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A reciprocity law of the power residue symbol.

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Let l be a positive rational prime number and k be an algebraic number field of finite degree, containing a primitive l-th root ζ of unity. Denote by ζ_n a primitive l^n-th root of unity, and set $k_{(n)}=k(\zeta_n)$.

Let $\mathfrak{p}$ be a prime ideal of k prime to l, and α be an element of k prime to $\mathfrak{p}$. For these α and $\mathfrak{p}$, we define the symbol $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ inductively as follows.

For $n=0$, we set always $\left[\frac{\alpha}{\mathfrak{p}}\right]_n=1$.

For $n\geqq 1$, this symbol is defined only when we have

$$l^n \mid N\mathfrak{p}-1 \quad \text{and} \quad \left[\frac{\alpha}{\mathfrak{p}}\right]_{n-1}=1, \tag{1}$$

and, if that is so, we set

$$\left[\frac{\alpha}{\mathfrak{p}}\right]_n=\zeta^x \tag{2}$$

whenever we have

$$\alpha^{\frac{N\mathfrak{p}-1}{l^n}} \equiv \zeta^x \pmod{\mathfrak{p}}. \tag{3}$$

Since every l-th root of unity is mutually incongruent modulo $\mathfrak{p}$, the value of $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ is uniquely determined in k by (2) and (3).

If $\mathfrak{m}$ is an ideal of k prime to α and to l with the prime ideal decomposition

$$\mathfrak{m}=\mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_r^{m_r},$$

then we set

$$\left[\frac{\alpha}{\mathfrak{m}}\right]_n=\left[\frac{\alpha}{\mathfrak{p}_1}\right]_n^{m_1}\cdots\left[\frac{\alpha}{\mathfrak{p}_r}\right]_n^{m_r}.$$

The symbol $\left[\frac{\alpha}{\mathfrak{m}}\right]_n$ is considered to generalize the Diriclet's 4-th power residue symbol and, as we see in the latter half of § 1, it is closely related to the "restricted Artin's symbol" in Rédei [4].

In Kuroda [3], a reciprocity law of the 4-th power residue symbol is given and, as an application, the decomposition law of rational primes in

some non-abelian field is obtained. In the present paper, we shall have a generalization of the reciprocity law of Kuroda [3].

§ 1. Fundamental properties of the power residue symbol.

By the definition, we have immediately the following

LEMMA 1. *We have*

$$\left[\frac{\alpha}{\mathfrak{m}_1}\right]_n\left[\frac{\alpha}{\mathfrak{m}_2}\right]_n=\left[\frac{\alpha}{\mathfrak{m}_1\mathfrak{m}_2}\right]_n$$

$$\left[\frac{\alpha_1}{\mathfrak{m}}\right]_n\left[\frac{\alpha_2}{\mathfrak{m}}\right]_n=\left[\frac{\alpha_1\alpha_2}{\mathfrak{m}}\right]_n \quad \textit{and}$$

$$\left[\frac{\alpha}{\mathfrak{m}}\right]_{n+d}^{l^d}=\left[\frac{\alpha}{\mathfrak{m}}\right]_n .$$

We have also

LEMMA 2. *We have* $\left[\frac{\alpha}{\mathfrak{p}}\right]_n=1$ *if and only if* $\mathfrak{p}$ *decomposes completely in* $k_{(n)}(\sqrt[l^n]{\alpha})$.

PROOF. In order that $\mathfrak{p}$ decomposes completely in $k_{(n)}(\sqrt[l^n]{\alpha})$, it is necessary and sufficient that we have $l^n \mid N\mathfrak{p}-1$ and $\alpha^{\frac{N\mathfrak{p}-1}{l^n}}\equiv 1 \pmod{\mathfrak{p}}$, which is, by definition, equivalent with $\left[\frac{\alpha}{\mathfrak{p}}\right]_n=1$.

