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On the partitions of a number into the powers of prime numbers.

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1. In **1953,** Szekeres [4] proved an asymptotic formula for the number $P(n,m)$ of the partitions of n into positive integers not exceeding m, for large n and m. The generating function of $P(n,m)$ is

$$F(w)=\prod_{\nu=1}^{m}(1-w^{\nu})^{-1}=\sum_{n=0}^{\infty}P(n,m)w^{n} \qquad (|w|<1)$$

and we have

$$P(n,m)=\frac{1}{2\pi}e^{n\rho}\int_{-\pi}^{\pi}F(e^{-\rho+i\theta})e^{-ni\theta}d\theta, \tag{1}$$

where ρ is the root of the equation

$$n=\sum_{\nu=1}^{m}\frac{\nu}{e^{\nu\rho}-1}.$$

The essential point in Szekeres's proof is this determination of ρ, by which it is shown that the integral over the neighborhood of the point $\theta=0$ in (1) gives the principal term of the asymptotic formula for $P(n,m)$.

In this paper, we shall prove, by a method partly analogous to Szekeres's proof, an asymptotic formula for the number $T(n,m;k)$ of the partitions of n into k-th $(k\geqq 1)$ powers of *prime numbers* not exceeding m. Our result is stated as follows:

THEOREM. *Let n and m be sufficiently large integers and $n^{1/k}\geqq m$. Then we have, uniformly in n and m,*

$$T(n,m;k)=\frac{1}{\sqrt{2\pi A_{2}}}e^{n\alpha+A_{1}}\left\{1+O\left(\max\left(n^{-\frac{k}{2(k+1)(k+2)}},m^{-\frac{k}{2(k+2)}}\right)\right)\right\},$$

where α is the root of equation

$$n=\sum_{p\leqq m}\frac{p^{k}}{e^{\alpha p^{k}}-1} \tag{2}$$

and

$$A_1 = -\sum_{p\leqq m} \log(1-e^{-\alpha p^k}), \tag{3}$$

$$A_2 = \sum_{p\leqq m} \frac{p^{2k}e^{\alpha p^k}}{(e^{\alpha p^k}-1)^2}. \tag{4}$$

In the summations (2), (3) *and* (4), *p is taken over all prime numbers $\leqq m$.*

The generating function of $T(n, m; k)$ is

$$G(w) = \prod_{p\leqq m} (1-w^{p^k})^{-1} = \sum_{n=0}^{\infty} T(n, m; k)w^n \qquad (|w|<1)$$

and we have by Cauchy's theorem

$$\begin{aligned} T(n, m; k) &= e^{n\alpha}\int_{-1/2}^{1/2} G(e^{-\alpha+2\pi i\theta})e^{-2\pi in\theta}d\theta \\ &= e^{n\alpha}G(e^{-\alpha})\int_{-1/2}^{1/2} \frac{G(e^{-\alpha+2\pi i\theta})}{G(e^{-\alpha})} e^{-2\pi in\theta}d\theta. \end{aligned} \tag{5}$$

We shall divide the last integral as follows:

$$\int_{-1/2}^{1/2} = \int_{-\theta_0}^{\theta_0} + \left\{\int_{\theta_0}^{1/2} + \int_{-1/2}^{-\theta_0}\right\} = I_1 + I_2,$$

where $\theta_0 = n^{-1}(n\alpha)^{(k+2)/(2k+3)}$. (Since $\alpha<1$, θ_0 becomes small).

The estimation of I_1 will be given in section 2 by an analogous method to that of Szekeres. In section 3, the estimation of I_2 will be considered, to which we shall apply, differing in this point from Szekeres's case, the estimation of a certain trigonometrical sum obtained by Vinogradov [5], Hua [1] and [2]. These estimations will lead to our Theorem.

In section 4, special cases will be treated. As the results, we shall have

COROLLARY 1. *Let n and m be sufficiently large and $m \geqq (n\log^2 n)^{1/(k+1)}$, then*

$$T(n, m; k) = \frac{1}{\sqrt{2\pi}}\, n^{-\frac{2k+1}{2(k+1)}}\left\{\left(1+\frac{1}{k}\right)^{-\frac{1}{k}} \Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)(\log n)^{-1}\right\}^{\frac{k}{2(k+1)}}$$
$$\times e^{n\alpha+A_1}\left(1-\frac{k\log\log n}{2(k+1)\log n}+O\left(\frac{1}{\log n}\right)\right),$$

where α and A_1 have the meaning mentioned in our Theorem, $\zeta(s)$ is Riemann zeta function, and we have asymptotically

$$\alpha=\left\{\frac{\Gamma\left(2+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)}{n\log n}\right\}^{\frac{k}{k+1}}\left(1-\frac{k\log\log n}{(k+1)\log n}+O\left(\frac{1}{\log n}\right)\right),$$

$$A_1=\left\{\Gamma\left(2+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{\frac{k}{k+1}}\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}}\left(1-\frac{k\log\log n}{(k+1)\log n}+O\left(\frac{1}{\log n}\right)\right).$$

COROLLARY 2. *Let n and m be sufficiently large and $m\leqq n^{\frac{1}{k+1}}$, then*

$$T(n,m;k)=\frac{1}{n}\left(\frac{m}{2\pi\log m}\right)^{1/2}e^{n\alpha+A_1}\left(1+O\left(\frac{1}{\log m}\right)\right),$$

where we have asymptotically

$$\alpha=\frac{m}{n\log m}\left(1+O\left(\frac{1}{\log m}\right)\right),$$

$$A_1=\frac{m}{\log m}\log\frac{n\log m}{m^{k+1}}\left(1+O\left(\frac{1}{\log m}\right)\right).$$

Throughout this paper, c denotes the positive constant which is independent of n and m but may depend on k. It does not always mean the same constant at every time it appears. When it is necessary to distinguish such constants, we shall use $c_1, c_2, \cdots$. We denote by p prime numbers and write $a=1/k$.

