Journal of the Mathematical Society of Japan Vol. 10, No. 1, January, 1958

On the local property of the absolute summability $|C, \alpha|$ for Fourier series.

By Mineo KIYOHARA[1]

(Received Oct. 1, 1957)

1. V. A. Magarik [4] has generalized Wiener's theorem[2] on the absolute convergence of Fourier series to the absolute summability $|C, \alpha|$. His assertion is as follows:

Let $f(x)$ be Lebesgue integrable in the interval $(-\pi, \pi)$ and periodic with period 2π. If at every point y on the closed interval $[-\pi, \pi]$ there are a function $g_y(x)$ and a $\delta>0$ such that (i) *$g_y(x)=f(x)$ for $|x-y|<\delta$, and* (ii) *both the Fourier series of $g_y(x)$ and its conjugate series are absolutely summable $|C, \alpha|$,*[3] *then the Fourier series of $f(x)$ is absolutely summable $|C, \alpha|$, where $\alpha \geqq 0$.*

For the case $\alpha=1$, W. C. Randels [5] proved this proposition without the condition on the absolute summability $|C, 1|$ for the conjugate series.

In the present note, we shall show that the condition on the absolute summability for the conjugate series is also superfluous for the general case; that is, the following theorem will be established.

THEOREM. *Let $f(x)$ be Lebesgue integrable in the interval $(-\pi, \pi)$ and periodic with period 2π. If at every point y on the closed interval $[-\pi, \pi]$ there are a function $g_y(x)$ and a $\delta>0$ such that* (i) *$g_y(x)=f(x)$ for $|x-y|<\delta$ and* (ii) *the Fourier series of $g_y(x)$ is absolutely summable $|C, \alpha|$, then the Fourier series of $f(x)$ is absolutely summable $|C, \alpha|$, where $\alpha \geqq 0$.*

2. The case for $\alpha>1$ of our theorem follows immediately from the known theorem of L. S. Bosanquet [1]:

The absolute summability $|C, \alpha|$, $\alpha>1$, for Fourier series of a Lebesgue integrable function with period 2π at a point $x=x_0$ depends only on the behaviour of the generating function in the neighbourhood of the point x_0.

On the other hand, L. S. Bosanquet and H. Kestelman [2] proved that *the mentioned result of L. S. Bosanquet does not hold for $\alpha=1$.*

Thus, it is the case $0 \leqq \alpha \leqq 1$ in which we are interested. However, it

1) The author wishes to thank Dr. S. Yano for his valuable advice during the preparation of this paper.

2) A. Zygmund [6], p. 140.

3) For the definition of absolute summability $|C, \alpha|$, see below.

will be sufficient to prove our theorem only for $0<\alpha<1$, because the cases for $\alpha=0$ and $\alpha=1$ were already proved by N. Wiener and W. C. Randels respectively.

We must now make some general remarks about the absolute summability $|C,\alpha|$.

Let α be any real number and put

$$(1)\qquad \begin{aligned} A_n^\alpha &= \binom{\alpha+n}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \qquad (n>0) \\ A_0^\alpha &= 1\,. \end{aligned}$$

Then, it is well known[4] that if $\alpha\neq-1,-2,\cdots$

$$(2)\qquad A_n^\alpha \cong \frac{n^\alpha}{\Gamma(\alpha+1)}\,.$$

For any given series $\sum_{n=0}^{\infty} x_n$ and any $n\geqq 0$ we write

$$(3)\qquad \sigma_n^\alpha = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^\alpha x_k\,.$$

The series $\sum_{n=0}^{\infty} x_n$ is said to be *absolutely summable* $|C,\alpha|$, if the series

$$(4)\qquad \sum_{n=1}^{\infty} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

is convergent.

We shall need the next lemmas which are due to M. E. Kogbetliantz [3].