Now, set $K^{(n)}=k_{(n)}(\sqrt[l^n]{\alpha})$, $\omega=\sqrt[l^n]{\alpha}$ and let ψ be the character of the Galois group $\mathfrak{A}$ of $K^{(n)}/k_{(n)}$ such that

$$\psi(\sigma)=\frac{\omega^\sigma}{\omega} \tag{4}$$

for every element σ of $\mathfrak{A}$. Then the character group of $\mathfrak{A}$ is generated by ψ. If $\mathfrak{P}$ is a prime ideal of $k_{(n)}$ and if we have $\left(\frac{K^{(n)}/k_{(n)}}{\mathfrak{P}}\right)=\sigma$, then, by the generalized Euler's criterion[1], we have

$$\frac{\omega^\sigma}{\omega}\equiv\alpha^{\frac{N\mathfrak{P}-1}{l^n}} \pmod{\mathfrak{P}}.$$

Moreover if $\mathfrak{p}$ is a prime ideal of k which is divisible by $\mathfrak{P}$ and satisfies the condition (1), then we have $N\mathfrak{P}=N\mathfrak{p}$ and we see that the right hand side of (4) is congruent to a l-th root of unity. Therefore, if we set

$$\psi(\mathfrak{P})=\psi\left(\left(\frac{K^{(n)}/k_{(n)}}{\mathfrak{P}}\right)\right)=\psi(\sigma), \tag{5}$$

1) See Hasse [1], II, p. 50.

then we have

$$\psi(\mathfrak{P})=\alpha^{\frac{N\mathfrak{p}-1}{l^n}} \pmod{\mathfrak{p}}.$$

By (2), (4) and (5), we have

LEMMA 3. *If* $l^n \mid N\mathfrak{p}-1$ *and* $\left[\frac{\alpha}{\mathfrak{p}}\right]_{n-1}=1$, *then we have*

$$\left[\frac{\alpha}{\mathfrak{p}}\right]_n=\psi(\mathfrak{P})=\left(\frac{\alpha}{\mathfrak{P}}\right)_{l^n},$$

where $\left(\frac{\alpha}{\mathfrak{P}}\right)_{l^n}$ *is the ordinary* l^n*-th power residue symbol in* $k_{(n)}$.

Now, for a prime ideal $\mathfrak{q}$ of k, let $\tilde{\mathfrak{q}}$ be the product of $\mathfrak{q}$ by all the infinite places of k, and call $\mathfrak{q}$ the finite part of $\tilde{\mathfrak{q}}$. For such a $\tilde{\mathfrak{q}}$, we define the symbol $\left[\frac{\alpha}{\tilde{\mathfrak{q}}}\right]_n$ by $\left[\frac{\alpha}{\tilde{\mathfrak{q}}}\right]_n=\left[\frac{\alpha}{\mathfrak{q}}\right]_n$ only when α is totally positive.

Assume that $\left[\frac{\alpha}{\tilde{\mathfrak{q}}}\right]_n$ is defined and that we have $\left[\frac{\varepsilon}{\mathfrak{q}}\right]_n=1$ for all totally positive units ε of k. Then we may set

$$\left[\frac{(\alpha)}{\tilde{\mathfrak{q}}}\right]_n=\left[\frac{\alpha}{\mathfrak{q}}\right]_n \tag{6}$$

because, for a principal ideal (α) where α is totally positive, the value of $\left[\frac{(\alpha)}{\tilde{\mathfrak{q}}}\right]_n$ is uniquely defined by (6).

For every $\tilde{\mathfrak{q}}$, all the principal ideals (α) with $\left[\frac{\alpha}{\tilde{\mathfrak{q}}}\right]_n=1$ form a congruence group $H^{(n)}$ of k defined modulo $\tilde{\mathfrak{q}}$. Thus we obtain the class field $A^{(n)}$ over $H^{(n)}$. In particular, $A^{(0)}$ is, for every $\tilde{\mathfrak{q}}$, the absolute class field over k.[2]

Denote by χ a character of the Galois group $\mathfrak{A}$ of $A^{(n)}/k$. Then, all the characters χ which are defined by means of a similar relation to (5) form the character group of the congruence group modulo $H^{(n)}$. Since the factor group of (α) modulo $H^{(n)}$ is cyclic, we can choose a character χ such that

$$\left[\frac{(\alpha)}{\tilde{\mathfrak{q}}}\right]_n=\chi(\alpha) \tag{7}$$

for every totally positive element of k which is prime to $\mathfrak{q}$. Let l^N be the order of this character χ, and $k_{(n)}$ the field obtained by adjoining to k all the l^N-th roots of unity. Denote by $A_\chi^{(n)}$ the cyclic subfield of $A^{(n)}$ over k corresponding to χ, and by $\overline{A_\chi^{(n)}}$ the composite field of $A_\chi^{(n)}$ and $k_{(N)}$. Then $\overline{A_\chi^{(n)}}$ is a Kummer field over $k_{(N)}$.