We shall need a summation formula for $\sum f(p)$, instead of Euler's summation formula often used in Szekeres's paper. It is formulated as follows. Writing $\pi(x)=\sum_{p\leqq x}1$, we obtain

$$\sum_{p\leqq m}f(p)=\sum_{\nu=2}^{m}f(\nu)(\pi(\nu)-\pi(\nu-1))=\pi(m)f(m)-\int_2^m\pi(t)f'(t)dt.$$

Therefore putting

$$\pi(x)=\int_2^x\frac{dt}{\log t}+\phi(x)$$

in the above formula, we have

$$\sum_{p\leqq m}f(p)=\int_2^m\frac{f(t)}{\log t}dt+f(m)\phi(m)-\int_2^m f'(t)\phi(t)dt, \tag{6}$$

where

$$\phi(x)=O(xe^{-c\sqrt{\log x}})$$

by the prime number theorem. (We shall use O-notation when the constants in it are independent of n and m). Furthermore, we shall consider the functions of the following forms

$$f_{r,s}(x)=\frac{x^{rk}}{(e^{\beta x^k}-1)^s} \qquad (x\geqq 0),$$

where $\beta>0$, r and s are integers and $r\geqq s\geqq 1$. The following properties of $f_{r,s}(x)$ are obvious;

$$f_{r,s}(x)=O(\beta^{-r})\cdot\min(1,(\beta x^k)^{r-s}),$$

$$xf_{r,s}(x)=O(\beta^{-r-a})\cdot\min(1,x\beta^a),$$

$$\int_0^x f_{r,s}(t)dt=O(\beta^{-r-a})\cdot\min(1,x\beta^a),$$

$$f'_{r,s}(x)=O(\beta^{-r+a}),\quad xf'_{r,s}(x)=O(\beta^{-r}),$$

$$\int_0^x t\,|f'_{r,s}(t)|\,dt=O(\beta^{-r-a})\cdot\min(1,x\beta^a).$$

Now we shall apply (6) to $f_{r,s}(x)$, then we have

$$f_{r,s}(m)\phi(m)=O(e^{-c\sqrt{\log m}})\cdot mf_{r,s}(m)=O(e^{-c\sqrt{\log m}})\cdot\beta^{-r-a}\min(1,m\beta^a),$$

$$\int_2^m f'_{r,s}(x)\phi(x)dx=\int_2^{m^{1/3}}+\int_{m^{1/3}}^m$$

$$=O(\beta^{-r}m^{1/3})+O(e^{-c\sqrt{\log m}})\cdot\beta^{-r-a}\min(1,m\beta^a)$$

and finally

$$\int_2^{m^{1/3}}\frac{f_{r,s}(x)}{\log x}dx=O(\beta^{-r}m^{1/3}).$$

Hence we have uniformly in β and m

$$\sum_{p\leqq m}\frac{p^{rk}}{(e^{\beta p^k}-1)^s}=\int_{m^{1/3}}^m\frac{x^{rk}}{(e^{\beta x^k}-1)^s\log x}dx+O(m^{1/3}\beta^{-r})$$

(7)

$$+O(e^{-c\sqrt{\log m}})\cdot\beta^{-r-a}\min(1,m\beta^a).$$

2. From now on we shall assume that n and m are sufficiently large and $n^{1/k}\geqq m$. We shall first prove the following lemma.

LEMMA 1. *If α is the root of equation*

$$n=\sum_{p\leqq m}\frac{p^k}{e^{\alpha p^k}-1},$$

then

(8)
$$\frac{1}{3n}\min\left\{\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}},\ \frac{m}{\log m}\right\}\leqq\alpha$$
$$\leqq\frac{6(2+k)}{n}\min\left\{\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}},\ \frac{m}{\log m}\right\}.$$

PROOF. We apply (7) to the case $r=s=1$ and put

$$\beta=\frac{b}{n}\min\left\{\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}},\ \frac{m}{\log m}\right\}\qquad(b>0).$$

Then we have $\beta=O(n^{-k/(k+1)})$, therefore

(9)
$$\sum_{p\leqq m}\frac{p^k}{e^{\beta p^k}-1}=\int_{m^{1/3}}^{m}\frac{x^k}{(e^{\beta x^k}-1)\log x}\,dx+\frac{1}{\beta^{1+a}}\min(1,m\beta^a)\cdot O(e^{-c\sqrt{\log m}}).$$

Since

$$te^{t/2}\leqq e^t-1\leqq te^t,$$
$$\int_0^t e^{-u}u^{a-1}du\leqq\min(\Gamma(a),kt^a),$$
$$\int_0^t e^{-u}u^{a-1}du\geqq\int_0^{\min(1,t)}(1-u)u^{a-1}du\geqq\frac{k}{2}\min(1,t^a)$$

for $t\geqq 0$, we have

$$\int_{m^{1/3}}^{m}\frac{x^k}{(e^{\beta x^k}-1)\log x}\,dx\leqq\frac{3}{\beta\log m}\int_0^m e^{-\frac{\beta}{2}x^k}dx$$
$$=\frac{3\cdot 2^a a}{\beta^{1+a}\log m}\int_0^{\frac{\beta}{2}m^k}e^{-u}u^{a-1}du\leqq\frac{6}{\beta^{1+a}\log m}\min(1,m\beta^a)$$

and

$$\int_{m^{1/3}}^{m}\frac{x^k}{(e^{\beta x^k}-1)\log x}\,dx\geqq\frac{1}{\beta\log m}\int_{m^{1/3}}^{m}e^{-\beta x^k}dx$$
$$=\frac{a}{\beta^{1+a}\log m}\int_{\beta m^{k/3}}^{\beta m^k}e^{-u}u^{a-1}du\geqq\frac{1}{2\beta^{1+a}\log m}\min(1,m\beta^a)-\frac{m^{1/3}}{\beta\log m}.$$

Thus we have

(10) $$\frac{1}{2\beta^{1+a}\log m}(1+\varphi_1)\min(1, m\beta^a) \leqq \sum_{p\leqq m}\frac{p^k}{e^{\beta p^k}-1}$$
$$\leqq \frac{6}{\beta^{1+a}\log m}(1+\varphi_2)\min(1, m\beta^a),$$

where

$$\varphi_1 = O(e^{-c\sqrt{\log m}}), \qquad \varphi_2 = O(e^{-c\sqrt{\log m}}).$$

Now, if

$$\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}} \geqq \frac{m}{\log m},$$

then

$$\frac{6}{\beta^{1+a}\log m}\min(1, m\beta^a) \leqq \frac{6m}{\beta\log m} = \frac{6n}{b}.$$

On the other hand, if

(11) $$\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}} < \frac{m}{\log m},$$

then

$$\frac{6}{\beta^{1+a}\log m}\min(1, m\beta^a) \leqq \frac{6}{\beta^{1+a}\log m} = \frac{6n\log n}{b^{1+a}\log m} \leqq \frac{6(2+k)n}{b^{1+a}},$$

since it follows from (11) that

$$\log m > \log m - \log\log m > \frac{1}{k+1}(\log n - k\log\log n) > \frac{1}{k+2}\log n$$

for large n. Furthermore we see that

$$\frac{1}{\log m}\min(1, m\beta^a) \geqq \frac{\min(1, b^a)}{n^a}\min\left\{\frac{n^a}{\log n}, \left(\frac{m}{\log m}\right)^{a(k+1)}\right\}$$
$$= \frac{\min(1, b^a)}{n^a}\left(\frac{n\beta}{b}\right)^{1+a}.$$

