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A note on one-parameter and monothetic groups.*

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An important technique in the study of locally compact groups is the construction of one-parameter subgroups within the group. Such subgroups are topological analogues of the cyclic subgroups, which play such an important role in abstract group theory. The principal fact about one parameter subgroups of locally compact groups is that they are either isomorphic and homeomorphic to the reals, or else they have compact closure. The corresponding fact about cyclic subgroups is almost identical: a cyclic subgroup of a locally compact group is either discrete or has compact closure. (This latter fact includes the usual facts about cyclic subgroups of abstract groups, since any group is a topological group if given the discrete topology.)

Several proofs of these results have been given in the literature (in particular, see [3] and [5]). The purpose of this note is to prove a slight generalization of these facts. The proof to be given is not, however, simply a generalization of the extant proofs. Rather, it is based on certain concepts introduced in an earlier paper [7], and thus becomes, in a sense, more intrinsic and perhaps more conceptual.

A remark is in order concerning the terminology employed in this note. The three definitions stated in the sequel will give meaning to certain expressions used throughout the paper. The reader is cautioned that this usage is not universal. In general, the common practice is to give a more restricted sense to some of these terms.

DEFINITION 1. Let R denote the additive group of real numbers, and let Y be any subgroup of R. A topological group H is called a one-parameter group if there exists a continuous homomorphism f of Y onto H. In this case, the group Y is called a parameter for H, and the mapping f is called a parametrization of H by Y. If a subgroup H of a topological group G is a one-parameter group, then

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H is called a one-parameter subgroup of G.

If the subgroup Y of R is closed, then either $Y=R$, or else Y is a cyclic subgroup of R and thus isomorphic (and homeomorphic) to the discrete group of integers. If, on the other hand, Y is not closed, then Y is dense in R. In this case, R can be considered as the completion of the group Y. If H is a one-parameter group having parameter Y and parametrization f, then f has a unique extension to a continuous homomorphism of R into the completion H^* of H. Thus, for many purposes, it is sufficient to consider only those one-parameter groups which have a closed parameter.

DEFINITION 2. A one-parameter group H with a closed parameter Y in R will be called a regular one-parameter group. If the parameter of such a group is all of R, then the group will be called a real one-parameter group; if the parameter is cyclic, the group will be called cyclic.

(The term "one-parameter group" usually refers to what we have called a "real one-parameter group".)

All one-parameter groups are abelian, so that we may apply the concepts introduced in [7]. We shall need the following facts. A non-empty subset S of an (additive) abelian group G is called a semigroup if $x, y \in S$ implies that $x+y \in S$. A semigroup S is called 0-proper if $0 \notin S$. A semigroup M is called a maximal 0-proper open semigroup if M is a semigroup which is open, 0-proper, and maximal with respect to these properties. If M is a maximal 0-proper open semigroup, then the set $-M$ of inverses of elements of M is also such a semigroup, and the complement $b(M)$ of $(M \cup -M)$ is a closed subgroup of G. This group $b(M)$ is called the residual subgroup of G belonging to M. A group G is called a radical group if there are no 0-proper open semigroups in G, and G is called maximally radical-free if $\{0\}$ is a residual subgroup in G. Clearly, if $b(M)$ is a residual subgroup in G, then the quotient group $G/b(M)$ is maximally radical-free.

The version of the theorems discussed above which we shall prove here is the following.

THEOREM 1. *A regular one-parameter group H is either a radical group, or else H is isomorphic and homeomorphic to its parameter, under the given parametrization.*

It is convenient to break the proof up into three lemmas.

LEMMA 1.1. *If the parametrization of a regular one-parameter* *group is not an algebraic isomorphism, then the group is compact.*

PROOF. Let H be a regular one-parameter group, with parameter Y and parametrization f. If f is not an algebraic isomorphism, then the kernel $K=f^{-1}\{0\}$ is a non-trivial closed subgroup of Y. It is well-known that f induces a continuous isomorphism of Y/K onto H, and since Y/K is compact, H is also compact.

This lemma has an obviously sharper form, which states that H is either isomorphic and homeomorphic to the circle group, or is cyclic of finite order. For the purposes of Theorem 1, however, it is sufficient to note that every compact abelian group is a radical group [6]. We may now restrict attention to those cases in which the parametrization is an algebraic isomorphism.

LEMMA 1.2. *Let H be a regular one-parameter group, with parameter Y and isomorphic parametrization f. If H is not a radical group, then H is maximally radical-free.*

PROOF. If H is not a radical group, then there exists a residual subgroup $b(M)$ in H, with $b(M)\neq H$. We show that, under the hypothesis of the theorem, it is necessarily true that $b(M)=\{0\}$. If we set $P=f^{-1}b(M)$, then P is a closed subgroup of Y. Since $b(M)\neq H$, it follows that $P\neq Y$. Hence P is a discrete subgroup of Y, whatever the closed subgroup Y may be.

