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Groups of projective transformations and groups of conformal transformations.

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In 1928, M. S. Knebelman [7] proved that a group of projective transformations in an n-dimensional affinely connected manifold preserves a projectively related affine connection if the group is of order $r\leqq n$. In this respect, it seems to be interesting to ask whether a group of conformal transformations of a Riemannian metric preserves or not another Riemannian metric. In § 1 we shall show that a group G of projective transformations of an affine connection leaves another projectively related affine connection invariant if G is compact. For a transitive group G we shall further prove that the same remains valid, if the isotropy group of G is compact, or, if the identity component of the linear isotropy group of G is irreducible and the space is projectively non-flat. In § 2 we shall obtain, concerning groups of conformal transformations, some results analogous to those proved in § 1.

On the other hand, the compactness, the completeness or the irreducibility of a Riemannian manifold implies strong restrictions on affine, conformal or isometric transformations [1, 3, 6, 8, 10, 11, 19, 20]. In this respect, in § 3 we shall study groups of projective transformations preserving the Ricci tensor in an affinely connected manifold and obtain the fact that such groups are affine in a space, which is complete, or, whose homogeneous holonomy group has no invariant hyperplane. In § 4 such groups will be discussed in a complete or compact Riemannian manifold. In § 5 we shall study groups of conformal transformations leaving the Ricci tensor invariant in a complete or compact Riemannian manifold and obtain some results analogous to those proved in § 4.

The last section is devoted to the proof of a lemma used in § 1 concerning groups of affine motions of the ordinary affine space.

§ 1. Groups of projective transformations.

Let M be an n-dimensional manifold[1] with an affine connection Γ^i_{jk} having no torsion[2], i. e. $\Gamma^i_{jk}=\Gamma^i_{kj}$[3]. If a transformation s of M onto itself carries a point p of M into a point p' of M, and if U and V are coordinate neighborhoods of p and p' respectively, then the restriction of s to $U\cap s^{-1}(V)$ can be represented by a coordinate transformation:

$$\bar{x}^i=\bar{x}^i(x^1, x^2,\cdots, x^n),$$

where (x) and $(\bar{x}^i)$ are systems of local coordinates in U and V respectively. Denoting by $(\Gamma^i_{jk})_q$ the components of the affine connection at a point $q\in U\cap s^{-1}(V)$ with respect to (x^i), we obtain a new connection $\bar{\Gamma}^i_{jk}$ in $s(U)\cap V$ whose components at $q'=s(q)$ are given by

$$(\bar{\Gamma}^a_{jk})_{q'}\frac{\partial x^i}{\partial \bar{x}^a}=\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}(\Gamma^i_{bc})_q+\frac{\partial^2 x^i}{\partial \bar{x}^j\partial \bar{x}^k}$$

with respect to $(\bar{x}^i)$, where all derivatives respresent their values at the point q'. It is easily seen that the affine connection $\bar{\Gamma}^i_{jk}$ thus locally constructed defines globally an affine connection in M. We have hence in M an affine connection denoted also by $\bar{\Gamma}^i_{jk}$, which is called the *affine connection induced from* Γ^i_{jk} *by* s.

In the manifold M the system of paths is defined by differential equations

$$\frac{d^2x^i}{d\sigma^2}+\Gamma^i_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}=0,$$

where Γ^i_{jk} is the affine connection of M. Such a parameter σ of a path is determined up to an affine transformation

$$\sigma'=a\sigma+b,$$

a and b being constant, and σ is called an *affine parameter* of the path. If a transformation s of M preserves the system of paths

1) Throughout the paper, we assume that manifolds of geometric objects, transformations etc. are differentiable and of class C^∞. Moreover, we suppose that $\dim M=n>1$. For simplicity we assume that any manifold is connected.

2) We consider only affine connections without torsion.

3) Indices a, b, c, i, j, k, l take the values in the range $\{1, 2,\cdots, n\}$. The usual summation convention is used with respect to this system of indices.

and the affine character of the parameter σ on each path, then s is called an ***affine transformation*** of the connection Γ^i_{jk} or simply of the manifold M, and we say that s leaves the connection Γ^i_{jk} invariant [14]. If s leaves the system of paths invariant, the affine character of the parameter σ being not necessarily preserved, then s is called a ***projective transformation*** of the affine connection Γ^i_{jk} or simply of M [14, 16].