Therefore we have from (10)

(12) $$\frac{\min(1, b^a)}{2b^{1+a}}n(1+\varphi_1) \leqq \sum_{p\leqq m}\frac{p^k}{e^{\beta p^k}-1} \leqq 6\max\left(\frac{1}{b}, \frac{2+k}{b^{1+a}}\right)n(1+\varphi_2),$$

which shows that we have for large n and m

$$\sum_{p \leqq m} \frac{p^k}{e^{\beta p^k}-1} \leqq n$$

if we put $b=6(2+k)$, and on the other hand

$$\sum_{p \leqq m} \frac{p^k}{e^{\beta p^k}-1} \geqq n$$

if we put $b=1/3$. Our lemma is thereby proved.

REMARK. We have from (8)

$$c_1 n\alpha^a \leqq \frac{1}{\alpha \log m} \min(1, m\alpha^a) \leqq c_2 n\alpha^a . \tag{13}$$

Furthermore, we may assume that $n\alpha$ is sufficiently large.

Now we shall consider

$$I_1 = \int_{-\theta_0}^{\theta_0} \frac{G(e^{-\alpha+2\pi i\theta})}{G(e^{-\alpha})} e^{-2\pi i n\theta} d\theta$$

$$= \int_{-\theta_0}^{\theta_0} \exp\left\{ -\sum_{p \leqq m} \log \frac{e^{\alpha p^k} - e^{2\pi i\theta p^k}}{e^{\alpha p^k}-1} - 2\pi i n\theta \right\} d\theta .$$

We divide the sum in this integrand into two parts,

$$\sum_{p \leqq m} = \sum_{p \leqq m_1} + \sum_{m_1 < p \leqq m} , \tag{14}$$

where $m_1 = \min(m, [n^{2/(2k+1)}])$. The second sum in (14) is empty if $m \leqq n^{2/(2k+1)}$. When it is not empty, $m > n^{2/(2k+1)}$ implies that $\alpha \geqq c(n \log n)^{-k/(k+1)}$, so that $\alpha m_1^k \geqq c(n^{1/(2k+1)} (\log n)^{-1})^{k/(k+1)} \geqq c(n\alpha)^{k/(2k+2)}$. Therefore

$$\sum_{m_1 < p \leqq m} \log \frac{e^{\alpha p^k} - e^{2\pi i\theta p^k}}{e^{\alpha p^k}-1} = O(\sum_{m_1 < p \leqq m} e^{-c\alpha p^k}) = O(\exp(-c(n\alpha)^{\frac{k}{2(k+1)}})) .$$

Since $|\theta p^k| \leqq \theta_0 m_1^k \leqq n^{-1}(n\alpha)^{(k+2)/(2k+3)} \cdot n^{2k/(2k+1)} \leqq c(\log n)^{-k/(2k+3)}$ and

$$\frac{e^{2\pi i\theta p^k}-1}{e^{\alpha p^k}-1} = O\left(\frac{\theta}{\alpha}\right) = O\left(\frac{\theta_0}{\alpha}\right) = O((n\alpha)^{-\frac{k+1}{2k+3}})$$

for $p \leqq m_1$, we have the following expansion of the first sum in (14);

$$\sum_{p \leqq m_1} \log \frac{e^{\alpha p^k} - e^{2\pi i\theta p^k}}{e^{\alpha p^k}-1} = \sum_{p \leqq m_1} \log\left(1 - \frac{e^{2\pi i\theta p^k}-1}{e^{\alpha p^k}-1}\right) \tag{15}$$

$$=\sum_{p\leqq m_1}\left\{-\frac{e^{2\pi i\theta p^k}-1}{e^{\alpha p^k}-1}-\frac{(e^{2\pi i\theta p^k}-1)^2}{2(e^{\alpha p^k}-1)^2}+O\left(\frac{\theta^3 p^{3k}}{(e^{\alpha p^k}-1)^3}\right)\right\}$$

$$=-2\pi i\theta\sum_{p\leqq m_1}\frac{p^k}{e^{\alpha p^k}-1}+2\pi^2\theta^2\sum_{p\leqq m_1}\frac{p^{2k}e^{\alpha p^k}}{(e^{\alpha p^k}-1)^2}+O\left(\theta^3\sum_{l=1}^{3}\sum_{p\leqq m_1}\frac{p^{3k}}{(e^{\alpha p^k}-1)^l}\right).$$

It follows from (7) and (13) that this error term is

$$O(\theta_0^3\alpha^{-3-a}(\log m)^{-1})\cdot\min(1, m\alpha^a)=O(\theta_0^3\alpha^{-2}n)=O((n\alpha)^{-\frac{k}{2k+3}}).$$

Furthermore we have, by the definitions of α and A_2,

$$(16)\quad \sum_{p\leqq m_1}\frac{p^k}{e^{\alpha p^k}-1}=\left\{\sum_{p\leqq m}-\sum_{m_1<p\leqq m}\right\}\frac{p^k}{e^{\alpha p^k}-1}=n+O(\exp(-c(n\alpha)^{\frac{k}{2(k+1)}})),$$

$$(17)\quad \sum_{p\leqq m_1}\frac{p^{2k}e^{\alpha p^k}}{(e^{\alpha p^k}-1)^2}=\left\{\sum_{p\leqq m}-\sum_{m_1<p\leqq m}\right\}\frac{p^{2k}e^{\alpha p^k}}{(e^{\alpha p^k}-1)^2}=A_2+O(\exp(-c(n\alpha)^{\frac{k}{2(k+1)}}))\,.$$

Collecting (15), (16) and (17), we have

$$I_1=\int_{-\theta_0}^{\theta_0}e^{-2\pi^2A_2\theta^2}d\theta\{1+O((n\alpha)^{-\frac{k}{2k+3}})\}\,.$$