LEMMA 1. *The series* $\sum_{n=0}^{\infty} x_n$ *is absolutely summable* $|C,\alpha|$ *if and only if the series*

$$(5)\qquad \sum_{n=0}^{\infty} \frac{1}{A_n^{\alpha+1}} \Big|\sum_{k=0}^{n} k A_{n-k}^{\alpha-1} x_k\Big|$$

is convergent, where $\alpha\neq-1,-2,\cdots$.

LEMMA 2. *If the series* $\sum_{n=0}^{\infty} x_n$ *is absolutely summable* $|C,\alpha|$, *then the series*

$$(6)\qquad \sum_{n=0}^{\infty} \frac{|x_n|}{(n+1)^\alpha}$$

is convergent, where $\alpha\geqq 0$.

In order to apply this definition to Fourier series in the exponential form we put

---

4) A. Zygmund [6], p. 42.

(7) $$x_n=(c_n e^{inx}+c_{-n}e^{-inx}).$$

From Lemma 2 it follows that *if a Fourier series is absolutely summable $|C, \alpha|$ over any interval (a, b), then the series*

(8) $$\sum_{n=-\infty}^{\infty} \frac{|c_n|}{(|n|+1)^{\alpha}}$$

is convergent, where c_n's are the Fourier coefficients and $\alpha \geqq 0$.

3. Let us proceed to the proof of Theorem. We may suppose $0<\alpha<1$ as it was remarked in the above section.

By the Heine-Borel covering theorem and the hypothesis of our theorem there exist a finite number of overlapping intervals (δ_i, δ_i') covering $(-\pi, \pi)$ and functions $g_i(x)$ such that the Fourier series of $g_i(x)$ is absolutely summable $|C, \alpha|$ and $g_i(x)=f(x)$ on (δ_i, δ_i'). These intervals may be chosen so that $\delta_i<\delta_{i-1}'<\delta_{i+1}<\delta_i'$.

The functions $h_i(x)$ are now defined by

$$h_i(x)=\begin{cases} A_i(x-\delta_i)^3+B_i(x-\delta_i)^2, & \delta_i\leqq x<\delta_{i-1}', \\ 1, & \delta_{i-1}'\leqq x<\delta_{i+1}, \\ 1-h_{i+1}(x), & \delta_{i+1}\leqq x<\delta_i', \\ 0, & x<\delta_i \text{ or } \delta_i'\leqq x, \\ h_i(x+2\pi), & \end{cases}$$

where A_i, B_i are defined by the relations

(9) $$3A_i(\delta_{i-1}'-\delta_i)+2B_i=0,$$
$$A_i(\delta_{i-1}'-\delta_i)^3+B_i(\delta_{i-1}'-\delta_i)^2=1.$$

The second relation of (9) implies that $h_i(x)$ is continuous, and by the first relation we see that

$$h_i'(\delta_i)=h_i'(\delta_{i-1}')=h_i'(\delta_{i+1})=h_i'(\delta_i)=0,$$

so that $h_i'(x)$ is absolutely continuous and $h''(x)$ is of bounded variation. Therefore the Fourier coefficients of $h_i(x)$ are

$$c_n(h_i)=\frac{1}{2\pi}\int_{-\pi}^{\pi} h_i(x)e^{-inx}dx$$
$$=-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h_i''(x)}{n^2}\, e^{-inx}dx$$

(10) $$=O(n^{-3}).$$

It is also clear that

$$\sum h_i(x)=1$$

and

(11) $$f(x)=\sum_i g_i(x)h_i(x).$$

The Fourier coefficients of $g_i(x)h_i(x)$ will be given by

(12) $$c_n(g_i\cdot h_i)=\sum_{m=-\infty}^{\infty} c_m(h_i)c_{n-m}(g_i)$$

where the series on the right hand side is convergent since $h_i(x)$ is of bounded variation. For convenience we put

$$c_n(g_i\cdot h_i)=c_n, \quad c_n(h_i)=b_n, \quad c_n(g_i)=a_n.$$

Then from Lemma 1 and (11) we have to consider

(13)
$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=0}^{n} kA_{n-k}^{\alpha-1}(c_ke^{ikx}+c_{-k}e^{-ikx})\right| \\
&=\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}c_ke^{ikx}\right| \\
&=\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}\sum_{m=-\infty}^{\infty}b_m a_{k-m}e^{ikx}\right| \\
&\leqq\sum_{m=-\infty}^{\infty}|b_m|\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}a_{k-m}e^{ikx}\right|.
\end{aligned}$$

