-e_0)2^{-n})\geqq 0$ for some $e_0(\lambda)\in E_0(\lambda)$ and some natural number n. Hence, we have $e_0(\lambda)(1-e_0-D(e))\geqq 0$ for any elementary projection e of order n with $e^{\natural}=e_0(\lambda)$. Thus, we get $1-e_0\geqq e_0(\lambda)(1-e_0)\geqq e_0(\lambda)D(e)=D(e)$.

We shall prove that there exists an orthogonal system $(e_n'; 1\leqq n<\infty)$ of projections of R satisfying $e_n\sim e_n'$ and $\bigoplus(e_n'; 1\leqq n<\infty)\leqq e^c$ by induction. For an n, suppose that we have construct already an orthogonal system $(e_i'; 1\leqq i\leqq n)$ of projections of R satisfying $e_i\sim e_i'$ $(1\leqq i\leqq n)$ and $\bigoplus(e_i'; 1\leqq i\leqq n)\leqq e^c$. Then, denoting $(e\bigoplus(\bigoplus(e_i'; 1\leqq i\leqq n))^c$ briefly by e', we have $D(e')=1-D(e)-\sum(D(e_i'); 1\leqq i\leqq n)=1-D(e)-\sum(D(e_i); 1\leqq i\leqq n)\geqq D(e_{n+1})$. Since e', e_{n+1} are expressed as a direct-sum of a finite number of elementary projections, for any spectre λ of R, we have $e_{n+1}\sim_\lambda e_{n+1}'\leqq e'$ for some $e_{n+1}'\in E$ by Prop. 3.7. Hence, we have $e_{n+1}\sim e_{n+1}'\leqq e'$ for some $e_{n+1}'\in E$ by Prop. 1.1. The proof for $n=1$ is obtained by making $e'=e^c$. Thus, we can construct the system $(e_n'; 1\leqq n<\infty)$ in question. From this it follows that $1=\bigoplus(e_n; 1\leqq n<\infty)\sim\bigoplus(e_n'; 1\leqq n<\infty)\leqq e^c\neq 1$. This is a contradiction. Therefore, we arrive at the assertion. q. e. d.

Now, we shall introduce the following

Definition 4.3. *An operator D from E into R_0 satisfying the following*

(4.11) $D(1)=1$,

(4.12) $D(\bigoplus(e_\iota; \iota\in I))=\sum(D(e_\iota); \iota\in I)$,

(4.13) $D(e_0e)=e_0D(e)$ *for* $e_0\in E_0$,

(4.14) $D(e_1)=D(e_2)$ *if and only if* $e_1\sim e_2$,

is called a relative dimension function of R (after J. Dixmier [5]). From this definition, it follows $D(e_0)=e_0$ for $e_0\in E_0$.

Proposition 4.3. *There exists one (and only one) relative dimension function of R if and only if R is finite.*

Proof. Necessity. If R is not finite, then we may find a projection e of R and a non-zero projection e_0 of R_0 with $e_0e\sim e_0\sim e_0e^c$ by Prop. 3.2. Thus, we have $e_0=D(e_0)=D(e_0e)+D(e_0e^c)=2e_0$ if R has a relative dimension function D. This is a contradiction. Sufficiency. For any projection e of R there is a decomposition $e=\bigoplus(e_\iota; \iota\in I)$ by the Corollary of Lemma 4.2, where each e_ι is an elementary projection of R. We denote $\sum(D(e_\iota); \iota\in I)$ by $D(e)$. For another decomposition $e=\bigoplus(e_{\iota'}'; \iota'\in I')$ combining with a decomposition $e^c=\bigoplus(e_{\iota''}''; \iota''\in I'')$, we have $\sum(D(e_\iota); \iota\in I)+\sum(D(e_{\iota''}''); \iota''\in I'')=(D(e_\iota); \iota'\in I')+\sum(D(e_{\iota''}''); \iota''\in I'')$, by Lemma 4.3. Hence, we have $\sum(D(e_\iota); \iota\in I)=\sum(D(e_{\iota'}'); \iota'\in I')$. This means that $D(e)$ is well-defined. It is easy to see (4.11)-(4.14). Uniqueness. For another relative dimension function D' of R, we have

$D(e_\iota)=D'(e_\iota)$ for an elementary projection e_ι of R. For an arbitrary projection e, we may find a decomposition $e=\oplus(e_\iota;\iota\in I)$, where each e_ι is an elementary projection of R by the Corollary of Lemma 4.2, and so we have $D(e)=D'(e)$ by (4.12). This completes the proof. q. e. d.

We now establish the local forms of Lemma 4.3, Def. 4.3, and Prop. 4.3.

LEMMA 4.4. *We have* $1=\sum(D_\lambda(e_\iota);\iota\in I)$ *for any maximal orthogonal system* $(e_\iota;\iota\in I)$ *of elementary projections* e_ι*'s of* R *with* $\lambda(e_\iota^\natural)=1$.