In this right hand side,

$$\int_{-\theta_0}^{\theta_0}e^{-2\pi^2A_2\theta^2}d\theta=\frac{1}{\sqrt{2\pi^2A_2}}\int_{-x_0}^{x_0}e^{-u^2}du=\frac{1}{\sqrt{2\pi^2A_2}}\left(\int_{-\infty}^{\infty}e^{-u^2}du-2\int_{x_0}^{\infty}e^{-u^2}du\right)$$

$$=\frac{1}{\sqrt{2\pi A_2}}\,(1+O(e^{-x_0^2}))\,,$$

where $x_0^2=2\pi^2A_2\theta_0^2$. Since

$$\frac{p^{2k}e^{\alpha p^k}}{(e^{\alpha p^k}-1)^2}=\frac{p^{2k}}{e^{\alpha p^k}-1}+\frac{p^{2k}}{(e^{\alpha p^k}-1)^2}\,,$$

we can apply the formula (7) to A_2, and we have

$$(18)\qquad \frac{c_3}{\alpha^{2+a}\log m}\min(1, m\alpha^a)\leqq A_2\leqq\frac{c_4}{\alpha^{2+a}\log m}\min(1, m\alpha^a)\,.$$

Hence by (13)

$$x_0^2\geqq c\theta_0^2 n\alpha^{-1}\geqq c(n\alpha)^{\frac{1}{2k+3}}\,.$$

Thus we obtain the estimation of I_1;

(19) $$I_1=\frac{1}{\sqrt{2\pi A_2}}\{1+O((n\alpha)^{-\frac{k}{2k+3}})\}.$$

3. We put

$$H(\theta)=\left|\frac{G(e^{-\alpha+2\pi i\theta})}{G(e^{-\alpha})}\right|=\exp\left(-\sum_{p\leqq m}\log\left|\frac{e^{\alpha p^k}-e^{2\pi i\theta p^k}}{e^{\alpha p^k}-1}\right|\right).$$

and assume first that $\theta_0<\theta\leqq\theta_1=(n\alpha)^{-k/(2k+3)}$.

Obviously we have

(20) $$H(\theta)\leqq\exp\left\{-\frac{1}{2}\sum_{p\leqq m_2}\log\left(1+\frac{2e^{\alpha p^k}(1-\cos 2\pi\theta p^k)}{(e^{\alpha p^k}-1)^2}\right)\right\},$$

where $m_2=\min(m,[\alpha^{-a}],[(2\theta)^{-a}])$. Since

$$1-\cos 2\pi\theta p^k\geqq 8\theta^2p^{2k},\quad \frac{1}{e^{\alpha p^k}-1}\geqq\frac{1}{2\alpha p^k}$$

for $p\leqq m_2$, it follows from (20) that

$$H(\theta)\leqq\exp\left\{-\frac{\pi(m_2)}{2}\log\left(1+\frac{4\theta^2}{\alpha^2}\right)\right\}.$$

If $\theta\leqq\alpha$, then

$$\frac{\pi(m_2)}{2}\log\left(1+\frac{4\theta^2}{\alpha^2}\right)\geqq c\pi(m_2)\frac{\theta^2}{\alpha^2}\geqq\frac{cm_2}{\log m_2}\cdot\frac{\theta^2}{\alpha^2}$$

$$\geqq\frac{c}{\log m}\min\left(m,\frac{1}{(2\alpha)^a}\right)\frac{\theta^2}{\alpha^2}\geqq\frac{c}{\log m}\min(1,m\alpha^a)\frac{\theta_0^{\ 2}}{\alpha^{2+a}}\geqq c(n\alpha)^{\frac{1}{2k+3}}.$$

Hence

(21) $$\int_{\theta_0}^{\alpha}H(\theta)d\theta=O(\alpha)\cdot\exp\{-c(n\alpha)^{\frac{1}{2k+3}}\}.$$

On the other hand, if $\theta_1\geqq\theta\geqq\alpha$, then

$$\log\left(1+\frac{4\theta^2}{\alpha^2}\right)\geqq\log\frac{2\theta}{\alpha},$$

so we have, putting $A=\pi(m_2)/2$,

$$\int_{\alpha}^{\theta_1}H(\theta)d\theta\leqq\int_{\alpha}^{\theta_1}\exp\left(-A\log\frac{2\theta}{\alpha}\right)d\theta=\frac{\alpha}{2}\int_2^{2\theta_1/\alpha}e^{-A\log x}dx$$

$$=\frac{\alpha}{2(A-1)}\{e^{(1-A)\log 2}-e^{(1-A)\log(2\theta_1/\alpha)}\}$$

$$\leqq c\alpha\{e^{(1-A)\log 2}-e^{(1-A)\log(2\theta_1/\alpha)}\}.$$

Since

$$m_2=\min(m,[(2\theta)^{-a}])\geqq\min(m,(3\theta_1)^{-a})\geqq c(n\alpha)^{\frac{1}{2k+3}}$$

and θ_1/α is large, we have

$$(22)\qquad \int_{\alpha}^{\theta_1}H(\theta)d\theta\leqq c\alpha\exp\{-c\pi(m_2)\}\leqq c\alpha\exp\{-c(n\alpha)^{\frac{1}{2k+4}}\}.$$

Thus we have by (21) and (22)

$$(23)\qquad \int_{\theta_0}^{\theta_1}H(\theta)d\theta=O(\alpha)\cdot\exp\{-c(n\alpha)^{\frac{1}{2k+4}}\}.$$

Finally we shall estimate $H(\theta)$ for $\theta_1<\theta\leqq 1/2$. Our starting point is the following expansion;

$$\log G(e^{-\alpha+2\pi i\theta})=\sum_{r=1}^{\infty}\frac{1}{r}\sum_{p\leqq m}e^{-\alpha rp^k+2\pi i\theta rp^k}.$$

Put now

$$S_\theta(t)=\sum_{p\leqq t}e^{2\pi i\theta p^k}$$

for integer $t\geqq 1$, then

$$\sum_{p\leqq m}e^{-\alpha rp^k+2\pi i\theta rp^k}=\sum_{t=2}^{m}(S_{r\theta}(t)-S_{r\theta}(t-1))e^{-\alpha rt^k}$$

$$=\sum_{t=2}^{m-1}S_{r\theta}(t)(e^{-\alpha rt^k}-e^{-\alpha r(t+1)^k})+S_{r\theta}(m)e^{-\alpha rm^k}.$$

Since $S_0(t)=\pi(t)$, we have

$$\log H(\theta)=\Re\log G(e^{-\alpha+2\pi i\theta})-\log G(e^{-\alpha})=\sum_{r=1}^{\infty}\frac{1}{r}g(r),$$

where

$$g(r)=\sum_{t=2}^{m-1}\{\Re S_{r\theta}(t)-\pi(t)\}(e^{-\alpha rt^k}-e^{-\alpha r(t+1)^k})+\{\Re S_{r\theta}(m)-\pi(m)\}e^{-\alpha rm^k}.$$

The following lemma follows from the results of Vinogradov and Hua.