A transformation s of M is affine, if and only if

$$\Gamma^i_{jk}(s)=\Gamma^i_{jk}, \tag{1.1}$$

where $\Gamma^i_{jk}(s)$ denotes the affine connection induced from Γ^i_{jk} by s [14]. The transformation s is projective in M, if and only if there exists a certain covariant vector field $\varphi(s)$ such that

$$\Gamma^i_{jk}(s)=\Gamma^i_{jk}+\delta^i_j\varphi_k+\delta^i_k\varphi_j, \; \delta^i_j=\begin{cases}1, & \text{if } i=j, \\ 0, & \text{if } i\neq j,\end{cases} \tag{1.2}$$

where φ_j denote the components of the field $\varphi(s)$ [14, 16]. By virtue of (1.2), if s and t are two projective transformations, we find[4]

$$\varphi(st)=\varphi(s)+\hat{s}\cdot\varphi(t). \tag{1.3}$$

When, for two affine connections Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$, there exists a certain covariant vector field ψ with components ψ_j such that

$$\overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j,$$

they are ***projectively related*** to each other, by definition. Let G be a group of projective transformations of the affine connection Γ^i_{jk}. If there exists a certain projectively related affine connection for which G is a group of affine transformations, then the group G is said to be ***essentially affine*** with respect to the connection Γ^i_{jk}.

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4) Let s be a transformation of M and X a tangent vector of M at a point $p\in M$. Then the differential mapping ds of s carries X into a tangent vector $ds\cdot X$ at the point $s(p)$. Since s is a homeomorphism, ds is an isomorphism $T_p\to T_{p'}$, where T_p and $T_{p'}$ are the tangent spaces of M respectively at p and $p'=s(p)$. Thus the dual mapping of ds is also an isomorphism. Then we denote simply by the simbol $\hat{s}$ the inverse of the dual mapping of ds. Therefore, for a covariant vector u at a point $p\in M$, $\hat{s}\cdot u$ is a covariant vector at the point $s(p)$.

Given a covariant vector field φ in M, we denote by $[\varphi]_p$ the value of the field φ at a point $p\in M$. We shall define a covariant vector field ψ by $[\psi]_{p'}=\hat{s}\cdot[\varphi]_p$ and denote by $\hat{s}\cdot\varphi$ the field ψ thus obtained, where p is an arbitrary point and $p'=s(p)$. If follows hence $[\hat{s}\cdot\varphi]_{s(p)}=\hat{s}\cdot[\varphi]_p$.

We shall now consider projective transformations of a Riemannian manifold. A transformation s of a Riemannian manifold[5] M with the metric tensor g_{ij} is called projective if it is projective with respect to the affine connection determined by the Christoffel symbols $\{{}^{\,i}_{jk}\}$ of g_{ij}. If s is affine with respect to $\{{}^{\,i}_{jk}\}$, it is called an affine transformation of M.

M. S. Knebelman [7] proved the following theorem:

If a given affine connection admits a finite continuous group G of projective or affine collineations, there exists projectively related affine connections for which G is a group of affine collineations, where G is of order $r \leqq n$. The determination of these projectively related connections depends upon n functions which are arbitrary in $n-r$ of the coordinates.

In this regard, we have the following

THEOREM 1. *Let G be a compact group[6] of projective transformations of an affinely connected manifold M. Then G is essentially affine with respect to the affine connection of M.*

PROOF. We shall construct an affine connection which is projectively related to the given connection Γ^i_{jk} and invariant under G. For this purpose, we define a covariant vector field ψ as follows. Denoting by Δ the total measure of the compact group G, we construct a covariant vector field ψ by

$$[\psi]_p = \frac{1}{\Delta}\int [\varphi(t)]_p dt ,$$

p being an arbitrary point of M, where $\varphi(t)$ is the covariant vector field given by (1.2) corresponding to an element t of G and the integral is extended over the whole group manifold of G. For the sake of simplicity let us put

$$\psi = \frac{1}{\Delta}\int \varphi(t) dt . \tag{1.4}$$

Denoting by ψ_j the components of the field ψ, we shall introduce an affine connection by

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j . \tag{1.5}$$

5) We consider only Riemannian manifolds whose metric tensor is positive definite.

6) We consider only Lie groups throughout the paper.