PROOF. It is easy to see that $1\geqq\sum(D(e_\iota);\iota\in I')$ for any finite subset I' of I. Hence, we may assume without loss of generality that I is a countable set (say $I=(n;1\leqq n<\infty)$). Thus, we have $1\geqq\sum_{n=1}^{\infty}D_\lambda(e_n)$. Moreover, we may assume without loss of generality that $D_\lambda(e_n)\geqq D_\lambda(e_{n+1})$ for $n\geqq 1$. If $\sum_{n=1}^{\infty}D_\lambda(e_n)<1$, we may find a natural number $k\geqq 2$ with $\sum_{n=1}^{\infty}D_\lambda(e_n)+2^{-(k-1)}<1$. There exists a system $(e_n^{(i)};1\leqq i\leqq 2^n,1\leqq n<\infty)$ of elementary projections of R, which satisfies the following conditions: (1) $D(e_n^{(i)})=2^{-n}$, (2) $e_n^{(i)}e_n^{(j)}=0$ for $i\neq j$, and $e_m^{(i)}e_n^{(j)}=0$ or $e_n^{(j)}$ for $m<n$. Then, it is easily seen that there exists an orthogonal subsystem $(e_n';1\leqq n<\infty)$ of the above system $(e_n^{(i)};1\leqq i\leqq 2^n,1\leqq n<\infty)$ satisfying $e_n\sim e_n''\leqq e_n'$ and $e_n'(e_k^{(1)}\oplus e_k^{(2)})=0$. Since $(\oplus(e_n;1\leqq n<\infty))^c$ is locally singular, we have $d_\lambda((\oplus(e_n;1\leqq n<\infty)^c)\leqq d_\lambda(e_k^{(1)})$. Thus, we have $d_\lambda(1)\leqq d_\lambda(e_k^{(2)c})$, which is a contradiction. q. e. d.

DEFINITION 4.4. *A functional* D_λ *of* E *satisfying the following*

(4.15) $D_\lambda(1)=1$,

(4.16) $D_\lambda(\oplus(e_\iota;\iota\in I))=\sum(D_\lambda(e_\iota);\iota\in I)$ *for mutually orthogonal elementary projections* e_ι*'s of* R *with* $\lambda(e_\iota^\natural)=1$ *except for at most a finite number of projections,*

(4.17) $D_\lambda(e)=D_\lambda(e_0(\lambda)e)$ *for* $e_0(\lambda)\in E_0(\lambda)$,

(4.18) $D_\lambda(e_1)=D_\lambda(e_2)$ *for* $e_1\sim_\lambda e_2$,

is called a local relative dimension function of R *(after F.J. Murray and J. v. Neumann* [1]).

PROPOSITION 4.4. *There exists one (and only one) local relative dimension function of* R *if and only if* R *is locally finite.*

PROOF. We can prove the assertion by a similar argument as in the proof of Prop. 4.3. q. e. d.

For any projection e of R, the center of eRe is isomorphic to that of $e^\natural R$. In fact, putting $R_1=e^\natural R$, $R_2=eRe$, and $A=eR$, we have $b^*a\in R_1$, $ab^*\in R_2$, and $R_2aR_1\subseteqq A$ for any $a,b\in A$, that is, we see that A satisfies the conditions of A in 2. Under the same terminologies as in § 2, we have $I_1^\#=e=I_2$ and $I_2^\#=e^\natural=I_1$. Hence, the center of $e^\natural R$ is isomorphic to that of eRe by $\#$ (cf. Prop. 2.8). Thus, $\#$ induces a homeomorphism between the spectrum of the center $e^\natural R_0$ of $e^\natural R$ and that of eRe. Therefore we may identify a spectre of $e^\natural R$ and its image by the induced homeomorphism.

We denote by $\mathfrak{R}$ the real number field, which we consider as being imbedded in N_0.

Now, we shall introduce the following important tool for the proof of Prop. 4.1.

Definition 4.5. *For an elementa of N, we write*

(4.19) $\bar{t}_\lambda(a, \sigma)=\sup(\alpha ; \sigma(a) \geqq_\lambda \alpha, \alpha \in \mathfrak{R})$, $\underline{t}_\lambda(a, \sigma)=\inf(\alpha ; \sigma(a) \leqq_\lambda \alpha, \alpha \in \mathfrak{R})$, *for* $\sigma \in \Sigma$,

(4.20) $\bar{t}_\lambda(a)=\sup(\bar{t}_\lambda(a, \sigma) ; \sigma \in \Sigma)$, $\underline{t}_\lambda(a)=\inf(\underline{t}_\lambda(a, \sigma) ; \sigma \in \Sigma)$.
Moreover, we use the notation $\bar{t}_\lambda(a, e)$ instead of $\bar{t}_\lambda(a)$ of eRe and the notation $\underline{t}_\lambda(a, e)$ instead of $\underline{t}_\lambda(a)$ of eRe for an element a of eRe.

In order to prove Prop. 4.1, we use the following three lemmas.