LEMMA 2. *Let N be sufficiently large integer and $1/2\geqq|\theta|\geqq(\log N)^{\sigma}\cdot N^{-k}$, where $\sigma=2^{6k+3}$, then we have*

$$(24)\qquad \Re S_\theta(N)-\pi(N)\leqq -c\frac{N}{\log N}.$$

PROOF. Since $S_{\theta'}(N)=S_\theta(N)$ for $\theta'=\theta+1$, it is sufficient to prove (24) for θ such that

$$\frac{(\log N)^\sigma}{N^k}\leqq\theta\leqq1-\frac{(\log N)^\sigma}{N^k}. \tag{25}$$

Putting $\tau=N^k(\log N)^{-\sigma}$, we pick up the subintervals

$$I_{h,q}=\left[\frac{h}{q}-\frac{1}{\tau},\ \frac{h}{q}+\frac{1}{\tau}\right]$$

from the interval $[\tau^{-1}, 1-\tau^{-1}]$, where h and q are integers such that $1\leqq h<q\leqq(\log N)^\sigma$, $(h, q)=1$.

If θ does not belong to any $I_{h,q}$, then we have uniformly in θ

$$S_\theta(N)=O\left(\frac{N}{\log^3 N}\right) \tag{26}$$

(Hua [2]).

If $\theta\in I_{h,q}$, then θ can be written in the form

$$\theta=\frac{h}{q}+z,\qquad |z|\leqq\frac{1}{\tau}$$

and we have

$$S_\theta(N)=\frac{W_{h,q}}{\varphi(q)}\int_2^N\frac{e^{2\pi izx^k}}{\log x}\,dx+O\left(\frac{N}{\log^3 N}\right), \tag{27}$$

where

$$W_{h,q}=\sum_{\substack{l=1\\(l,q)=1}}^{q-1}e^{2\pi i\frac{h}{q}l^k}$$

and the constants in the error term of (27) are independent of q and h. (This formula (27) can be easily obtained in the same manner as in Vinogradov [5]). Since $W_{h,q}=O(q^{3/4})$ (Hua [1], [2]) and $1/\varphi(q)=O(q^{-7/8})$ (Landau [3], Satz 245), there exists an integer q_1 depending on k alone and

$$\frac{|W_{h,q}|}{\varphi(q)}\leqq\frac{1}{2}$$

for $q\geqq q_1$. Therefore, noting that $|W_{h,q}|<\varphi(q)$ for $q>2$, we see that

$$b_0=\max_{q>2}\frac{|W_{h,q}|}{\varphi(q)}\leqq\max_{q_1>q>2}\frac{|W_{h,q}|}{\varphi(q)}<1$$

and consequently

(28) $$|S_\theta(N)| \leqq \frac{1+b_0}{2} \cdot \frac{N}{\log N}$$

for $\theta \in I_{h,q}$ with $q \geqq 3$.

When $\theta = 1/2 + z$, $|z| \leqq \tau^{-1}$, then

(29) $$\Re S_\theta(N) = -\int_2^N \frac{\cos 2\pi z x^k}{\log x}\, dx + O\left(\frac{N}{\log^3 N}\right),$$

since $W_{1,2} = -1$, $\varphi(2) = 1$.

Assume first $0 \leqq z \leqq (4N^k)^{-1}$, then $\cos 2\pi z x^k \geqq 0$ for $2 \leqq x \leqq N$, which means that

(30) $$\Re S_\theta(N) \leqq \frac{N}{\log^2 N}$$

for large N.

Next assume $z > (4N^k)^{-1}$. Since $n^{a-1}/\log u$ is a decreasing function of u for $u > 1$, we have

(31) $$-\int_2^N \frac{\cos 2\pi z x^k}{\log x}\, dx = -\int_{2^k}^{N^k} \frac{u^{a-1} \cos 2\pi z u}{\log u}\, du$$
$$\leqq -\int_{\frac{1}{4z}}^{\xi} \frac{u^{a-1} \cos 2\pi z u}{\log u}\, du,$$

where $\xi = \min\left(N^k, \dfrac{3}{4z}\right)$. Furthermore, the application of the second mean value theorem for integral to the right hand side of (31) gives

$$-\int_{\frac{1}{4z}}^{\xi} \frac{u^{a-1} \cos 2\pi z u}{\log u}\, du = -\frac{(4z)^{1-a}}{\log \dfrac{1}{4z}} \int_{\frac{1}{4z}}^{\eta} \cos 2\pi z u\, du$$
$$= \frac{4^{1-a}}{\log \dfrac{1}{4z}} \cdot \frac{1 - \sin 2\pi z \eta}{2\pi z^a} \leqq \frac{4^{1-a}}{\log(\tau/4)} \cdot \frac{1 - \sin 2\pi z \eta}{2\pi z^a},$$

where $\eta \leqq \xi$.

Now, if $N^k \leqq (2z)^{-1}$, then $2\pi z \eta \leqq \pi$, so that

$$\frac{1 - \sin 2\pi z \eta}{2\pi z^a} \leqq \frac{1}{2\pi z^a} \leqq \frac{4^a}{2\pi} N.$$

On the other hand, $N^k > (2z)^{-1}$ implies that

$$\frac{1-\sin 2\pi z\eta}{2\pi z^a} \leqq \frac{1}{\pi z^a} < \frac{2^a}{\pi} N.$$

Thus we have

$$(32) \qquad -\int_{\frac{1}{4z}}^{\xi} \frac{u^{a-1}\cos 2\pi z u}{\log u} du$$

$$\leqq \frac{2^{2-a}}{\pi} \cdot \frac{N}{k\log N - \log(4(\log N)^\sigma)} \leqq \frac{2N}{3\log N}$$

and it follows from (30) and (32) combined with (29) and (31) that

$$(33) \qquad \Re S_\theta(N) \leqq \frac{3N}{4\log N}$$

for $\theta \in I_{1,2}$.

Collecting the results (26), (28) and (33), we see that there exists a positive constant $c_0 < 1$ independent of N such that

$$(34) \qquad \Re S_\theta(N) \leqq c_0 \frac{N}{\log N}$$

for sufficiently large N and $1/2 \geqq |\theta| \geqq \tau^{-1}$.

Our lemma follows then from (34) at once.

Now we shall apply Lemma 2 to the estimation of $\log H(\theta)$. Let M_θ be the set of integers $r \geqq 1$ such that $\min(r\theta - [r\theta], 1+[r\theta] - r\theta) \geqq m_3^{-k}(\log m_3)^\sigma$, where we put $m_3 = [m^{1/(2k+1)}]$. Since m is sufficiently large, it follows from Lemma 2 that

$$\Re S_{r\theta}(t) - \pi(t) \leqq -c\frac{t}{\log t}$$

for $t \geqq m_3$ and $r \in M_\theta$.