For an element s of G we denote by $\bar{\Gamma}^i_{jk}(s)$ the affine connection induced from $\bar{\Gamma}^i_{jk}$ by s. Then, it follows that s is a projective transformation of $\bar{\Gamma}^i_{jk}$, i. e.

$$\bar{\Gamma}^i_{jk}(s)=\bar{\Gamma}^i_{jk}+\delta^i_j\psi_k(s)+\delta^i_k\psi_j(s) \tag{1.6}$$

with a covariant vector field $\psi(s)$ having $\psi_j(s)$ as its components, since the new connection $\bar{\Gamma}^i_{jk}$ is projectively related to the original connection Γ^i_{jk} for which s is projective. As a consequence of (1.2), (1.5) and (1.6), we have easily

$$\psi(s)=\varphi(s)+\hat{s}\cdot\psi-\psi\,,$$

where $\varphi(s)$ is the covariant vector field defined by (1.2).

Substituting (1.4) in the second term of the right-hand side, we obtain

$$\psi(s)=\frac{1}{\Delta}\int(\varphi(s)+\hat{s}\cdot\varphi(t))dt-\psi\,,$$

because we have from the definitions

$$\hat{s}\cdot\int\varphi(t)dt=\int\hat{s}\cdot\varphi(t)dt\,.$$

Therefore, by virtue of (1.3) it follows

$$\psi(s)=\frac{1}{\Delta}\int\varphi(st)dt-\psi\,.$$

On the other hand, we have

$$\int\varphi(st)\,dt=\int\varphi(t)dt\,,$$

since the invariant measure over the compact group is two-sided invariant. Consequently, we can conclude

$$\psi(s)=0\,.$$

This shows by virtue of (1.6) that the affine connection $\bar{\Gamma}^i_{jk}$ is invariant under the group G. The proof of Theorem 1 is therefore completed.

We shall, from now on, consider transitive groups of projective transformations. First we have the following

THEOREM 2. *Let G be a transitive group of projective transformations in an affinely connected manifold M. If the isotropy group of G at a point of M is compact, then G is essentially affine with respect to* *the affine connection of M.*

PROOF. Since the isotropy group H of G at a point O is compact, as a consequence of Theorem 1, we may suppose that the given affine connection Γ^i_{jk} is invariant under the group H. Thus the covariant vector field $\varphi(s)$ defined by (1.2) corresponding to any $s \in H$ vanishes. Moreover, if $s's^{-1} \in H$, we have

$$\varphi(s) = \varphi(s').$$

In fact, putting $s's^{-1} = t$, we find from (1.3)

$$\varphi(s') = \varphi(st) = \varphi(s) + \hat{s} \cdot \varphi(t).$$

This implies immediately $\varphi(s) = \varphi(s')$ because $\varphi(t) = 0$.

We shall construct a projectively related affine connection which is invariant under G. For this purpose, we define a covariant vector field ψ as follow. For a point p of M we put[7]

$$[\psi]_p = [\varphi(s)]_p, \tag{1.7}$$

where s is an element of G such that $s(O) = p$. The right-hand side of (1.7) is independent of the choice of s such as $s(O) = p$, since we have $\varphi(s) = \varphi(s')$ for s and s' such as $s's^{-1} \in H$. If we put

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j,$$

ψ_j being the components of the field ψ defined by (1.7), then we can assign to any element s of G a covariant vector field $\psi(s)$ defined by (1.6), where

$$\psi(s) = \varphi(s) + \hat{s} \cdot \psi - \psi.$$

Thus the connection $\bar{\Gamma}^i_{jk}$ is invariant under G, if $\psi(s) = 0$ for any s of G.

Let q be an arbitrary point of M and t an element of G such that $t(O) = q$. From (1.7) we have

$$[\psi]_q = [\varphi(t)]_q$$

and then

$$[\hat{s} \cdot \psi]_{s(q)} = [\hat{s} \cdot \varphi(t)]_{s(q)}$$

for any $s \in G$. Thus it follows

$$[\varphi(s) + \hat{s} \cdot \psi]_{s(q)} = [\varphi(s) + \hat{s} \cdot \varphi(t)]_{s(q)}.$$

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7) See the footnote 4).

Therefore, by virtue of (1.3) we obtain

$$[\varphi(s)+\hat{s}\cdot\psi]_{s(q)}=[\varphi(st)]_{s(q)}=[\varphi(st)]_{st(o)}=[\psi]_{s(q)}\,.$$

It follows consequently

$$[\psi(s)]_{s(q)}=[\varphi(s)+\hat{s}\cdot\psi]_{s(q)}-[\psi]_{s(q)}=0\,.$$

This shows that the field $\psi(s)$ vanishes identically, since q is an arbitrary point of M. Thus the proof of Theorem 2 is completed.