Now let us put

$$\rho(m)=\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}a_{k-m}e^{ikx}\right|.$$

If $m>0$, we have

$$\rho(m)=\sum_{n=0}^{2m}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}a_{k-m}e^{ikx}\right|+\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}a_{k-m}e^{ikx}\right|$$

(14) $$=I_1+I_2, \quad \text{say.}$$

Since a_m, being Fourier coefficients of g_i, is bounded, we get

(15)
$$\begin{aligned}
I_1&=\sum_{n=0}^{2m}\frac{1}{A_n^{\alpha+1}}\left|\sum_{k=1}^{n} kA_{n-k}^{\alpha-1}(a_{k-m}e^{i(k-m)x}+a_{-k-m}e^{i(-k-m)x})\right| \\
&\leqq\sum_{n=0}^{2m}\frac{1}{A_n^{\alpha+1}}\left\{\sum_{k=1}^{n} kA_{n-k}^{\alpha-1}(|a_{k-m}|+|a_{-k-m}|)\right\} \\
&=O\left\{\sum_{n=0}^{2m}\frac{1}{A_n^{\alpha+1}}\sum_{k=1}^{n} kA_{n-k}^{\alpha-1}\right\} \\
&=O\left\{\sum_{n=0}^{2m}\frac{1}{(n+1)^{\alpha+1}}(n+1)\sum_{k=1}^{n}\frac{1}{(n-k+1)^{1-\alpha}}\right\} \\
&=O\left\{\sum_{n=0}^{2m}\frac{1}{(n+1)^{\alpha+1}}(n+1)(n+1)^{\alpha}\right\} \\
&=O(m).
\end{aligned}$$

Proceeding to the estimation of the sum I_2, we divide its inner sum into four parts:

$$\sum_{k=-n}^{n} |k| A_{n-|k|}^{\alpha-1} a_{k-m} e^{i(k-m)x}$$

$$= \sum_{k=-n}^{-n+m} |k| A_{n-|k|}^{\alpha-1} a_{k-m} e^{i(k-m)x} + \sum_{k=-n+m}^{n+m} |k-m| A_{n-|k-m|}^{\alpha-1} a_{k-m} e^{i(k-m)x}$$

$$+ \sum_{k=-n+m}^{n+m} |k-m| (A_{n-|k|}^{\alpha-1} - A_{n-|k-m|}^{\alpha-1}) a_{k-m} e^{i(k-m)x}$$

$$+ \sum_{k=-n+m}^{n+m} (|k|-|k-m|) A_{n-|k|}^{\alpha-1} a_{k-m} e^{i(k-m)x}$$

$$= J_1 + J_2 + J_3 + J_4, \quad \text{say,}$$

where we set

(16) $$A_k^{\alpha-1} = 0 \quad \text{if} \quad k = -1, -2, \cdots.$$

Then we have

$$I_2 = \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |J_1 + J_2 + J_3 + J_4|$$

$$\leqq \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} (|J_1| + |J_2| + |J_3| + |J_4|)$$

(17) $$= K_1 + K_2 + K_3 + K_4, \quad \text{say.}$$

The estimation of K_1 will go as follows:

$$K_1 = \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} \left| \sum_{k=-n}^{-n+m} |k| A_{n-|k|}^{\alpha-1} a_{k-m} e^{i(k-m)x} \right|$$

$$\leqq \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} \sum_{k=n-m}^{n} k A_{n-k}^{\alpha-1} |a_{-k-m}|$$

$$= O\{ \sum_{n=2m+1}^{\infty} \frac{1}{n^{\alpha+1}} \sum_{k=n-m}^{n} k(n-k+1)^{\alpha-1} |a_{-k-m}| \}$$