Lemma 4.5. *For $a, b \in N$, $e_i \in E (i=1,2,3)$ and $\alpha \in \mathfrak{R}$, we have*

(1) $\bar{t}_\lambda(s^* a s)=\bar{t}_\lambda(a)$ *for any unitary element s of R,*

(2) $\bar{t}_\lambda(\alpha a)=\alpha \bar{t}_\lambda(a)$ *for* $\alpha \geqq 0$, (3) $\bar{t}_\lambda(a) \geqq 0$ *for* $a \geqq 0$,

(4) $\bar{t}_\lambda(a-\alpha)=t_\lambda(a)-\alpha$, (5) $\bar{t}_\lambda(a+b) \leqq \bar{t}_\lambda(a)+\bar{t}_\lambda(b)$,

(6) $\bar{t}_\lambda(a) \geqq \underline{t}_\lambda(a)$, (7) $\bar{t}_\lambda(e_1, e_2) \bar{t}_\lambda(e_2, e_3) \leqq \bar{t}_\lambda(e_1, e_3)$ *for* $e_1 \leqq e_2 \leqq e_3$, (8) $\bar{t}_\lambda(e_1)/D_\lambda(e_1) \leqq \bar{t}_\lambda(e_2)/D_\lambda(e_2)$ *for locally elementary projections e_i $(i=1,2)$ with* $0<D_\lambda(e_2) \leqq D_\lambda(e_1)$, *and hence* $\bar{t}_\lambda(e_1)=\bar{t}_\lambda(e_2)$ *when* $D_\lambda(e_1)=D_\lambda(e_2)$, *and* (9) $\bar{t}_\lambda(e)=0$ *for every locally singular projection e of N.*

Proof. It is easy to see (1)-(4). In order to show (5), we may find $\sigma_i \in \Sigma$ $(i=1,2,3)$ and $\alpha_j \in \mathfrak{R}$ $(j=1,2)$ for a positive number ε such that $\bar{t}_\lambda(a+b)-\varepsilon \leqq_\lambda \sigma_1(a+b)$ by the definition of $\bar{t}_\lambda$, $\|\sigma_2 \sigma_1(a)-\alpha_1\|_\lambda \leqq \varepsilon$ by Prop. 4.2, and $\|\sigma_3 \sigma_2 \sigma_1(b)-\alpha_2\|_\lambda \leqq \varepsilon$ by Prop. 4.2. Hence, we have $\bar{t}_\lambda(a+b)-\varepsilon \leqq \alpha_1+\alpha_2+2\varepsilon$ by an easy computation. Since $\alpha_1-\varepsilon \leqq \bar{t}_\lambda(a)$ and $\alpha_2-\varepsilon \leqq \bar{t}_\lambda(b)$, we get $\bar{t}_\lambda(a+b) \leqq \bar{t}_\lambda(a)+\bar{t}_\lambda(b)+4\varepsilon$. Making $\varepsilon \downarrow 0$, we have (5).

In order to see (6), we may find $\sigma \in \Sigma$ and $\alpha \in \mathfrak{R}$ for a positive number ε such that $\|\sigma(a)-\alpha\|_\lambda \leqq \varepsilon$ by Prop. 4.2. Therefore, we have $\alpha-\varepsilon \leqq \bar{t}_\lambda(a)$ and $\alpha+\varepsilon \geqq \underline{t}_\lambda(a)$. Making $\varepsilon \downarrow 0$, we obtain (6).

In order to see (7), we may find $\sigma_2 \in \Sigma$ with $\sigma_2(e_2)=e_2$, $\sigma_2(e_3)=e_3$ such that $(\bar{t}_\lambda(e_1, e_2)-\varepsilon) e_2 \leqq_\lambda \sigma_2(e_1)$ by Prop. 4.2 (cf. the proof of Prop. 4.2) and $\sigma_3 \in \Sigma$ with $\sigma_3(e_3)=e_3$ such that $(\bar{t}_\lambda(e_2, e_3)-\varepsilon) e_3 \leqq_\lambda \sigma_3(e_2)$ by Prop. 4.2. Hence, we have $(\bar{t}_\lambda(e_1, e_2)-\varepsilon)(\bar{t}_\lambda(e_2, e_3)-\varepsilon) \leqq \bar{t}_\lambda(e_1, e_3)$. Making $\varepsilon \downarrow 0$, we get (7).

In order to see (8), we may find $\sigma_1 \in \Sigma$ such that $\sigma_1(e_2)={}_\lambda(D_\lambda(e_2)/D_\lambda(e_1)) e_1$ by using the local elementarity of e_1, e_2 and $D_\lambda(e_1) \geqq D_\lambda(e_2)$, and $\sigma \in \Sigma$ such that $\bar{t}_\lambda(e_1)-\varepsilon \leqq_\lambda \sigma(e_1)$ by the definition of t_λ. Hence, we have $(\bar{t}_\lambda(e_1)-\varepsilon) D_\lambda(e_2)/D_\lambda(e_1) \leqq_\lambda \sigma \sigma_1(e_2)$. Thus, we have $(\bar{t}_\lambda(e_1)-\varepsilon) D_\lambda(e_2)/D_\lambda(e_1) \leqq \bar{t}_\lambda(e_2)$. Making $\varepsilon \downarrow 0$, we obtain (8).