Let L and $\tilde{L}$ be respectively the group of all non-singular matrices of the form

$$\sigma=\begin{pmatrix} 1 & a^{o}_{\cdot j}(\sigma) \\ 0 & a^{i}_{\cdot j}(\sigma) \end{pmatrix} \tag{1.8}$$

and the group of all non-singular (n, n)-matrices. The correspondence $\sigma\to(a^{i}_{\cdot j}(\sigma))$ defines a homomorphism β of L onto $\tilde{L}$.

In the previous paper [4] the author has proved the following propositions:

(i) If G is a group of projective transformations in an affinely connected manifold M, then the isotropy group H of G at a point O is homomorphic to a group K_o of matrices of the form (1.8). If G is effective on M, two groups H and K_o are isomorphic to each other. The homomorphism of H onto K_o is denoted by α: $H\to K_o$.

(ii) Let s be an element of H and $\varphi(s)$ the covariant vector field defined by (1.2) corresponding to s. Putting

$$\alpha(s)=\begin{pmatrix} 1 & a^{o}_{\cdot j}(s) \\ 0 & a^{i}_{\cdot j}(s) \end{pmatrix}, \qquad s\in H,$$

we see that the covariant vector $[\varphi(s)]_o$, the value of the field $\varphi(s)$ at O, has $a^{o}_{\cdot j}(s)$ as its components in the natural way.

(iii) The linear isotropy group[8] $\tilde{H}$ of G at O is given by $\tilde{H}=\beta(K_o)$. Then, for an element s of H the linear transformation

$$\beta\circ\alpha(s)=(a^{i}_{\cdot j}(s))$$

of the tangent space T_o of M at O is nothing but the linear trans-

8) The isotropy group H of G at a point O induces a group $\tilde{H}$ of linear transformations of the tangent space T_0 at O. The linear group H is called as usual the linear isotropy group of G at O.

formation induced in T_o by the differential mapping of s.

Denoting by N the kernel of the homomorphism $\beta: L \rightarrow \tilde{L}$, we have obtained in [4] the following

LEMMA 1. *Let G be a transitive group of projective transformations in an affinely connected manifold M. Then the affine connection of M is projectively flat, if* $\dim N \cap K_o > 0$.

We need here the following lemma for the later use.

LEMMA 2. *Let Γ be a subgroup of the group L and $\tilde{\Gamma}$ the subgroup of $\tilde{L}$ defined by $\tilde{\Gamma} = \beta(\Gamma)$. If the kernel $N \cap \Gamma$ of the homomorphism β in Γ is discrete, and, if the identity component of $\tilde{\Gamma}$ is an irreducible group of matrices, then there exists a matrix*

$$T = \begin{pmatrix} 1 & \xi_j \\ 0 & \delta^i_j \end{pmatrix} \tag{1.9}$$

such that

$$T \sigma T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & a^{i}_{\cdot j}(\sigma) \end{pmatrix} \tag{1.10}$$

for any matrix σ of Γ.

The proof of Lemma 2 will be given in § 6.

From (1.10) it follows easily that

$$\xi_j = a^{a}_{\cdot j}(\sigma) \xi_a + a^{o}_{\cdot j}(\sigma)$$

for any matrix σ of Γ. This implies the following

LEMMA 2'. *Let G be a group of projective transformations in an affinely connected manifold M. If the kernel $N \cap K_o$ of the homomorphism $\beta: K_o \rightarrow \tilde{H}$ is discrete, and, if the identity component of $\tilde{H}$ is irreducible in the tangent space T_o, then there exists a covariant vector $u = (u_1, u_2, \cdots, u_n)$ at the point O such that*

$$u_j = a^{a}_{\cdot j}(s) u_a + a^{o}_{\cdot j}(s)$$

or equivalently

$$u = [\varphi(s)]_o + \hat{s} \cdot u \tag{1.11}$$

for all $s \in H$, where $a^{o}_{\cdot j}(s)$ and $a^{i}_{\cdot j}(s)$ are coefficients of the matrix $\alpha(s)$.