Clearly, there is a continuous isomorphism of Y/P onto $H/b(M)$, induced by f. As was noted above, $H/b(M)$ is maximally radical-free. On the other hand, if $P\neq\{0\}$, then Y/P is compact, so that $H/b(M)$ is compact. But a compact group is a radical group, and hence not maximally radical-free. Thus $P=\{0\}$, and consequently $b(M)=\{0\}$, which proves the lemma.

LEMMA 1.3. *Let H be a maximally radical-free regular one-parameter group, with parameter Y and parametrization f. Then f is a homeomorphism.*

PROOF. By virtue of Lemmas 1.1 and 1.2, these hypotheses guarantee that f is one-to-one, so that it only remains to show that f is an open mapping. We must now distinguish the two possibilities for the parameter Y.

If $Y=R$, then H is connected, and it follows from [7: Theorem 5.2] that there is a continuous isomorphism of H onto R. Following this with a suitable automorphism of R will yield a continuous isomorphism of H onto R which is the inverse of f. Thus f is indeed a homeomorphism.

If Y is discrete, it is sufficient to show that H is also discrete. Now H has two topologies making it a topological group: the given topology, and the interval topology given by the linear order induced by the maximal 0-proper open semigroup M for which $b(M)=\{0\}$. (Recall that H is maximally radical-free. See the discussion in [7] for the relevant facts about this interval topology.) The latter is weaker than the former, so that if H is not originally discrete, it is not discrete in the interval topology. However, if $a>0$ is a generator of H, and if H is not discrete in the interval topology, then there exists an integer k such that $0<ka<a$. Then $(k-1)a<0$, and hence [7: Lemma 3.2], $a<0$. This contradiction shows that H is indeed discrete in the interval topology, and hence discrete in the original topology. This completes the proof of Lemma 1.3.

Theorem 1 now follows at once from these three lemmas.

This result enables us to pass at once to the consideration of a slightly larger class of groups, which have been intensively studied, and which are of importance in many fields, in ergodic theory, for example.

DEFINITION 3. A topological group G will be called homothetic if G contains a dense one-parameter subgroup; G will be called a regular homothetic group if it contains a dense regular one-parameter subgroup.

A homothetic group with a dense cyclic subgroup has been called monothetic [1,4]. The compact homothetic groups with a dense real one-parameter subgroup are called solenoids [1]. These groups are discussed in detail in [1,4], but the matters treated there are complementary to (and deeper than) the subject matter of this paper.

Let G be a regular homothetic group, and let H be a dense regular one-parameter subgroup of G. If H is a radical group in its own right, then it is a radical subgroup of G, and hence $G=\overline{H}$ is also a radical group [7: Lemma 4.3]. Otherwise, H is isomorphic and homeomorphic to its parameter. Hence H is a locally compact subgroup of G, so that H is closed [2: page 27]. This yields at once.

THEOREM 2. *A regular homothetic group is either a radical group, or is itself a regular one-parameter group isomorphic and homeomorphic to its parameter.*

Specializing now to the locally compact case, we obtain the standard theorems [5: page 96] and [3: page 6]. Let G be a locally compact regular homothetic group, and let H be a dense regular one-parameter subgroup of G. If G is a radical group, then the component K of G is compact [7: Theorem 8.7]. If H is a real one-parameter group, then H is connected, so that G is also connected and therefore compact. On the other hand, if H is a dense cyclic subgroup of G, let x be a generator of H. If $x \in K$, then $H \subset K$, and therefore $G = K$ is compact. If $x \notin K$, then G/K is a totally disconnected regular homothetic group. Since K is compact, G will be compact if G/K is compact. It therefore suffices to consider a totally disconnected G, where G has a dense cyclic subgroup and is a radical group. Let C be any compact open subgroup of G. Then G/C is a discrete torsion group [7: Theorem 4.1] and is obviously a cyclic group. Hence G/C is compact, and since C is compact, it follows that G is compact.

Thus we have shown that a locally compact regular homothetic group is compact if it is a radical group. If it is not a radical group, then Theorem 2 applies. Thus we have completed a proof of the theorem of [5: page 96], which we state here in the present terminology.

THEOREM 3. *A locally compact regular homothetic group is either compact or is itself a regular one-parameter group whose parametrization is a homeomorphism.*

We may conclude with some remarks about homothetic groups which are not regular. In such a case, there is a continuous homomorphism of a dense subgroup Y of R onto a dense subgroup H of the homothetic group G. If G is complete, it may be regarded as the completion of H, and by the previous remarks, the parametrization can be extended to all of R. This gives at once

THEOREM 4. *The completion of a homothetic group is a regular homothetic group.*

Since every locally compact abelian group is complete, this shows that the hypothesis of regularity in Theorem 3 is superfluous.

THEOREM 5. *A locally compact homothetic group is a regular homothetic group.*

Theorems 3 and 5, taken together, constitute the content of [2: page 6, No. 2].

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