$$= O\{ \sum_{n=2m+1}^{\infty} \frac{1}{n^{\alpha+1}}\, n \sum_{k=0}^{m} (m+1-k)^{\alpha-1} |a_{-(k+n)}| \}$$

$$= O\left\{ \sum_{k=0}^{m} (m+1-k)^{\alpha-1} \sum_{n=2m+1}^{\infty} \frac{|a_{-(k+n)}|}{n^{\alpha}} \right\}$$

$$= O\left\{ \sum_{k=0}^{m} (m+1-k)^{\alpha-1} \sum_{n=2m+1}^{\infty} \frac{|a_{-(k+n)}|}{(k+n)^{\alpha}} \left(\frac{k+n}{n} \right)^{\alpha} \right\}.$$

Since $k \leqq 2m < n$, $\left(\frac{k+n}{n}\right)^{\alpha}$ does not exceed a constant not depending on k, m and n. Therefore

$$K_1=O\Big\{\sum_{k=0}^{m}(m+1-k)^{\alpha-1}\sum_{n=2m+k+1}^{\infty}\frac{|a_{-n}|}{n^\alpha}\Big\}$$

$$=O\Big\{\sum_{k=0}^{m}(m+1-k)^{\alpha-1}\sum_{n=1}^{\infty}\frac{|a_{-n}|}{n^\alpha}\Big\}$$

$$=O\Big\{m^\alpha\sum_{n=1}^{\infty}\frac{|a_{-n}|}{n^\alpha}\Big\}.$$

The absolute summability $|C,\alpha|$ for Fourier series of $g_i(x)$ gives

$$\sum_{n=1}^{\infty}\frac{|a_{-n}|}{n^\alpha}<\infty$$

by (8). Hence we get

$$K_1=O(m^\alpha). \tag{18}$$

The sum K_2 will be easily estimated by using the assumption of $g_i(x)$ and Lemma 1:

$$K_2=\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\Big|\sum_{k=-n+m}^{n+m}|k-m|A_{n-|k-m|}^{\alpha-1}a_{k-m}e^{i(k-m)x}\Big|$$

$$=\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\Big|\sum_{k=-n}^{n}|k|A_{n-|k|}^{\alpha-1}a_ke^{ikx}\Big|$$

$$=O(1). \tag{19}$$

To estimate the sum K_3, its inner sum J_3 will be further divided into two parts:

$$J_3=\sum_{k=-n+m}^{n+m}|k-m|(A_{n-|k|}^{\alpha-1}-A_{n-|k-m|}^{\alpha-1})a_{k-m}e^{i(k-m)x}$$

$$=\sum_{k=0}^{2n}|k-n|(A_{n-|k-n+m|}^{\alpha-1}-A_{n-|k-n|}^{\alpha-1})a_{k-n}e^{i(k-n)x}$$

$$=\sum_{k=0}^{n}+\sum_{k=n+1}^{2n}=L_1+L_2,\qquad\text{say.}$$

Then we have

$$K_3\leqq\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}|L_1|+\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}|L_2|. \tag{20}$$

We shall first estimate the first sum on the right hand side of this inequality:

$$\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}|L_1|\leqq\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\sum_{k=0}^{n}|k-n|\,|A_{n-|k-n+m|}^{\alpha-1}-A_{n-|k-n|}^{\alpha-1}|\,|a_{k-n}|$$

$$=\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\sum_{k=0}^{n-m}(n-k)\,|A_{k+m}^{\alpha-1}-A_k^{\alpha-1}|\,|a_{k-n}|+$$

$$+\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\sum_{k=n-m+1}^{n}(n-k)\,|A_{2n-m-k}^{\alpha-1}-A_k^{\alpha-1}|\,|a_{k-n}|$$

$$=M_1+M_2, \qquad \text{say.}$$

Since

(21) $$|A_{k+m}^{\alpha-1}-A_k^{\alpha-1}|=|\sum_{j=0}^{m-1}A_{k+1+j}^{\alpha-2}|=O\Big(\frac{m}{(k+1)^{2-\alpha}}\Big),$$