THEOREM 3. *Let G be a transitive group of projective transformations of a manifold with an affine connection which is not projectively flat. If the identity component of the linear isotropy group of G at a* *point is irreducible in the tangent space at the point, then G is essentially affine with respect to the affine connection.*

PROOF. Keeping notations as above, we see by virtue of Lemma 1 that the kernel $N\cap K_o$ of $\beta: K_o\rightarrow\tilde{H}$ is discrete, since G is transitive and the affine connection Γ^i_{jk} is not projectively flat. Further, because the identity component of $\tilde{H}$ is irreducible, Lemma 2′ implies that there exists at the point O a covariant vector u satisfying (1.11) for all $s\in H$.

We shall now define a covariant vector field ψ in M as follows. Taking an arbitrary point p of M and an element s of G such that $s(O)=p$, we put

$$[\psi]_p=[\varphi(s)]_p+\hat{s}\cdot u\,, \tag{1.12}$$

where $\varphi(s)$ is the field defined by (1.2) corresponding to s. Here, the sum in the right-hand side is independent of the choice of s such that $s(O)=p$. In fact, if s' is another element of G such that $s'(O)=p$, then $s's^{-1}=t\in H$. Consequently, we have

$$\begin{aligned}[\varphi(s')]_p+\hat{s}'\cdot u&=[\varphi(st)]_p+\widehat{(st)}\cdot u\\&=[\varphi(s)+\hat{s}\cdot\varphi(t)]_p+\hat{s}\circ\hat{t}\cdot u\\&=[\varphi(s)]_p+\hat{s}\cdot\{[\varphi(t)]_o+\hat{t}\cdot u\}\,.\end{aligned}$$

On the other hand, since the covariant vector u satisfies (1.11), it follows

$$u=[\varphi(t)]_o+\hat{t}\cdot u\,,$$

where t belongs to H. Thus we find

$$[\varphi(s)]_p+\hat{s}\cdot u=[\varphi(s')]_p+\hat{s}'\cdot u\,.$$

Hence, we can define a covariant vector field ψ by (1.12).

The covariant vector field ψ thus defined satisfies the equation

$$\psi=\varphi(s)+\hat{s}\cdot\psi \tag{1.13}$$

for any $s\in G$. In fact, let q be an arbitrary point of M and t an element of G such that $t(O)=q$. Then we have from (1.12)

$$[\psi]_q=[\varphi(t)]_q+\hat{t}\cdot u$$

and hence for any $s\in G$

$$[\hat{s}\cdot\psi]_{s(q)}=[\hat{s}\cdot\varphi(t)]_{s(q)}+\widehat{(st)}\cdot u\,.$$

It follows thus

$$[\varphi(s)+\hat{s}\cdot\psi]_{s(q)}=[\varphi(s)+\hat{s}\cdot\varphi(t)]_{s(q)}+(\widehat{st})\cdot u .$$

Therefore, as a consequence of (1.13), we obtain

$$[\varphi(s)+\cdot\hat{s}\,\psi]_{s(q)}=[\varphi(st)]_{s(q)}+(\widehat{st})\cdot u=[\psi]_{st(o)}=[\psi]_{s(q)} .$$

This implies the required relation (1.13), since the point q is arbitrarily chosen.

Next, we construct an affine connection

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j ,$$

where ψ_j are the components of the covariant vector field ψ defined by (1.12). Then the connection Γ^i_{jk} is invariant under G. In fact, if $\psi(s)$ is the covariant vector field defined by (1.6) corresponding to an element $s\in G$, then we obtain

$$\psi(s)=\varphi(s)+\hat{s}\cdot\psi-\psi .$$

Therefore, from (1.13) it follows

$$\psi(s)=0 .$$

This means that the connection $\bar{\Gamma}^i_{jk}$ is invariant under G. Thus the proof of Theorem 3 is completed.

It is well known that in a Riemannian manifold M the affine connection determined by its Christoffel symbols is projectively flat, if and only if M has constant sectional curvature.[9] Thus the following corollary holds good.

COROLLARY. *Let G be a transitive group of projective transformations in a Riemannian manifold whose sectional curvature is not constant. If the identity component of the linear isotropy group of G at a point is irreducible in the tangent space at the point, then G is essentially affine with respect to the affine connection determined by the Christoffel symbols.*

§ 2. Groups of conformal transformations in a Riemannian manifold.