2) This means the absolute class field in the narrow sense, i.e., the class field over the ideal group consisting of all the principal ideals generated by totally positive numbers.

For this character χ of the Galois group $\mathfrak{A}$ of $A^{(n)}/k$, let $\bar{\chi}$ stand for the character of the Galois group $\bar{\mathfrak{A}}$ of $\overline{A^{(n)}}/k_{(N)}$ which is induced by

$$\bar{\chi}(\bar{\sigma})=\chi(\sigma), \tag{8}$$

where σ is the restriction of $\bar{\sigma}$ to $A_\chi^{(n)}$. Denote by ω the Kummer generator of $\overline{A_\chi^{(n)}}$ over $k_{(N)}$ corresponding to $\bar{\chi}$, then we have

$$\bar{\chi}(\bar{\sigma})=\frac{\omega^{\bar{\sigma}}}{\omega}$$

for every $\bar{\sigma}$ of $\bar{\mathfrak{A}}$. If we set $w=\omega^{l^N}$, then w is an element of $k_{(N)}$, and, for an ideal $\mathfrak{m}$ prime to the conductor of $\overline{A_\chi^{(n)}}/k_{(N)}$, we have

$$\bar{\chi}(\mathfrak{m})=\left(\frac{w}{\mathfrak{m}}\right)_{l^N}, \tag{9}$$

where the right hand side of (9) stands for the ordinary l^N-th power residue symbol in $k_{(N)}$. In particular, if $N_{k_{(N)}/k}\mathfrak{m}=(\alpha)$, then, by (8), we have

$$\bar{\chi}(\mathfrak{m})=\bar{\chi}\left(\frac{\overline{A_\chi^{(n)}}/k_{(n)}}{\mathfrak{m}}\right)=\chi\left(\frac{A_\chi^{(n)}/k}{N_{k_{(n)}/k}\mathfrak{m}}\right)=\chi(\alpha). \tag{10}$$

By (7), (9) and (10), we have

LEMMA 4. *If* $\left[\frac{(\alpha)}{\mathfrak{q}}\right]_n$ *is defined, and if, for a sufficiently large* N, *we have* $(\alpha)=N_{k_{(N)}/k}\mathfrak{m}$ *with some ideal* $\mathfrak{m}$ *of* $k_{(N)}$, *then we have*

$$\left[\frac{(\alpha)}{\mathfrak{q}}\right]_n=\chi(\alpha)=\left(\frac{w}{\mathfrak{m}}\right)_{l^N}.$$

By lemma 3 and lemma 4, we have the following fact.

Let π and κ be two distinct prime numbers of k prime to l. Suppose that all the totally positive units ε of k are l^n-th power residue modulo κ, and that $N\pi-1$ is divisible by a sufficiently large power l^N of l. Then we have

$$\left[\frac{\pi}{\kappa}\right]_n\left[\frac{\kappa}{\pi}\right]_n=\left(\frac{w\kappa^{l^{N-n}}}{\mathfrak{p}}\right)_{l^N}$$

where $\mathfrak{p}$ is a prime divisor of π in $k_{(N)}$.

§ 2. Reciprocity law in the rational number field.

From now on, we consider the case where $l=2$ and the ground field is the rational number field P. We denote by ζ_n a primitive 2^n-th root of unity and set $P_{(n)}=P(\zeta_n)$.

For a positive rational odd prime number p and a positive rational

integer a prime to p, the symbol $\left[\frac{a}{p}\right]_n$ is simply determined as follows.

$\left[\frac{a}{p}\right]_1$ is equal to the ordinary quadratic power residue symbol $\left(\frac{a}{p}\right)$ in P.

$\left[\frac{a}{p}\right]_n$ is defined only when $p-1$ is divisible by 2^n and a is 2^{n-1}-th power residue modulo p, and, if that is so, $\left[\frac{a}{p}\right]_n$ is equal to 1 or -1 according as a is 2^n-th power residue modulo p or not.

We now extend this symbol to the case of $p=2$ as follows.

