Journal of the Mathematical Society of Japan Vol. 9, No. 4, October, 1957

On Prüfer rings.

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(Received March 25, 1957)

To find a necessary and sufficient condition for an integral domain Λ to satisfy the following condition (C): (C) *If A and B are torsion-free Λ-modules, then $A\otimes_\Lambda B$ is also a torsion-free Λ-module.* This is a problem recently proposed by M. Nagata.[1]

We know, following J. Dieudonné,[2] that (C) is satisfied by any Dedekind ring, and more generally by any Prüfer ring, as is shown by H. Cartan and S. Eilenberg in their recent publication.[3] In this paper, we shall prove conversely that a ring satisfying (C) is necessarily a Prüfer ring (Theorem 2). This will solve the above problem completely, and at the same time yield a characterization of Prüfer rings.[4]

Let Λ denote an integral domain (with an identity). Instead of $A\otimes_\Lambda B$, $\mathrm{Tor}_n^\Lambda(A, B)$, $\mathrm{Hom}_\Lambda(A, B)$, $\mathrm{Ext}_\Lambda^n(A, B)$, we shall use simplified notations $A\otimes B$, $\mathrm{Tor}_n(A, B)$, $\mathrm{Hom}(A, B)$, $\mathrm{Ext}^n(A, B)$, A and B being Λ-modules. (See *HA*, for the definition of these *functors*).

LEMMA 1. *For a finitely (Λ-) generated torsion-free Λ-module A, there exists a free Λ-module F on finite basis containing A and such that the residue class module F/A is a torsion module.*

PROOF. We have only to modify the proof of *HA*, Prop. VII. 2.4: Let Q be the quotient field of Λ, then A is a submodule of $Q\otimes A$, and a system of Λ-generators $\{a_1,\cdots, a_r\}$ is also a system of Q-generators of the vector space $Q\otimes A$ over Q. Hence the set $\{a_1\cdots, a_r\}$ contains a Q-basis of $Q\otimes A$, say $\{a_1,\cdots, a_s\}$. If

$$a_i=\sum_{j=1}^{s} q_{ij}a_j\,, \qquad i=1,\cdots, r\,, \qquad q_{ij}\in Q\,,$$

1) Sûgaku, vol. 6.1 (July, 1954), Problem 6.1.13.

2) J. Dieudonné, *Sur les produits tensoriels*, Ann. de l'Ecole Norm. Sup. LXIV (1947), pp. 101–117. Théorème 3.

3) H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press (1956). Prop. VII. 4.5. In the following we shall refer to this book by *HA*.

4) The author published this result already in Sûgaku, vol. 8. (1957).

take a non-zero $\lambda \in \Lambda$ such that every $\lambda q_{ij} \in \Lambda$. Then the Λ-submodule F of $Q \otimes A$ generated by $\{\lambda^{-1}a_1, \cdots, \lambda^{-1}a_s\}$ satisfies the conditions of Lemma 1.

The following theorem, which will be used later, will be of interest in itself.

THEOREM 1. *For any two Λ-modules A and B,* $\mathrm{Tor}_n(A, B)$ *are all torsion modules* $(n=1, 2, \cdots)$.

PROOF. As the functor Tor_n commutes with the direct limit (*HA*, Prop. VI. 1.3), the Theorem follows if we prove it under the assumption that A is finitely generated. Now, if A, assumed to be finitely generated, is moreover torsion-free, then Lemma 1 may be applied, and we have the isomorphism

$$\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_{n+1}(F/A, B)\,, \qquad n \geqq 1\,,$$

since F is a free Λ-module. As F/A is a torsion module, so is $\mathrm{Tor}_{n+1}(F/A, B)$ (*HA*, Prop. VI. 1.7), and hence $\mathrm{Tor}_n(A, B)$ is a torsion module $(n=1, 2, \cdots)$. If A has the torsion tA, we have the following exact sequence

$$\mathrm{Tor}_n(tA, B) \rightarrow \mathrm{Tor}_n(A, B) \rightarrow \mathrm{Tor}_n(A/tA, B)\,.$$

We know already that both left and right terms of this sequence are torsion modules, hence the middle term $\mathrm{Tor}_n(A, B)$ is also a torsion module.