Let M be a Riemannian manifold with the metric tensor g_{ij}. A transformation s of M onto itself induces naturally a tensor field

9) See, for example, Kurita [9].

$g_{ij}(s)$ from the metric tensor g_{ij}. When we have especially

$$g_{ij}(s)=\rho(s)g_{ij} \tag{2.1}$$

with a positive scalar field $\rho(s)$ determined by s, we call s a *conformal transformation* of M [14, 16]. That is to say, a conformal transformation of M preserves the angle of intersection of any two curves. When $\rho(s)$ is a constant field, s is called a *homothetic transformation* of M, which is affine as well as conformal in M [18]. If $\rho(s)$ is identically equal to 1 in M, so s is obviously an isometry of M. From (2.1) it follows easily

$$\rho(st)=\rho(s)(\hat{s}\cdot\rho(t)) \tag{2.2}$$

for any two conformal transformations s and t.[10]

Let $\bar{g}_{ij}$ be another Riemannian metric in M, for which there exists a field of positive scalars ρ such that

$$\bar{g}_{ij}=\rho g_{ij}\,;$$

then we say that $\bar{g}_{ij}$ is *conformally related* to the original metric tensor g_{ij}. Let G be a group of conformal transformations of M. If there exists in M a certain conformally related Riemannian metric which is invariant under G, then we say that G is *essentially isometric* with respect to g_{ij}. We have now the following

THEOREM 4. *Let G be a compact group of conformal transformations of a Riemannian manifold M with the metric tensor g_{ij}. Then G is essentially isometric with respect to g_{ij}.*

PROOF. For an element s of G we denote by $\rho(s)$ the scalar field defined by (2.1). We shall now construct a scalar field λ in M as follows. Denoting by Δ the total measure of the compact group G, we put

$$[\lambda]_p=\frac{1}{\Delta}\int[\rho(t)]_p dt\,,$$

p being an arbitrary point of M, where the integral is extended over the whole group manifold of G. For the sake of simplicity, let us put

10) Let s be a transformation of M and λ a scalar at a point p of M. Then we can define a scalar λ' at the point $s(p)$ by $\lambda'=\lambda$. The scalar λ' is denoted by $\hat{s}\cdot\lambda$. If ρ is a scalar field in M, denoting by $[\rho]_p$ the value of the field ρ at a point $p\in M$, we can define a scalar field ρ' in M by $[\rho']_{s(p)}=\hat{s}\cdot[\rho]_p$. Denoting by $\hat{s}\cdot\rho$ the field ρ' thus introduced, we have $[\hat{s}\cdot\rho]_{s(p)}=\hat{s}\cdot[\rho]_p$.

$$\lambda = \frac{1}{\Delta} \int \rho(t) dt . \tag{2.3}$$

Defining a metric tensor field $\bar{g}_{ij}$ by

$$\bar{g}_{ij} = \lambda g_{ij} ,$$

we shall prove that G is a group of isometries with respect to $\bar{g}_{ij}$.

Let $\bar{g}_{ij}(s)$ be the tensor field induced from $\bar{g}_{ij}$ by $s \in G$. We find then

$$\bar{g}_{ij}(s) = \bar{\rho}(s) \bar{g}_{ij} ,$$

where $\bar{\rho}(s)$ is given by

$$\bar{\rho}(s) = \frac{1}{\lambda} \rho(s)(\hat{s} \cdot \lambda) .$$

That is, s is a conformal transformation with respect to $\bar{g}_{ij}$, since s is a conformal transformation with respect to g_{ij} which is conformally related to $\bar{g}_{ij}$. Hence, we have $\bar{\rho}(s) = 1$ for any element s of G. In fact, substituting (2.3) in the above equation, we find

$$\bar{\rho}(s) = \frac{1}{\lambda} \frac{1}{\Delta} \int \rho(s)(\hat{s} \cdot \rho(t)) dt ,$$

because we have from the definitions

$$\hat{s} \cdot \int \rho(t)\, dt = \int \hat{s} \cdot \rho(t) dt .$$

Thus, from (2.2) it follows

$$\bar{\rho}(s) = \frac{1}{\lambda} \frac{1}{\Delta} \int \rho(st) dt .$$

Further, the above equation implies

$$\bar{\rho}(s) = \frac{1}{\lambda} \frac{1}{\Delta} \int \rho(t)\, dt = \frac{1}{\lambda} \lambda = 1 ,$$

since the invariant measure over the compact group is two-sided invariant. Since $\bar{\rho}(s)$ is equal to 1, we have $\bar{g}_{ij}(s) = \bar{g}_{ij}$ for any s of G. This proves the theorm.