Hence we have for $r \in M_\theta$

$$g(r) \leqq \sum_{t=m_3}^{m-1} \{\Re S_{r\theta}(t) - \pi(t)\}(e^{-\alpha r t^k} - e^{-\alpha r (t+1)^k}) + \{\Re S_{r\theta}(m) - \pi(m)\} e^{-\alpha r m^k}$$

$$\leqq -\frac{c}{\log m}\left\{\sum_{t=m_3}^{m-1} t(e^{-\alpha r t^k} - e^{-\alpha r(t+1)^k}) + m e^{-\alpha r m^k}\right\}$$

$$= -\frac{c}{\log m}\left(m_3 e^{-\alpha r m_3{}^k} + \sum_{t=m_3+1}^{m} e^{-\alpha r t^k}\right) \leqq -\frac{c}{\log m}\int_{m_3}^{m} e^{-\alpha r x^k} dx$$

and consequently

$$\text{(35)} \qquad \log H(\theta) \leqq \sum_{r \in M_\theta} \frac{1}{r} g(r)$$

$$\leqq -\frac{c}{\log m} \sum_{r \in M_\theta} \frac{1}{r} \int_{m_3}^{m} e^{-\alpha r x^k} dx\,.$$

Since

$$|\theta| \geqq \theta_1 \geqq c\left(\frac{\log m}{m}\right)^{\frac{k}{2k+3}} \geqq \frac{(\log m_3)^\sigma}{m_3^k}$$

for large m, it follows that $1 \in M_\theta$. Therefore we have

$$\text{(36)} \qquad \log H(\theta) \leqq -\frac{c}{\log m} \int_{m_3}^{m} e^{-\alpha x^k} dx \leqq -\frac{c}{\alpha^a \log m} \min(1, m\alpha^a)$$

$$\leqq -cn\alpha \leqq -c(n\alpha)^{2/3} - \log n$$

for $m \geqq n^{\frac{1}{4k}}$.

When $m < n^{\frac{1}{4k}}$, we must consider the following inequality derived from (35):

$$\log H(\theta) \leqq -\frac{c}{\log m} \sum_{r \in M'} \frac{1}{r} \int_{m_3}^{m} e^{-\alpha r x^k} dx\,,$$

where M' is the set of r such that $r \in M_\theta$, $1 \leqq r \leqq (m\alpha^a)^{-1}$. (Note that $m\alpha^a$ is very small in this case). Since $\alpha r m^k \leqq 1$ for $r \in M'$, we have

$$\text{(37)} \qquad \log H(\theta) \leqq -\frac{c}{\log m} \sum_{r \in M'} \frac{1}{r} e^{-1}(m - m_3) \leqq -\frac{cm}{\log m} \sum_{r \in M'} \frac{1}{r}$$

$$= -\frac{cm}{\log m}\left(\sum_{r=1}^{[(m\alpha^a)^{-1}]} \frac{1}{r} - \sum_{r \in M''} \frac{1}{r}\right)$$

$$\leqq -\frac{cm}{\log m}\left(\log \frac{1}{m\alpha^a} - \sum_{r \in M''} \frac{1}{r}\right),$$

where M'' is the set of r such that $r \notin M_\theta$, $1 \leqq r \leqq (m\alpha^a)^{-1}$.

We shall consider here Farey dissection of order $\tau_1/4$, where $\tau_1 = m_3^k (\log m_3)^{-\sigma}$. Let h/q, h'/q' be the consecutive points of this dissection such that $h/q \geqq \theta \geqq h'/q'$. We see then $hq' - h'q = 1$ and $q \neq q'$. Furthermore $q, q' \geqq 2$, since

$$\theta \geqq \theta_1 \geqq c\left(\frac{\log m}{m}\right)^{\frac{k}{2k+3}} \geqq \frac{4}{\tau_1}$$

for large m.

Now we shall define, in x-y plane, for a pair of non-negative integers s and t, a parallelogram $B(s,t)$ and two segments I and I' contained in $B(s,t)$ as follows;

$$B(s,t)=\{(uq+vq',\ uh+vh');\ s\leqq u<s+1,\ t\leqq v<t+1\},$$
$$I=\{((s+1)q+tq',\ (s+1)h+th'-z);\ 0<z\leqq\tau_1^{-1}\},$$
$$I'=\{(sq+(t+1)q',\ sh+(t+1)h'+z);\ 0<z\leqq\tau_1^{-1}\}.$$

The area of $B(s,t)$ is equal to 1 and only one point with integral coordinates is contained in $B(s,t)$. Furthermore we denote by l_θ a straight line $y=\theta x$.

Let $(b,\theta b)$, $(b+1,\theta(b+1)),\cdots,(b',\theta b')$ be all points in $B(s,t)\cap l_\theta$, of which x-coordinates are integers. This set may be empty. When it is not empty, we see that $r\notin M_\theta$ for $b\leqq r\leqq b'$ if and only if r satisfies one of the following three conditions;

(i) $r=b=sq+tq'$.

(ii) $r=(s+1)q+tq'$, $(r,\theta r)\in I\cap l_\theta$.

(iii) $r=sq+(t+1)q'$, $(r,\theta r)\in I'\cap l_\theta$.

We shall show here that l_θ cannot have common points with I and I' at the same time. If there exist the points $(x,y)\in I\cap l_\theta$ and $(x',y')\in I'\cap l_\theta$, then

$$\theta=\frac{y-y'}{x-x'}=\frac{h-h'-z-z'}{q-q'},$$

where $0<z,z'\leqq\tau_1^{-1}$. Since $(z+z')\max(q,q')\leqq 2\tau_1^{-1}\cdot\tau_1/4=1/2$ and $hq'-h'q=1$, we have

$$\theta-\frac{h}{q}=\frac{1-(z+z')q}{q(q-q')}>0 \qquad (\text{when } q>q'),$$

$$\frac{h'}{q'}-\theta=\frac{1-(z+z')q'}{q'(q'-q)}>0 \qquad (\text{when } q'>q),$$

which is contrary to $h/q\geqq\theta\geqq h'/q'$. Therefore at least one of $l_\theta\cap I$ and $l_\theta\cap I'$ is empty. It follows then that the number of r such that $r\notin M_\theta$, $b\leqq r\leqq b'$ is at most two. Furthermore we have for these $r\notin M_\theta$

$$(38) \qquad r \geqq \begin{cases} (s+t)q_0 & (\text{when } r=b=sq+tq'), \\ (s+t+1)q_0 & (\text{when } (r, \theta r) \in I \text{ or } I'), \end{cases}$$

where $q_0 = \min(q, q') \geqq 2$.