In order to prove (9), we notice that $e_1 \cup e_2$ is locally singular with e_i $(i=1,2)$ because of $d_\lambda(e_1 \cup e_2-e_1) \leqq d_\lambda(e_2)$ (cf. Lemma 2.5 and the Corollary 1 of Prop. 3.6.). From this, it follows easily that (9) holds. q. e. d.

LEMMA 4.6. *If there exists a locally non-zero and locally elementary projection e_1 of R with $\bar{t}_\lambda(e)<1$, then there exists a projection e_1 of R and a state f of e_1Re_1 such that it holds $f(e_1')=0$ if and only if e_1' is a locally singular projection of e_1Re_1, where we say that a linear functional f of R is a state of R if it satisfies that* (1) $f(1)=1$, (2) $f(\alpha a)=\alpha f(a)$ *for* $a\in R$ *and* α *being any complex number,* (3) $f(a^*)=\overline{f(a)}$ *for* $a\in R(\overline{f(a)}=$*the conjugate complex number of* $f(a)$), (4) $f(a^*a)\geqq 0$ *for* $a\in R$.

PROOF. Let E_L be the set of elementary projections e_1's of R with $e_1^\natural=1$ and let G_L be the set of supremum of orthogonal system of projections of E_L. We may assume without loss of generality that e is a projection of E_L. We note that $D(e')=D_\lambda(e')\cdot 1$ when $e'^\natural=1$. So we shall consider henceforth $D(e')$ (when $e'^\natural=1$) as a real number and then the above equality can be written as $D(e')=D_\lambda(e')$ when $e'^\natural=1$. Then, we can find a maximal chain G of projections of G_L containing e and satisfying $(D(e');\, e'\in G)=[0,1]$ (the closed interval of real numbers between 0 and 1). In fact, we can find a system $G_0=(e_n(\beta_0);\, 0\leqq\beta_0\leqq 1, 2^n\beta_0$ a natural number with n) of projections of E_L such that (1) $e\in G_0$, (2) $D(e_n(\beta_0))=2^{-n}$, and (3) $e_n(\beta_0)=e_{n+1}(\beta_0-2^{-(n+1)})\oplus e_{n+1}(\beta_0)$ for $0<\beta_0\leqq 1$ and $2^n\beta_0$ is a natural number. For any real number β with $0\leqq\beta\leqq 1$, we have an expression $\beta=\sum_{\nu=0}^{\infty}\varepsilon_\nu 2^{-\nu}(\varepsilon_\nu=0$ or $1)$. We put $\beta_n=\sum_{\nu=0}^{n}\varepsilon_\nu 2^{-\nu}$ and $e(\beta)=\oplus(e_n(\beta_n)$; for n such that $2^n\beta_n$ is odd) for $\beta>0$ and $e(0)=0$ for $\beta=0$. Then, $e(\beta)$ is independent on the expression of β. We denote $(e(\beta);\, 0\leqq\beta\leqq 1)$ by G. Then, G is a chain of projections of G_L in question. It is easy to see that $D(e(\beta))=\beta$ and $\lim_{\beta\downarrow\alpha} e(\beta)=e(\alpha)$. We denote by e_1 the supremum of $(e';\, t_\lambda(e')<1, e'\in G)$. Then, we have $e_1\in G$ and $D(e)\leqq D(e_1)$.

If $\bar{t}_\lambda(e_1)=1$, then we have $\bar{t}_\lambda(e', e_1)<1$ for any projection e' of G with $D(e')<D(e_1)$ by (7) of Lemma 4.5. For any elementary projection e_1' of e_1Re_1 with $\lambda(e_1'^\natural)=1$, we may find a projection e' of G such that $D_\lambda(e_1e_1'^c)<D(e')<D(e_1)$. Hence, we have $\bar{t}_\lambda(e_1e_1'^c, e_1)<1$. By the extention theorem of Hahn and Banach, there exists a state f of e_1Re_1 with $f(a)\leqq t_\lambda(a, e_1)$ for any hermitian element a of e_1Re_1 by virtue of (2), (4) and (5) of Lemma 4.5. Therefore, we have $f(e_1')=1-f(e_1e_1'^c)\geqq 1-t_\lambda(e_1e_1'^c)<0$, for any elementary projection e_1' of e_1Re_1 with $\lambda(e_1'^\natural)=1$.