$\left[\frac{a}{2}\right]_n$ is defined only when $a\equiv 1 \pmod{2^{n+1}}$, and, if that is so, $\left[\frac{a}{2}\right]_n$ is equal to 1 or -1 according as $a\equiv 1 \pmod{2^{n+2}}$ or not.

Now, let $\mathfrak{l}_n$ be a prime ideal in $P_{(n)}$ dividing 2. Denote by P_2 and by $P_{(n),\mathfrak{l}_n}$ the 2-adic and the $\mathfrak{l}_n$-adic completion of P and of $P_{(n)}$ respectively. Then we have

LEMMA 5. *If an element a, prime to 2, is a 2^n-th power in $P_{(n),\mathfrak{l}_n}$, then a is already a 2^n-th power in P_2.*

PROOF. If U is the unit group of P_2, then the factor group U/U^{2^n} is generated by -1 and 5. On the other hand $\mathfrak{l}_n$ is not decomposed in $P_{(n),\mathfrak{l}_n}(\sqrt[2^n]{-1})/P_{(n),\mathfrak{l}_n}$, ramifies in $P_{(n),\mathfrak{l}_n}(\sqrt{5})/P_{(n),\mathfrak{l}_n}$ and therefore ramifies in $P_{(n),\mathfrak{l}_n}(\sqrt[2^n]{-1\sqrt{5}})/P_{(n),\mathfrak{l}_n}$. Hence, none of -1, 5 and -5 is a 2^n-th power in $P_{(n),\mathfrak{l}_n}$. Our assertion is thereby proved.

THEOREM 1. *We have $\left[\frac{a}{p}\right]_n=1$ if and only if a prime ideal $\mathfrak{p}$ of $P_{(n)}$ which divides p decomposes completely in $P_{(n)}(\sqrt[2^n]{a})/P_{(n)}$.*

PROOF. In the case where p is odd, the assertion is already proved by lemma 2. We shall prove the case of $p=2$. A prime ideal $\mathfrak{l}_n$ of $P_{(n)}$ dividing 2 decomposes completely in $P_{(n)}(\sqrt[2^n]{a})$ if and only if a is a 2^n-th power in $P_{(n),\mathfrak{l}_n}$. By lemma 5, this condition is equivalent with the condition that a is a 2^n-th power in P, because a is an element of P prime to 2. Moreover, a is a 2^n-th power in P_2 if and only if $a\equiv 1 \pmod{2^{n+2}}$, and the latter condition is equivalent with $\left[\frac{a}{2}\right]_n=1$. Thus the theorem is proved.

Now, in the case where the ground field is the rational number field, the number w of the previous section is explicitly written by means of the Gaussian sum.

Let q be a rational positive odd prime number such that $2^n \mid q-1$, and let $\chi_{(n)}$ be a character of order 2^n of the residue class group modulo q.

Denote by $\tau(\chi_{(n)})$ the Gaussian sum for such $\chi_{(n)}$; namely

$$\tau(\chi_{(n)})=\sum_{x \bmod q}\chi_{(n)}(x)\zeta^x$$

where ζ is a primitive q-th root of unity. Then we have[3]

(11) $$\tau(\chi_{(n)})^{2^n}\in P$$

and

(12) $$\overline{A^{(n)}}=A^{(n)}P_{(n)}=P_{(n)}(\tau(\chi_{(n)}))\,,$$

where $A^{(n)}$ is the class field over P corresponding to $H^{(n)}$ which consists of all (a) such that $\left[\frac{a}{q}\right]_n=1$, $a>0$, $a\in P$. If we set

$$\pi(\chi_{(\mu)},\ \chi_{(\nu)})=\sum_{x+y\equiv 1 \bmod q}\chi_{(\mu)}(x)\chi_{(\nu)}(y)\,,$$

then we have

(13) $$\begin{aligned}\tau(\chi_{(n)})^2&=\tau(\chi_{(n)}^2)\pi(\chi_{(n)},\ \chi_{(1)})\\&=(\tilde{\chi}_{(n)}(2))^2\tau(\chi_{(n-1)})\pi(\chi_{(n)},\ \chi_{(1)})\qquad \text{for } n\geqq 2 \text{ and}\end{aligned}$$