we have

$$M_1=O\Big\{\sum_{n=2m+1}^{\infty}\frac{1}{n^{\alpha+1}}\sum_{k=0}^{n-m}(n-k)\frac{m}{(k+1)^{2-\alpha}}|a_{k-n}|\Big\}$$

$$=O\Big\{m\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2-\alpha}}\sum_{n=k+m}^{\infty}\frac{(n-k)}{n^{\alpha+1}}|a_{k-n}|\Big\}$$

$$=O\Big\{m\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2-\alpha}}\sum_{n=0}^{\infty}\frac{(m+n)^{\alpha+1}|a_{-m-n}|}{(k+m+n)^{\alpha+1}(m+n)^{\alpha}}\Big\}$$

$$=O\Big\{m\sum_{k=0}^{\infty}\frac{1}{(k+1)^{2-\alpha}}\sum_{n=1}^{\infty}\frac{|a_{-n}|}{n^{\alpha}}\Big\}$$

$$=O\Big\{m\sum_{n=1}^{\infty}\frac{|a_{-n}|}{n^{\alpha}}\Big\}.$$

For the sum M_2, we have

$$M_2=\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\sum_{k=n-m+1}^{n}(n-k)\,|A_{2n-m-k}^{\alpha-1}-A_k^{\alpha-1}|\,|a_{k-n}|$$

$$=\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}\sum_{k=1}^{m}(m-k)\,|A_{n-k}^{\alpha-1}-A_{n-m+k}^{\alpha-1}|\,|a_{-m+k}|$$

$$=O\Big\{\sum_{n=2m+1}^{\infty}\frac{1}{n^{\alpha+1}}\sum_{k=1}^{m}(m-k)\frac{1}{(n-m+k)^{1-\alpha}}|a_{-m+k}|\Big\}$$

$$=O\Big\{\sum_{n=2m+1}^{\infty}\frac{1}{n^{\alpha+1}}\sum_{k=1}^{m}\frac{(m-k+1)^{\alpha+1}|a_{-m+k}|}{(n-m+k)^{1-\alpha}(m-k+1)^{\alpha}}\Big\}$$

$$=O\Big\{m^{\alpha+1}\sum_{n=2m+1}^{\infty}\frac{1}{n^{\alpha+1}(n-m)^{1-\alpha}}\sum_{k=1}^{\infty}\frac{|a_{-k}|}{k^{\alpha}}\Big\}$$

$$=O\Big\{m^{\alpha+1}\sum_{n=2m+1}^{\infty}\frac{1}{(n-m)^2}\sum_{k=1}^{\infty}\frac{|a_{-k}|}{k^{\alpha}}\Big\}$$

$$=O\Big\{m^{\alpha}\sum_{n=1}^{\infty}\frac{|a_{-k}|}{k^{\alpha}}\Big\}.$$

By the assumption of $g_i(x)$ and (8) we have

$$M_1=O(m),$$
$$M_2=O(m^{\alpha})=O(m).$$

Hence we get

(22) $$\sum_{n=2m+1}^{\infty}\frac{1}{A_n^{\alpha+1}}|L_1|=O(m).$$

Similarly

(23) $$\sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |L_2| = O(m).$$

Thus (20), (22) and (23) give

(24) $$K_3 = O(m).$$

Finally, let us estimate the sum K_4. Taking into account that

$$\begin{aligned} |J_4| &= \Big|\sum_{k=-n+m}^{n+m} (|k| - |k-m|) A_{n-|k|}^{\alpha-1} a_{k-m} e^{i(k-m)x}\Big| \\ &\leqq m \sum_{k=-n+m}^{n+m} A_{n-|k|}^{\alpha-1} |a_{k-m}| \\ &= m \sum_{k=-n}^{n} A_{n-|m+k|}^{\alpha-1} |a_k| \\ &= m \sum_{k=0}^{n-m} A_{n-(m+k)}^{\alpha-1} |a_k| + m \sum_{k=1}^{n} A_{n-|m-k|}^{\alpha-1} |a_{-k}| \\ &= N_1 + N_2, \quad \text{say,} \end{aligned}$$

we obtain

(25) $$K_4 \leqq \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |N_1| + \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |N_2|.$$