If $\bar{t}_\lambda(e_1)<1$, denoting by e_n'' the projection of G with $D(e_n'')=(1+2^{-n})D(e_1)$ for large n say $n\geqq n_0$, there exists a state f_n of $e_n''Re_n''$ with $f_n(a)\leqq\bar{t}_\lambda(a, e_n'')$ for any hermitian element a of $e_n''Re_n''$. Since $\bar{t}_\lambda(e_1, e_n'')<1$ by (7) of Lemma 4.5, we have $f_n(e_1')<0$ for any elementary projection e_1' of e_1Re_1 with $D_\lambda(e_1')=2^{-n}$. We put $f(a)=\varepsilon^{-1}\sum_{n=n_0}^{\infty}2^{-n}f_n(a)$ for any element a of e_1Re_1, where $\varepsilon=\sum_{n=n_0}^{\infty}2^{-n}f_n(e_1)$. Then, f is a state of e_1Re_1 and $f(e_1')>0$ for any elementary projection e_1' of e_1Re_1 with $\lambda(e_1'^\natural)=1$. This complete the proof. q. e. d.

Lemma 4.7. *A locally finite (locally non-discrete) AW*-algebra R has a local trace, if eRe has a local trace for a locally non-singular projection e of R.*

Proof. Since e is a locally non-singular projection of R, there exists a locally non-zero and locally elementary projection e_1 of order 2^n with $e_1 \leqq e$. Then, we may find a local decomposition $1 = {}_\lambda\bigoplus(e_j; 1 \leqq j \leqq 2^n)$ with $e_1 \sim_\lambda e_j$ $(1 \leqq j \leqq 2^n)$ and local partial isometries u_j $(1 \leqq j \leqq 2^n)$ with $e_*(u_j) = e_j$ and $e(u_j) = e_1$. We put $e_{ij} = u_i{}^* u_j$ $(1 \leqq i, j < 2^n)$. Then, $(e_{ij}; 1 \leqq i, j \leqq 2^n)$ is a system of matrix units of R. Put $\varphi(a) = \sum(u_j{}^* a u_j; 1 \leqq j \leqq 2^n)$ for each $a \in e_1 R e_1$. Then, φ is an isomorphism from $e_1 R e_1$ onto $\varphi(e_1 R e_1)$. It is easily seen that $\varphi(a) e_{ij} = e_{ij} \varphi(a)$ $(1 \leqq i, j \leqq 2^n)$ for each $a \in e_1 R e_1$ and $R = \sum(\varphi(e_1 R e_1) e_{ij}; 1 \leqq i, j \leqq 2^n)$, the full matric algebra of order 2^n over $e_1 R e_1$. Since a local trace of eRe is that of $e_1 R e_1$ by restriction with neglect to constant multiplier, we may assume without loss of generality that $\varphi(e_1 R e_1)$ has a local trace (say t_λ'). We put $t_\lambda(a) = \sum(t_\lambda'(a_{jj}); 1 \leqq j \leqq 2^n)$, where $a = \sum(a_{ij} e_{ij}; 1 \leqq i, j \leqq 2^n, a_{ij} \in \varphi(e_1 R e_1))$. Then, it is easy to see that t_λ is a local trace of R. q. e. d.

The proof of Prop. 4.1. We need only to prove the sufficiency. From Lemma 4.6, and Lemma 4.7, we may assume without loss of generality that there exists a state f_λ of R such that $f_\lambda(e) = 0$ if and only if e is locally singular.

First we shall prove that, for any positive number ε, there exists a local state g_λ of R such that it holds

(4.21) $|g_\lambda(a) - g_\lambda(\sigma(a))| \leqq 2\varepsilon \| a \|_\lambda$