In the previous paper we have obtained some properties of the group of conformal transformations. In this regard, as a consequence of Theorem 4, we can prove the following theorem in an analogous manner as in the proof of Theorem 2.

THEOREM 5. *Let G be a transitive group of conformal transformations of a Riemannian manifold M with the metric tensor field g_{ij}. If the isotropy group of G at a point is compact, then G is essentially isometric with respect to g_{ij}.*

§ 3. Groups of projective transformations leaving the Ricci tensor invariant.

We shall consider in this section groups of projective transformations preserving the Ricci tensor in an affinely connected manifold. We shall first, for the sake of simplicity, introduce some notations as follows:

For an affinely connected manifold or a Riemannian manifold M,

- $A(M)$ is the group of all affine transformations of M;
- $P(M)$ is the group of all projective transformations of M;
- $P^*(M)$ is the group of all projective transformations of M which preserve the Ricci tensor.

For a Riemannian manifold M,

- $I(M)$ is the group of all isometries of M;
- $H(M)$ is the group of all homothetic transformations of M;
- $C(M)$ is the group of all conformal transformations of M;
- $C^*(M)$ is the group of all conformal transformations of M which leave the Ricci tensor invariant.

$A_0(M)$ means the identity component of the group $A(M)$ and analogous notations are introduced for the other groups.

It is easily seen that in an affinely connected manifold or a Riemannian manifold M

$$P(M) \supset P^*(M) \supset A(M)$$

holds and that in a Riemannian manifold M the following inequalities hold:

$$P(M) \supset P^*(M) \supset A(M) \supset H(M) \supset I(M),$$

$$C(M) \supset C^*(M) \supset H(M) \supset I(M).$$

We shall now give some remarks concerning the projective transformation which leaves the Ricci tensor invariant. Let s be a projective transformation in an n-dimensional affinely connected manifold M with the affine connection Γ^i_{jk}. Denote by φ_j the covariant vector field defined by (1.2) corresponding to s. The curvature tensor

$\overline{R}^i_{jkl}$ of the affine connection $\overline{\Gamma}^i_{jk}$ induced from Γ^i_{jk} by s is given as follows [14]:[11]

$$\overline{R}^i_{jkl}=R^i_{jkl}+\delta^i_j(\varphi_{k;\,l}-\varphi_{l;\,k})+\delta^i_k\varphi_{j;\,l}-\delta^i_l\varphi_{j;\,k}+\delta^i_l\varphi_j\varphi_k-\delta^i_k\varphi_j\varphi_l,$$

where the curvature tensor R^i_{jkl} of Γ^i_{jk} has been defined by

$$R^i_{jkl}=\frac{\partial\Gamma^i_{jk}}{\partial x^l}-\frac{\partial\Gamma^i_{jl}}{\partial x^k}+\Gamma^a_{jk}\Gamma^i_{al}-\Gamma^a_{jl}\Gamma^i_{ak}.$$

Contracting indices i and l, we find

$$\overline{R}_{jk}=R_{jk}+\varphi_{k;\,j}-n\varphi_{j;\,k}+(n-1)\varphi_j\varphi_k, \tag{3.1}$$

where $R_{jk}=R^a_{jka}$ and $\overline{R}_{jk}=\overline{R}^a_{jka}$ are the Ricci tensors of Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ respectively. Suppose that s preserves the Ricci tensor, i. e. $\overline{R}_{jk}=R_{jk}$. Then, from (3.1) it follows

$$\varphi_{k;\,j}-n\varphi_{j;\,k}+(n-1)\varphi_j\varphi_k=0$$

and hence

$$(n^2-1)\varphi_{k;\,j}=(n^2-1)\varphi_k\varphi_j.$$

Provided $n\neq 1$, it follows thus

$$\varphi_{j;\,k}=\varphi_j\varphi_k. \tag{3.2}$$

Summing up, we have the following proposition:

Let s be a projective transformation preserving the Ricci tensor in an affinely connected manifold, then the covariant vector field φ_j corresponding to s satisfies (3.2). This implies that *a projective transformation preserves the Ricci tensor, if and only if it leaves the curvature tensor invariant.*

LEMMA 3. *If there exists a non-trivial covariant vector field φ_j satisfying* (3.2) *in an affinely connected manifold M, then the homogeneous holonomy group of M has an invariant hyper-plane and the restricted homogeneous holonomy group*[12] *of M has an invariant covariant vector.*

PROOF. Let $C: x^i=x^i(\tau)$ $(0\leqq\tau\leqq 1)$ be an arbitrary curve in M. Denote by p_0 the end point of C corresponding to $\tau=0$. We shall

11) The covariant differentiation with respect to Γ^i_{jk} is denoted by semi-colon followed by an index.