For the first sum on the right hand side of (25) we have

$$\begin{aligned} \sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |N_1| &= O\Big\{m \sum_{n=2m+1}^{\infty} \frac{1}{(n+1)^{\alpha+1}} \sum_{k=0}^{n-m} (n-m-k+1)^{\alpha-1} |a_k|\Big\} \\ &= O\Big\{m \sum_{k=0}^{\infty} |a_k| \sum_{n=m+k}^{\infty} \frac{1}{(n+1)^{\alpha+1}}\Big\} \\ &= O\Big\{m \sum_{k=0}^{\infty} \frac{|a_k|}{(m+k)^{\alpha}}\Big\} \end{aligned}$$

(26) $$= O\Big\{m \sum_{k=1}^{\infty} \frac{|a_k|}{k^{\alpha}}\Big\}.$$

Similarly

(27) $$\sum_{n=2m+1}^{\infty} \frac{1}{A_n^{\alpha+1}} |N_2| = O\Big\{m \sum_{k=1}^{\infty} \frac{|a_k|}{k^{\alpha}}\Big\}.$$

But by the assumption the series $\sum_{k=1}^{\infty} \frac{|a_k|}{k^{\alpha}}$ converges. Hence from (25), (26) and (27) we get

(28) $$K_4 = O(m).$$

Combining the above estimations (17), (18), (19), (24) and, (28) we obtain

(29) $$I_2 = O(m).$$

Therefore, if $m>0$, we have

(30) $$\rho(m)=O(m)$$

by (14), (15) and (29).

A similar result for $m\leqq 0$ can be proved in exactly the same manner, and therefore by (10)

$$\sum_{m=-\infty}^{\infty}|b_m|\,\rho(m)=O\Big\{\sum_{m=-\infty}^{\infty}|m|^{-3}|m|\Big\}$$

$$=O\Big\{\sum_{m=-\infty}^{\infty}|m|^{-2}\Big\}<\infty\,,$$

so that by (13)

(31) $$\sum_{n=0}^{\infty}\frac{1}{A_n^{\alpha+1}}\Big|\sum_{k=0}^{n}kA_{n-k}^{\alpha-1}(c_k e^{ikx}+c_{-k}e^{-ikx})\Big|<\infty\,.$$

From this and Lemma 1 it follows that the Fourier series of $h_i(x)g_i(x)$ is absolutely summable $|C,\alpha|$, and so by (11) $f(x)$, being the sum of a finite number of functions having Fourier series which are absolutely summable $|C,\alpha|$, must also have a Fourier series which is absolutely summable $|C,\alpha|$, where $0<\alpha<1$.

This completes the proof of our theorem.

Mathematical Institute,
Tokyo Metropolitan University.

References

[1] L. S. Bosanquet, The absolute Cesàro summability of Fourier series, Proc. London Math. Soc., **11** (1936), pp. 517-528.

[2] L. S. Bosanquet and H. Kestelman, The absolute convergence of series of integrals, Proc. London Math. Soc., (2) **45** (1939), pp. 88-97.

[3] M. E. Kogbetliantz, Sur les séries absolument sommables par la méthode des moyennes arithmétiques, Bull. Sci. Math., (2) **49** (1925), pp. 234-256.

[4] V. A. Magarik, On the summability $|C, \alpha|$ of Fourier series, Moskv. Gos. Univ. Uc. Zap. 181, Mat., **8** (1956), pp. 183-196. (in Russian).

[5] W. C. Randels, On the absolute summability of Fourier Series, Duke Math. J., **7** (1940), pp. 204-207.

[6] A. Zygmund, Trigonometrical series, Warszawa, (1935).