for any a of N and for any $\sigma \in \Sigma$, where we say that g_λ is a local state of R if it is a state of R with $g_\lambda(a) = g_\lambda(e_0(\lambda) a)$ for any $e_0(\lambda) \in E_0(\lambda)$. By a similar argument as in [2], we can find a locally non-zero and locally elementary projection e of R satisfying $f_\lambda(e_1) \leqq D_\lambda(e_1)$ for each locally elementary projection e_1 of R with e_1 of R with $e_1 \leqq e$. We shall prove this fact below. If we have a decomposition $1 = \bigoplus(e_\iota; \iota \in I) + e'$ such that $f_\lambda(e_\iota) > D_\lambda(e_\iota)$ $(\iota \in I)$ and that e' is locally singular, then, using (4.16) and the fact that $f_\lambda(e') = 0, D_\lambda(e') = 0$, we have $1 = f_\lambda(1) \geqq (f_\lambda(e_\iota); \iota \in I) > (D_\lambda(e_\iota); \iota \in I) = D_\lambda(1) = 1$. This leads to a contradiction. Hence, we get the above-mensioned fact. For a positive number θ, we denote by $e \leqq_p \theta$ (or $e \geqq_p \theta$) if $f_\lambda(e_1) \leqq \theta D_\lambda(e_1)$ (or $f_\lambda(e_1) \geqq \theta D_\lambda(e_1)$) for any projection e_1 of eRe after F. J. Murray-J. v. Neumann [2], J. Dixmier [5], Ti. Yen [8], and M. Goldman [9]. Then we can write $e \leqq_p 1$. In this case, moreover, for any decomposition (4.16) in eRe, we have $f_\lambda(e) = \sum(f_\lambda(e_\iota); \iota \in I)$. In fact, by the definition of the sum of a infinite number of positive numbers, for any $\varepsilon > 0$, we can find a finite number of projections (say $e_1, e_2, \cdots, e_n$) of $(e_\iota; \iota \in I)$ such that $D_\lambda(e) < \sum(D_\lambda(e_i); 1 \leqq i \leqq n) + \varepsilon$. Since $D_\lambda(e(\bigoplus(e_i; 1 \leqq i \leqq n))^c) = D_\lambda(e) - \sum(D_\lambda(e_i); 1 \leqq i \leqq n) < \varepsilon$, we can find a positive number β_0 such that $D_\lambda(e(\bigoplus(e_\iota; 1 \leqq i \leqq n))^c) \leqq \beta_0 < \varepsilon$ and that $2^n \beta_0$ is a natural number for some natural number n. In this case, we can find a projection $e^{(0)}$ satisfying that $e(\oplus(e_i; 1 \leqq i \leqq n))^c \leqq e^{(0)}$ and that $D_\lambda(e^{(0)}) = \beta_0$. Since $D_\lambda(e^{(0)}) = \beta_0$, we can find a local decomposition $e^{(0)} = \oplus(e^{(j)}; 1 \leqq j \leqq m)$ for some natural number m, where each $e^{(j)}$ is locally elementary. Hence, we have $f_\lambda(e(\oplus(e_i; 1 \leqq i \leqq n))^c) \leqq f_\lambda(e^{(0)}) = \sum(f_\lambda(e^{(j)}); 1 \leqq j \leqq m) = \sum(D_\lambda(e^{(j)}; 1 \leqq i \leqq m) = D_\lambda(e^{(0)}) = \beta_0 < \varepsilon$. This means that $f_\lambda(e) = \sum(f_\lambda(e_i); 1 \leqq i \leqq n) + f_\lambda(e(\oplus(e_i; 1 \leqq i \leqq n))^c) < \sum(f_\lambda(e_i); 1 \leqq i \leqq n) + \varepsilon$. On the other hand, of course, we have $f_\lambda(e) \geqq \sum(f_\lambda(e_\iota); \iota \in I)$. Thus, we get the desired equality $f_\lambda(e) = \sum(f_\lambda(e_\iota); \iota \in I)$. Put $\theta = \inf(\theta'; e \leqq_p \theta')$. Then, for any positive number η, we may find a locally non-zero and locally elementary projection e_η of R with $\theta - \eta \leqq_p e_\eta$ by a similar argument as in [2]. Hence, we have $|f_\lambda(e_1) - \theta D_\lambda(e_1)| \leqq \eta D_\lambda(e_1)$ for any locally elementary projection e_1 of R with e_1 of R with $e_1 \leqq e_\eta$. Therefore, we get $|f_\lambda(e^{(1)}) - \theta D_\lambda(e^{(1)})| \leqq \eta D_\lambda(e^{(1)})$ for any projection $e^{(1)}$ of R with $e^{(1)} \leqq e_\eta$. For any element a of N, we denote by $(e_\alpha(a); -\infty < \alpha < \infty)$ the resolution of the unit for a. We put $t_\lambda{}^0(a) = \int_{-\infty}^{\infty} \alpha dD_\lambda(e_\alpha(a))$. Then, we have $|f_\lambda(a) - \theta t_\lambda{}^0(a)| \leqq \eta t_\lambda{}^0(\sqrt{a^2}) \leqq \eta \|a\|_\lambda$ for $a \in e_\eta R e_\eta$ by an easy computation. Since $t_\lambda{}^0(a^*a) = t_\lambda{}^0(aa^*)$ by (3.6) of Prop. 3.6, we have $|f_\lambda(a^*a) - f_\lambda(aa^*)| \leqq 2\eta t_\lambda{}^0(a^*a)$, that is, $f_\lambda(aa^*) \leqq (1+\varepsilon) f_\lambda(a^*a)$, where $\varepsilon = 2\eta(\theta - \eta)^{-1} (0 < \eta < \theta)$. We denote the local order of e_η by 2^n (that is, $D_\lambda(e_\eta) = 2^{-n}$). Then, we may find a system $(e_{ij}; 1 \leqq i, j \leqq 2^n)$ of matrix units of R with $e_\eta = e_{11}$. We shall write $g_\lambda(a) = \sum_{j=1}^{2^n} f_\lambda(e_{j1}{}^* a e_{j1})$ for any element a of R. Then, g_λ is a local state of R. Moreover, for any element a of N with $a \geqq 0$ and for any unitary element s of R, we have $g_\lambda(s^*as) = \sum_{i,j=1}^{2^n} f_\lambda(e_{j1}{}^* s^* \sqrt{a}\, e_{i1} e_{i1}{}^* \sqrt{a}\, s e_{j1}) \leqq (1+\varepsilon) \sum_{i,j=1}^{2^n} f_\lambda(e_{i1}{}^* \sqrt{a}\, s e_{j1} e_{j1}{}^* s^* \sqrt{a} e_{i1}) = (1+\varepsilon) g_\lambda(a)$. Thus, we have $|g_\lambda(a) - g_\lambda(s^*as)| \leqq \varepsilon \|a\|_\lambda$. Hence, we get $|g_\lambda(a) - g_\lambda(s^*as)| \leqq 2\varepsilon \|a\|_\lambda$ for any element a of N. Therefore, we obtain (4.19).