(14) $$\tau(\chi_{(1)})^2=\chi_{(1)}(-1)q=\left(\frac{-1}{q}\right)q,$$

where $\left(\frac{-1}{q}\right)$ is the ordinary quadratic power residue symbol in P, and $\tilde{\chi}$ is the complex conjugate of χ. Moreover, if we set

$$\pi_{(n)}=(\tilde{\chi}_{(n)}(2))^2\pi(\chi_{(n)},\ \chi_{(1)})\qquad \text{for } n\geqq 2, \text{ and}$$

$$\omega_q^{(n)}=\pi_{(2)}\pi_{(3)}^2\cdots\pi_{(n)}^{2^{n-2}}\,,$$

then, by (13), we have

$$\tau(\chi_{(n)})^2=\tau(\chi_{(n-1)})\pi_{(n)}\qquad \text{for } n\geqq 2\,.$$

By repeating this process, it follows from (14) that we have

(15) $$\tau(\chi_{(n)})^{2^n}=\left(\frac{-1}{q}\right)q\omega_q^{(n)2}\,.$$

Now we prove

THEOREM 2.[4] *Let p and q be positive rational odd prime numbers. If $\left[\frac{p}{q}\right]_{n-1}=\left[\frac{q}{p}\right]_{n-1}=1$, then we have*

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3) As for these properties of the Gaussian sum, see Hasse [2], § 20.

4) In the case where $n=1$, this is the reciprocity law of the quadratic power residue symbol, and, in the case where $n=2$, that of Kuroda [3]; namely in the latter case we have $\omega_q^{(2)}=\pi_{(2)}=A\pm 2Bi$, $q=A^2+4B^2$ with $A\equiv 1 \pmod 4$, $A, B\in P$.

(16) $$\left[\frac{q}{p}\right]_n\left[\frac{p}{q}\right]_n=\left(\frac{\left(\frac{-1}{q}\right)}{p}\right)\left(\frac{\omega_q^{(n)}}{\mathfrak{p}_n}\right)_{2^{n-1}}=\left(\frac{\left(\frac{-1}{p}\right)}{q}\right)\left(\frac{\omega_p^{(n)}}{\mathfrak{q}_n}\right)_{2^{n-1}}.$$

If $\left[\frac{2}{p}\right]_{n-1}=\left[\frac{p}{2}\right]_{n-1}=1$, *then, for* $n\geqq 2$, *we have*

(17) $$\left[\frac{2}{p}\right]_n\left[\frac{p}{2}\right]_n=\left(\frac{1-i}{\mathfrak{p}_n}\right)_{2^{n-1}}=\left(\frac{\omega_p^{(n)}}{1-\zeta_n}\right)_{2^{n-1}}$$

PROOF. First we rewrite the left hand side of (16) by using the above properties of the Gaussian sum and lemma 4. Namely we choose a character χ such that we have

$$\chi(\sigma)=\frac{\tau(\chi)^\sigma}{\tau(\chi)}$$

for every element σ of the Galois group $\mathfrak{G}(\overline{A^{(n)}}/P_{(n)})$ of $\overline{A^{(n)}}/P_{(n)}$. Then, by (11) and (12), χ is a generating character of order 2^n of $\mathfrak{G}(\overline{A^{(n)}}/P_{(n)})$. Since $\left[\frac{p}{q}\right]_n=\pm 1$, it follows from lemma 4 and (15) that, under the condition of the theorem, we have

(18) $$\left[\frac{p}{q}\right]_n=\left(\frac{\tau(\chi_{(n)})^{2^n}}{\mathfrak{p}_n}\right)_{2^n}=\left(\frac{\left(\frac{-1}{q}\right)q\omega_q^{(n)^2}}{\mathfrak{p}_n}\right)_{2^n}.$$

Next, applying lemma 3 to $K^{(n)}=P_{(n)}(\sqrt[2^n]{q})$, we have under the condition of the theorem

(19) $$\left[\frac{q}{p}\right]_n=\left(\frac{q}{p}\right)_{2^n}.$$

Since we have $\left(\frac{-1}{q}\right)=1$ for $n\geqq 2$, it follows from (18), (19) and lemma 1 that we have

$$\left[\frac{q}{p}\right]_n\left[\frac{p}{q}\right]_n=\left[\frac{q}{p}\right]_n^2\left(\frac{\left(\frac{-1}{q}\right)}{\mathfrak{p}_n}\right)_{2^n}\left(\frac{\omega_q^{(n)}}{\mathfrak{p}_n}\right)_{2^{n-1}}=\left(\frac{\left(\frac{-1}{q}\right)}{p}\right)\left(\frac{\omega_q^{(n)}}{\mathfrak{p}_n}\right)_{2^{n-1}}.$$

Therefore the first equality (16) is proved.