Let $(s_1, t_1), (s_2, t_2), \cdots, (s_\mu, t_\mu)$ be all pairs of integers with $s_1 \leqq s_2 \leqq \cdots \leqq s_\mu$ and $t_1 \leqq t_2 \leqq \cdots \leqq t_\mu$, for which $B(s_i, t_i) \cap l_\theta$ are not empty and $s_i q + t_i q' \leqq (m\alpha^a)^{-1}$ $(i=1, 2, \cdots, \mu)$. Then we have from (38)

$$(39) \qquad \sum_{r \in M''} \frac{1}{r} \leqq \frac{1}{q_0} \sum_{i=1}^{\mu} \frac{1}{s_i + t_i + 1} + \frac{1}{q_0} \sum_{j \in J} \frac{1}{s_j + t_j},$$

where J is the set of integers j such that

$$1 \leqq j \leqq \mu, \ (s_j q + t_j q', \ s_j h + t_j h') \in l_\theta .$$

After simple examination, we see that

$$(40) \qquad \begin{aligned} & s_{i+1} = s_i + 1, \quad t_{i+1} = t_i + 1 && (\text{when } i+1 \in J) \\ & s_{i+1} + t_{i+1} = s_i + t_i + 1 && (\text{when } i+1 \notin J) \end{aligned}$$

for $i = 1, 2, \cdots, \mu - 1$. Since $s_\mu + t_\mu \leqq \nu = [(q_0 m\alpha^a)^{-1}]$, it follows from (39) and (40) that

$$(41) \qquad \begin{aligned} \sum_{r \in M''} \frac{1}{r} &\leqq \frac{1}{q_0} \sum_{s=1}^{\nu+1} \frac{1}{s} \leqq \frac{1}{q_0} (\log(\nu+1) + 1) \\ &\leqq \frac{1}{2} \log \left(\frac{1}{2m\alpha^a} + 1 \right) + \frac{1}{2} \leqq \frac{2}{3} \log \frac{1}{m\alpha^a} . \end{aligned}$$

Considering that m is sufficiently large, we have by (41) and (37)

$$\begin{aligned} \log H(\theta) &\leqq -\frac{cm}{\log m} \log \frac{1}{m\alpha^a} \leqq -\frac{cm}{\log m} \log \frac{n^a \log^a m}{m^{1+a}} \\ &\leqq -\frac{cm}{\log m} \log n \leqq -\frac{cm}{\log m} - \log n \end{aligned}$$

for $m < n^{\frac{1}{4k}}$.

Thus we have, by (36) and the result just proved,

$$(42) \qquad \log H(\theta) \leqq -c(n\alpha)^{2/3} - \log n$$

for $\theta_1 < \theta \leqq 1/2$. Furthermore it follows from (23) and (42) that

$$\int_{\theta_0}^{1/2} H(\theta) d\theta = O\left(\alpha + \frac{1}{n} \right) \cdot \exp\{ -c(n\alpha)^{\frac{1}{2k+4}} \}$$

$$=O\left(\frac{1}{n}\right)\cdot\exp\{-c(n\alpha)^{\frac{1}{2k+4}}\}.$$

We have similar result for $-\theta_0>\theta\geqq-1/2$ and finally

$$(43)\qquad |I_2|\leqq\left\{\int_{\theta_0}^{1/2}+\int_{-1/2}^{-\theta_0}\right\}H(\theta)d\theta=O\left(\frac{1}{n}\right)\cdot\exp\{-c(n\alpha)^{\frac{1}{2k+4}}\}.$$

Since we have $\sqrt{A_2}=O(\sqrt{n/\alpha})=O(n)$ by (18) and (13), so it follows from (19) and (43) that

$$(44)\qquad I_1+I_2=\frac{1}{\sqrt{2\pi A_2}}\{1+O((n\alpha)^{-\frac{k}{2k+3}})\}+O(n^{-1})\cdot\exp\{-c(n\alpha)^{\frac{1}{2k+4}}\}$$

$$=\frac{1}{\sqrt{2\pi A_2}}\{1+O((n\alpha)^{-\frac{k}{2k+3}})\}.$$

Putting this result (44) in (5), we now complete the proof of our Theorem.

4. We shall now consider special cases.

First assume that $m\geqq(n\log^2 n)^{1/(k+1)}$, then

$$\frac{m}{\log m}\geqq\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}},\ c(n\log n)^{-\frac{k}{k+1}}\geqq\alpha\geqq c(n\log n)^{-\frac{k}{k+1}},$$

$$\alpha m^k\geqq c(\log n)^{\frac{k}{k+1}}.$$

Applying (7) to the case $r=s=1$ and $\beta=\alpha$, we obtain

$$\sum_{p\leqq m}\frac{p^k}{e^{\alpha p^k}-1}=\int_2^m\frac{x^k}{(e^{\alpha x^k}-1)\log x}dx+O(e^{-c\sqrt{\log m}})\cdot\alpha^{-1-a},$$

where

$$\int_2^m\frac{x^k}{(e^{\alpha x^k}-1)\log x}dx=\int_2^{\alpha^{-a/2}}+\int_{\alpha^{-a/2}}^m=O(\alpha^{-1-a/2})$$

$$+\frac{1}{\alpha^{1+a}}\int_{\alpha^{1/2}}^{\alpha m^k}\frac{u^a}{(e^u-1)\log(u/\alpha)}du.$$

We transform the integral in the right hand side as follows;

$$\int_{\alpha^{1/2}}^{\alpha m^k}=\int_{\alpha^{-1/2}}^{\infty}+\int_{\alpha^{1/2}}^{\alpha^{-1/2}}-\int_{\alpha m^k}^{\infty}=O(e^{-c\alpha^{-1/2}})+\int_{\alpha^{1/2}}^{\alpha^{-1/2}}+O(e^{-\frac{1}{2}\alpha m^k})$$

$$=O(\exp(-c(\log n)^{\frac{k}{k+1}}))+\int_{\alpha^{1/2}}^{\alpha^{-1/2}}\frac{u^a}{(e^u-1)\log(u/\alpha)}du.$$