An integral domain Λ is called a *Prüfer ring* if every finitely generated ideal of Λ is inversible in the sense of the multiplicative ideal theory. Obviously this notion is a generalization of that of a Dedekind ring. As the projectivity and the inversibility of an ideal are equivalent, Λ is a Prüfer ring if and only if every finitely generated ideal of Λ is projective (see *HA*, Chap. VII).

THEOREM 2. *For an integral domain Λ, the following conditions are equivalent:*

a) *Λ is a Prüfer ring.*

b) $\mathrm{Tor}_2(A, B)=0$ *for every pair of Λ-modules A and B.*

c) $\mathrm{Tor}_1(X, A)$ $(\cong \mathrm{Tor}_1(A, X))=0$ *for every Λ-module X, if A is torsion-free.*

d) *$A \otimes B$ is torsion-free, if both A and B are torsion-free.* (Condition (C) at the beginning of this paper).

PROOF. *a)$\Rightarrow$b).* See *HA*, Prop. VI. 2.9.

b)$\Rightarrow$c). As in the Proof of Theorem 1, we may assume that A

is finitely generated. We then apply Lemma 1, and obtain an isomorphism

$$\mathrm{Tor}_1(X, A) \cong \mathrm{Tor}_2(X, F/A).$$

But the right hand side vanishes under the assumption *b*).

c) $\Rightarrow$ *d*). See the proof of *HA*, Prop. VII. 4.5.

d) $\Rightarrow$ *c*). Take an exact sequence

$$0 \to P' \to P \to X \to 0$$

where P is a Λ-projective module. As P' is torsion-free, so is $P' \otimes A$ by the assumption *d*), and hence $\mathrm{Tor}_1(X, A)$, which is a submodule of $P' \otimes A$, is also torsion-free. But $\mathrm{Tor}_1(X, A)$ is at the same time a torsion module by Theorem 1. It follows that $\mathrm{Tor}_1(X, A) = 0$.

The proof of the last step *c*) $\Rightarrow$ *a*) will be preceded by two lemmas on the Λ-homomorphic mapping

$$\sigma: \quad \mathrm{Hom}(B, C) \otimes A \to \mathrm{Hom}(\mathrm{Hom}(A, B), C)$$

defined by

$$\sigma(f \otimes a)(g) = f(ga), \qquad f \in \mathrm{Hom}(B, C), \quad g \in \mathrm{Hom}(A, B),$$

where A, B, C are arbitrary Λ-modules. It is known that σ is an isomorphism if A is a finitely generated Λ-projective module (*HA*, Prop. VI. 5.2).

LEMMA 2. *If A is a finitely generated module, and C is Λ-injective, then σ is an epimorphism.*

PROOF. There exists an exact sequence

$$0 \to P' \to P \to A \to 0$$

where P is a finitely generated Λ-projective module. We consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(B, C) \otimes P & \longrightarrow & \mathrm{Hom}(B, C) \otimes A & \longrightarrow & 0 \\
\downarrow \sigma & & \downarrow \sigma & & \\
\mathrm{Hom}(\mathrm{Hom}(P, B), C) & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}(A, B), C) & \longrightarrow & 0
\end{array}$$

The top row is exact obviously, and so is moreover the bottom row, since $\mathrm{Hom}(A, B) \to \mathrm{Hom}(P, B)$ is a monomorphism and C is injective. The Lemma follows immediately since the map σ of left hand side is an isomorphism as is remarked above.