In (18), it is not necessary that p is odd. Furthermore, $(1-\zeta_n)$ is a prime divisor of 2 in $P_{(n)}$, and we have $\left(\frac{-1}{p}\right)=1$ for $n\geqq 2$. Therefore, for $n\geqq 2$, we have

(20) $$\left[\frac{2}{p}\right]_n=\left(\frac{p\omega_p^{(n)^2}}{1-\zeta_n}\right)_{2^n}.$$

On the other hand, since we have $\left[\frac{p}{2}\right]_{n-1}=1$, theorem 1 implies

$$\left(\frac{p}{1-\zeta_n}\right)_{2^{n-1}}=\left(\frac{p}{N_{P_{(n)}/P_{(n-1)}}(1-\zeta_n)}\right)_{2^{n-1}}=\left(\frac{p}{1-\zeta_{n-1}}\right)_{2^{n-1}}=1.$$

Since $\left[\frac{p}{2}\right]_n$ is equal to 1 or -1, we have

$$\left[\frac{p}{2}\right]_n=\left(\frac{p}{1-\zeta_n}\right)_{2^n}. \tag{21}$$

By (20) and (21), we obtain

$$\left[\frac{2}{p}\right]_n\left[\frac{p}{2}\right]_n=\left(\frac{\omega_p^{(n)}}{1-\zeta_n}\right)_{2^{n-1}} \quad \text{for } n\geqq 2.$$

Now, in (19), it is not necessary that q is odd. Therefore we have

$$\left[\frac{2}{p}\right]_n=\left(\frac{2}{\mathfrak{p}_n}\right)_{2^n}. \tag{22}$$

On the other hand, $\left(\frac{i}{\mathfrak{p}_n}\right)_{2^n}=1$ holds if and only if $N\mathfrak{p}_n=p\equiv 1 \pmod{2^{n+2}}$. Hence by the definition of $\left[\frac{p}{2}\right]_n$, we see that, if $\left[\frac{p}{2}\right]_{n-1}=1$, then we have

$$\left[\frac{p}{2}\right]_n=\left(\frac{i}{\mathfrak{p}_n}\right)_{2^n}. \tag{23}$$

By (22) and (23), we have

$$\left[\frac{2}{p}\right]_n\left[\frac{p}{2}\right]_n=\left(\frac{2i}{\mathfrak{p}_n}\right)_{2^n}=\left(\frac{-(1-i)^2}{\mathfrak{p}_n}\right)_{2^n}=\left(\frac{-1}{\mathfrak{p}_n}\right)_{2^n}\left(\frac{1-i}{\mathfrak{p}_n}\right)_{2^{n-1}}.$$

Since, however $\left[\frac{p}{2}\right]_{n-1}=1$ holds and therefore $p\equiv 1 \pmod{2^{n+1}}$, we have $\left(\frac{-1}{\mathfrak{p}_n}\right)_{2^n}=1$ for $n\geqq 2$, whence

$$\left[\frac{2}{p}\right]_n\left[\frac{p}{2}\right]_n=\left(\frac{1-i}{\mathfrak{p}_n}\right)_{2^{n-1}}.$$

Our theorem is thus completely proved.

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References

[1] H. Hasse, Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper, I, Ia, II, Jber. Deutsch. Math. Verein., **35** (1926).
[2] ———, Vorlesungen über Zahlentheorie, Grund. Math. Wiss., **59** (1950).
[3] S. Kuroda, Über die Zerlegung rationaler Primzahlen in gewissen nicht-abelschen galoisschen Korpern, J. Math. Soc. Japan, **3**, 1, Takagi commemoration number, (1951), pp. 148-156.
[4] L. Rédei, Bedingtes Artinsches Symbol mit Anwendung in der Klassenkörpertheorie, Acta Math. Acad. Sci. Hungaricae, **4** (1953), pp. 1-29.