Since

$$\frac{1}{\log(u/\alpha)}=\frac{1}{\log 1/\alpha}\left(1+O\left(\left|\frac{\log u}{\log \alpha}\right|\right)\right)$$

for $\alpha^{1/2}\leqq u\leqq \alpha^{-1/2}$, we have

$$\int_{\alpha^{1/2}}^{\alpha^{-1/2}}\frac{u^a}{(e^u-1)\log(u/\alpha)}\,du=\frac{1}{\log 1/\alpha}\int_{\alpha^{1/2}}^{\alpha^{-1/2}}\frac{u^a}{e^u-1}\,du+O\left(\frac{1}{|\log \alpha|^2}\right)$$

$$=\frac{1}{\log 1/\alpha}\int_0^{\infty}\frac{u^a}{e^u-1}\,du\left(1+O\left(\frac{1}{\log n}\right)\right)=\frac{\Gamma(1+a)\zeta(1+a)}{\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right).$$

Therefore we have

$$\int_2^m\frac{x^k}{(e^{\alpha x^k}-1)\log x}\,dx=\frac{\Gamma(1+a)\zeta(1+a)}{\alpha^{1+a}\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right)$$

and consequently

$$\sum_{p\leqq m}\frac{p^k}{e^{\alpha p^k}-1}=\frac{\Gamma(1+a)\zeta(1+a)}{\alpha^{1+a}\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right).$$

As a consequence of our definition of α, it must satisfy the relation

$$n=\frac{\Gamma(1+a)\zeta(1+a)}{\alpha^{1+a}\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right),$$

which gives an asymptotic expansion of α stated in Corollary **1**.

Now applying (6) and (7) to A_1 and A_2, we have generally

$$A_1=-\int_2^m\frac{\log(1-e^{-\alpha x^k})}{\log x}\,dx+O(|\log(1-e^{-\alpha m^k})|\,me^{-c\sqrt{\log m}})$$
$$+O(e^{-c\sqrt{\log m}})\cdot\alpha^{-a}\min(1,m\alpha^a). \tag{45}$$

$$A_2=\int_2^m\frac{x^{2k}e^{\alpha x^k}}{(e^{\alpha x^k}-1)^2\log x}\,dx+O(e^{-c\sqrt{\log m}})\cdot\alpha^{-2-a}\min(1,m\alpha^a). \tag{46}$$

After similar calculations as above, we have

$$-\int_2^m\frac{\log(1-e^{-\alpha x^k})}{\log x}\,dx=\frac{\Gamma(1+a)\zeta(1+a)}{\alpha^a\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right).$$

$$\int_2^m\frac{x^{2k}e^{\alpha x^k}}{(e^{\alpha x^k}-1)^2\log x}\,dx=\frac{\Gamma(2+a)\zeta(1+a)}{\alpha^{2+a}\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right).$$

Therefore

$$A_1=\frac{\Gamma(1+a)\zeta(1+a)}{\alpha^a \log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right)$$

$$=\{\Gamma(2+a)\zeta(1+a)\}^{\frac{k}{k+1}}\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}}\left(1-\frac{k\log\log n}{(k+1)\log n}+O\left(\frac{1}{\log n}\right)\right).$$

$$A_2=\frac{\Gamma(2+a)\zeta(1+a)}{\alpha^{2+a}\log 1/\alpha}\left(1+O\left(\frac{1}{\log n}\right)\right)$$

$$=(1+a)n^{\frac{2k+1}{k+1}}\left\{\frac{\log n}{\Gamma(2+a)\zeta(1+a)}\right\}^{\frac{k}{k+1}}\left(1+\frac{k\log\log n}{(k+1)\log n}+O\left(\frac{1}{\log n}\right)\right).$$

Thus we obtain Corollary 1.

Assume now $m\leqq n^{\frac{1}{k+1}}$, then

$$\frac{m}{\log m}\leqq\left(\frac{n}{\log^k n}\right)^{\frac{1}{k+1}},\quad \alpha=O\left(\frac{m}{n\log m}\right),\quad \alpha m^k=O\left(\frac{1}{\log m}\right).$$

In this case, we have

$$\sum_{p\leqq m}\frac{p^k}{e^{\alpha p^k}-1}=\int_2^m\frac{x^k}{(e^{\alpha x^k}-1)\log x}\,dx+O(e^{-c\sqrt{\log m}})\cdot m\alpha^{-1},$$

where

$$\int_2^m\frac{x^k}{(e^{\alpha x^k}-1)\log x}\,dx=\frac{1}{\alpha}\int_2^m\frac{1+O(\alpha x^k)}{\log x}\,dx$$

$$=\frac{m}{\alpha\log m}\left(1+O\left(\frac{1}{\log m}\right)\right).$$

Therefore we have

$$\sum_{p\leqq m}\frac{p^k}{e^{\alpha p^k}-1}=\frac{m}{\alpha\log m}\left(1+O\left(\frac{1}{\log m}\right)\right)$$

and consequently

$$\alpha=\frac{m}{n\log m}\left(1+O\left(\frac{1}{\log m}\right)\right).$$

Since

$$-\int_2^m\frac{\log(1-e^{-\alpha x^k})}{\log x}\,dx=-\int_2^m\frac{\log(\alpha x^k)+O(1)}{\log x}\,dx$$

$$=\frac{m}{\log m}\log\frac{1}{\alpha}\cdot\left(1+O\left(\frac{1}{\log m}\right)\right)-km,$$

$$\int_2^m \frac{x^{2k}\,e^{\alpha x^k}}{(e^{\alpha x^k}-1)^2\log x}\,dx=\frac{1}{\alpha^2}\int_2^m\frac{1+O(\alpha x^k)}{\log x}\,dx$$

$$=\frac{m}{\alpha^2\log m}\left(1+O\left(\frac{1}{\log m}\right)\right),$$

we have by (45) and (46)

$$A_1=\frac{m}{\log m}\log\frac{1}{\alpha}\cdot\left(1-\frac{k\log m}{\log 1/\alpha}+O\left(\frac{1}{\log m}\right)\right)$$

$$=\frac{m}{\log m}\log\frac{n\log m}{m^{k+1}}\cdot\left(1+O\left(\frac{1}{\log m}\right)\right),$$

$$A_2=\frac{m}{\alpha^2\log m}\left(1+O\left(\frac{1}{\log m}\right)\right)=\frac{n^2\log m}{m}\left(1+O\left(\frac{1}{\log m}\right)\right).$$

Corollary 2 is thereby proved.

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Note

An asymptotic formula for $T(n,n;1)$ was obtained by Haselgrave and Temperley (Cambridge Phil. Soc., 50 (1954), 225-241). Their method is very different from ours. They make use of a function with two variables as their generating function and contour integrals.