LEMMA 3. *Let Λ satisfy the condition c) of Theorem 2. If A is a finitely generated torsion-free module, and B is an Λ-injective module (or* *more generally a divisible module), then σ is a monomorphism.*

PROOF. By Lemma 1, there exists a free Λ-module F on finite basis containing A. The exact sequence

$$0 \to A \to F \to F/A \to 0$$

yields the following commutative diagram with an exact top row:

$$\begin{tikzcd} \mathrm{Tor}_1(\mathrm{Hom}(B,C), F/A) \arrow[r] & \mathrm{Hom}(B,C)\otimes A \arrow[r] \arrow[d, "\sigma"] & \mathrm{Hom}(B,C)\otimes F \arrow[d, "\sigma"] \\ & \mathrm{Hom}(\mathrm{Hom}(A,B),C) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}(F,B),C) \end{tikzcd}$$

If B is divisible (especially injective), $\mathrm{Hom}(B, C)$ is torsion-free, hence $\mathrm{Tor}_1(\mathrm{Hom}(B, C), F/A)=0$ by the condition c). As the map σ of the right hand side is an isomorphism, σ of the left hand side is an monomorphism, as desired.

Now, we shall conclude the proof of Theorem 2 by showing the implication $c) \Rightarrow a)$. Let A be a fnintely generated torsion-free Λ-module, and Y an arbitrary Λ-module. There exists an exact sequence

$$0 \to Y \to B \to X \to 0$$

with B Λ-injective, and this yields the following exact sequence:

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, X) \to \mathrm{Ext}^1(A, Y) \to 0$$

Let C be an Λ-injective module, then we have also the exact sequence

$$0 \to \mathrm{Hom}(X, C) \to \mathrm{Hom}(B, C) \to \mathrm{Hom}(Y, C) \to 0$$

We consider the following commutative diagram, which is derived from these two exact sequences:

$$\begin{tikzcd} & \mathrm{Tor}_1(\mathrm{Hom}(Y,C), A) \arrow[r] & \mathrm{Hom}(X,C)\otimes A \arrow[r] \arrow[d, "\sigma"] & \mathrm{Hom}(B,C)\otimes A \arrow[d, "\sigma"] \\ 0 \arrow[r] & \mathrm{Hom}(\mathrm{Ext}^1(A,Y),C) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}(A,X),C) \arrow[r] & \mathrm{Hom}(\mathrm{Hom}(A,B),C) \end{tikzcd}$$

Both rows of this diagram are clearly exact, σ of the left hand side is an epimorphism by Lemma 2, σ of the right hand side is a monomorphism (actually an isomorphism) by Lemma 3, and finally $\mathrm{Tor}_1(\mathrm{Hom}(Y, C), A)=0$ by the condition c). Hence we have $\mathrm{Hom}(\mathrm{Ext}^1(A, Y), C)=0$. As C is an arbitrary injective module, this implies $\mathrm{Ext}^1(A, Y)=0$ (take e. g. C containing $\mathrm{Ext}^1(A, Y)$). As this holds for every Λ-module Y, A is Λ-projective. Since an ideal of Λ is obviously torsion-free, this implies a).

A ring Λ is said to be of ***weak global dimension*** $\leqq 1$, if $\mathrm{Tor}_2^{\Lambda}(A, B) = 0$ for every pair of Λ-modules A, B. So the equivalence of $a)$ and $b)$ may be formulated as follows:

COROLLARY. *An integeral domain Λ is of weak global dimension $\leqq 1$, if and only if Λ is a Prüfer ring.*

A ring Λ is called *left (right) semi-hereditary*, if every finitely generated left (right) ideal of Λ is projective as a Λ-module. If Λ is commutative, the distinction of 'left' and 'right' has no meaning, and a Prüfer ring is nothing but a semi-hereditary integral domain. The implication $a) \Rightarrow b)$ is a special case of the corresponding proposition for an arbitrary left or right semi-hereditary ring. It will be of interest to ask for the validity of the converse of this proposition. As the notion of weak global dimension is of symmetrical character, if the converse is true in general, it will then follow the equivalence of the left and the right semi-hereditarity, solving a problem of H. Cartan and S. Eilenberg (*HA*, p. 15). The above Corollary shows that the converse is true for every integral domain. But we have not succeeded in general case.

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