Next, we shall prove that $\bar{t}_\lambda(a) = \underline{t}_\lambda(a)$ for any element a of N (with $\|a\|_\lambda \leqq 1$). For, otherwise, putting $\bar{t}_\lambda(a) - \underline{t}_\lambda(a) = 7\varepsilon > 0$, we may find $\sigma_i \in \Sigma$ $(i = 1, 2)$ such that $\sigma_1(a) \geqq_\lambda \bar{t}_\lambda(a) - \varepsilon$ and $\underline{t}_\lambda(a) + \varepsilon \geqq_\lambda \sigma_2(a)$ by the definition of $\bar{t}_\lambda, \underline{t}_\lambda$. Since $|g_\lambda(\sigma_1(a)) - g_\lambda(\sigma_2(a))| \leqq 4\varepsilon$ by (4.21), we get $6\varepsilon \geqq t_\lambda(a) - t_\lambda(a)$. This leads to a contradiction. Hence, we have $\bar{t}_\lambda(a) = \underline{t}_\lambda(a)$.

We shall write $t_\lambda(a) = t_\lambda(\Re e(a)) + i t_\lambda(\Im m(a))$, where $\Re e(a) = \frac{1}{2}(a + a^*)$ and $\Im m(a) = \frac{1}{2i}(a - a^*)$. Then, it is easy to see that t_λ satisfies (4.1)-(4.7) by Lemma 4.5. The uniqueness of the local trace of R is an immediate consequence of Prop. 4.2. q. e. d.

As a global form of Def. 4.1, we shall introduce the following

Definition 4.6. *An operator t from R onto R_0 is called a trace of R if it satisfies the following*

(4.22) $t(1) = 1$,

(4.23) $t(a+b)=t(a)+t(b)$ *for* $a, b\in R$,

(4.24) $t(a^*)=t(a)^*$ *for* $a\in R$,

(4.25) $t(a)\geqq 0$ *for* $a\geqq 0$,

(4.26) $t(a_0a)=a_0t(a)$ *for* $a\in R$ *and* $a_0\in R$,

(4.27) $t(ab)=t(ba)$ *for* $a, b\in R$.

It is normal as a property in our sence that, for a projection e, eRe has a trace. The property that R has a local trace with respect to a spectre λ of R is not the local property corresponding to the property that R has a trace, but we have the following

PROPOSITION 4.5. *R has a trace if and only if R has a local trace with respect to any spectre of R.*

PROOF. Necessity. It is easily verified that $\lambda(t(a))$ $(a\in R)$ satisfies the conditions (4.1)-(4.7) of the local trace of R. Sufficiency. By Prop. 4.2, for any a of N and for any n, there exists an element $a_0^{(n)}$ of N_0 and $\sigma_n\in\Sigma$ such that $\|\sigma_n(a)-a_0^{(n)}\|\leqq 1/n$. Then, we have $\|t_\lambda(a)-\lambda(a_0^{(n)})\|\leqq 1/n$ for any spectre λ of R. This means that $a_0^{(n)}$ converges uniformly to an element (say $t(a)$) of R_0. Then, t satisfies (4.20)-(4.24). q. e. d.

We denote by Ω_0 the set of spectres λ's of R such that R has a local trace with respect to λ.

LEMMA 4.8. *Ω_0 is a closed subset of Ω.*

PROOF. Let μ be a limit of a hypersequence Λ of spectres of Ω_0. With the same notations as in the proof of Prop. 4.5, we have $|t_\lambda(a)-\lambda(a_0^{(n)})|\leqq 1/n$ for any spectre λ of Λ. This means that $t_\lambda(a)$ converges to a scalar (say $t_\mu(a)$). Then, t_μ satisfies (4.1)-(4.7). q. e. d.

LEMMA 4.9. *Let (P) be a global condition of R. We assume that R has at least one local trace if R has the condition (P). Then, R has a trace if R has the condition (P).*

PROOF. If $\Omega_0^c\neq\phi$, we may find a non-zero projection e_0 of R_0 such that $\lambda\in\Omega_0$ implies $\lambda(e_0)=0$. Hence, e_0R has no local trace. This contradicts the property of (P). q. e. d.

THEOREM 4.1. *Let R be a finite AW*-algebra with a faithful representation φ on a separable Hilbert space H. Then, there exists a trace of R.*

PROOF. Let $(f_n; 1\leqq n<\infty)$ be an orthogonal basis of H. We denote by the same f_n the state of R defined by $f_n(a)=\langle(a)f_n, f_n\rangle$ for $a\in R$ indifferently, where we denote by $\langle f, g\rangle$ the inner product of f and g in H. We put $f(a)=\sum_{n=1}^{\infty}2^{-n}f_n(a)$. Then, f is a state of R satisfying that $a=0$ if and only if $f(a^*a)=0$. Hence, there exists a locally non-zero and locally elementary projection e of R such that $f(e_1)\leqq D\lambda(e_1)$ for any locally elementary projection e_1 of R with $e_1\leqq e$. We shall prove this fact in the below. If there is no projection such as e_1 in the above, we have a decomposition $1=\oplus(e_\iota;\iota\in I)\oplus e'$ such that $f(e_\iota)>D_\lambda(e_\iota)$ $(\iota\in I)$ and that e' is locally singular. Then, using (4.16) and the fact that $D_\lambda(e')=0$, we have $I\geqq f(1)\geqq\sum(f(e_\iota);\iota\in I)>\sum(D_\lambda(e_\iota);\iota\in I)=D(1)=1$. This is a contradiction. Hence, we get the desired projection e. As we have only to prove the existence of a local trace by Prop. 4.5 and as, for that sake, we need to show it for eRe by Lemma 4.7, we may assume that $e=1$. And moreover, from what we have seen just above, we may assume that $\theta f(\sigma(e))\leqq D_\lambda(e)$ (θ: some constant) for any $\sigma\in\Sigma$ and for any locally elementary projection e of R.

If R has no local trace with respect to any spectre of R, then, for any elementary projection e with $e^\natural=1$ and for any $\varepsilon>0$, we may find an operator σ of Σ such that $\sigma(e)\geqq 1-\varepsilon$. Hence, we get $D_\lambda(e)\geqq\theta(1-\varepsilon)$. As we can make $D_\lambda(e)\downarrow 0$, this leads to a contradiction. Thus, R has at least one local trace. Since it is a global condition that R has a faithful representation on a separable Hilbert space, we arrive at the assertion by Lemma 4.9. q.e.d.

Ti. Yen [8] proved that a finite AW^*-algebra has a trace if it has a complete set of p-normal states. The following theorem contains this result of Ti. Yen.

THEOREM 4.2. *Let R be a finite AW^*-algebra with a complete set of states f's of R such that, for any orthogonal system E_1 of projections of R, $f(\oplus(e_1;e_1\in E_1))=0$ follows from $f(e_1)=0$ for each $e_1\in E_1$. Then, R has a trace.*

PROOF. Let Γ be a complete set of states of R. For any $f\in\Gamma$, by the assumption for f, there exists the minimal projections $e(f)$ of E and $e_0(f)$ of E_0 fixing f (cf. Lemma 1.3, [15]), where we say that a projection e fixes f if $f(e^c)=0$. It is easy to see that f is a state of $e(f)Re(f)$ satisfying that $a=0$ if and only if $f(a^*a)=0$. Hence, there exists a trace of $e(f)Re(f)$ by the proof of Theorem 4.1. Hence, there exists a local trace of $e(f)Re(f)$ with respect to any spectre of $e(f)Re(f)$ by Prop. 4.5. From this, together with Lemma 4.2, Lemma 4.7, it follows that there exists a local trace of $e_0(f)Re_0(f)$ with respect to some spectre of $e_0(f)Re_0(f)$, and hence there exists a local trace of R with respect to some spectre of R.

For any projection e_0 of R_0 and for any state f of Γ, we denote by f_{e_0} the state of e_0Re_0 defined by $f_{e_0}(a)=f(a)$ for any $a\in e_0Re_0$. Then, it is obvious that the system $(f_{e_0};f\in\Gamma)$ forms a complete set of states of e_0Re_0 satisfying the condition stated in the theorem. This means that the present condition imposed on R is a global condition. Thus, we arrive at the assertion by Lemma 4.9. q. e. d.

M. Goldman [9] proved that a finite AW^*-algebra R has a trace if it has a complete set of p-normal C-valued states, where C is a commutative AW^*-algebra contained in the center of R. The following theorem contains this result of M. Goldman.

THEOREM 4.3. *Let R be a finite AW^*-algebra and C be a commutative AW^*-algebra contained in the center R_0 of R. Further, we assume that R has a complete set of C-valued states f's such that f is completely additive on E_0 and that, for any orthogonal system E_1 of projections of R, $f(\oplus(e_1; e_1 \in E_1))=0$ follows from $f(e_1)=0$ for each $e_1 \in E_1$. Then, R has a trace.*

PROOF. Let Γ be a complete set of such states of R. For any $f \in \Gamma$, there exist the minimal projections $e(f)$ of E and $e_0(f)$ of E_0 fixing f (by the similar method as in the proof of Lemma 1.3, [15]), where we say that a projection e fixes f if $f(e^c)=0$. It is easy to see that f is a state of $e(f)Re(f)$ satisfying that $a=0$ if and only if $f(a^*a)=0$. By a similar argument as in the proof of theorem 4.2, there exists a trace of $e(f)Re(f)$. Hence, R has a trace by the same argument as in the proof of theorem 4.2. q. e. d.

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