12) In the homogeneous holonomy group the subgroup consisting of all elements corresponding to a closed curve, which is homotopic to zero, is the restricted homogeneous holonomy group.

define a function on C by

$$\varphi(\tau)=\int_0^\tau \varphi_a dx^a \qquad (0\leqq\tau\leqq 1),$$

which is taken along C. From (3.2) it follows that the relation

$$\frac{\delta\varphi_j}{d\tau}=\varphi_j\frac{d\varphi}{d\tau}$$

holds at any point of C, where the left-hand side has been defined by

$$\frac{\delta\varphi_j}{d\tau}=\frac{d\varphi_j}{d\tau}-\Gamma_b{}^a{}_j\varphi_a\frac{dx^b}{d\tau}.$$

If we put along C $\psi_j=e^{-\varphi}\varphi_j$, by virtue of (3.2) we find easily

$$\frac{\delta\psi_j}{d\tau}=0.$$

This means that the family of vectors ψ_j is parallel along C and $\psi_j=\varphi_j$ at p_0.

When the curve C is closed, we denote by α the element associated to C in the homogeneous holonomy group Φ. Denoting by $[\varphi]_0$ the value of the field φ_j at p_0, we have then by means of the above discussions

$$\alpha\cdot[\varphi]_0=e^{-b}[\varphi]_0, \tag{3.3}$$

where b denotes the line-integral extended along the whole curve C, i. e.

$$b=\int_C \varphi_a dx^a. \tag{3.3$'$}$$

Since the curve C is arbitrarily chosen, by means of (3.3) the group Φ leaves invariant a hyperplane defined by the covariant vector $[\varphi]_0$ in the tangent space at p_0. The proof of the first part in Lemma 3 is thereby completed.

To prove the second part, we suppose now that the closed curve C is homotopic to zero. Then the line-integral (3.3)$'$ vanishes, since the field φ_j satisfies

$$\varphi_{j;k}-\varphi_{k;j}=0$$

as a consequence of (3.2) and C is homotopic to zero. Taking account of this fact, we see from (3.3) that the transformation α leaves the covariant vector $[\varphi]_0$ invariant. Therefore the restricted homogeneous holonomy group of M preserves the covariant vector $[\varphi]_0$ in the tangent space at p_0. Lemma 3 is thereby proved completely.

If M in Lemma 3 is especially a Riemannian manifold, then the factor e^{-b} in (3.3) must be equal to 1, that is, for any closed curve C

$$\int_C \varphi_a dx^a = 0\,,$$

since the homogeneous holonomy group of a Riemannian manifold is a group of orthogonal transformations. This means that there exists a function φ in M such as

$$\frac{\partial \varphi}{\partial x^j} = \varphi_j\,.$$

Thus the vector field defined by $\psi_j = e^{-\varphi}\varphi_j$ is parallel by virtue of (3.3).

Lemma 3 implies the following

THEOREM 6. *Let M be an affinely connected manifold. If the homogeneous holonomy group of M has no invariant hyper-plane, or, if the restricted homogeneous holonomy group of M has no invariant covariant vector, then $P^*(M) = A(M)$. If, moreover, the Ricci tensor of M vanishes identically, then $P(M) = A(M)$.*

An affinely connected manifold M is said to be *complete*, if any affine parameter on any path of M takes any values in the range $(-\infty, +\infty)$. Now, we have the following lemma concerning the complete affinely connected manifold.

LEMMA 4. *In a complete affinely connected manifold, if a covariant vector field φ_j satisfying*

(3.2) $$\varphi_{j;k} = \varphi_j \varphi_k$$

has no singularity at any point, then φ_j vanishes identically.

PROOF. Let $g: x^i = x^i(\sigma)$ be an arbitrary path in the given affinely connected manifold M, where σ is an affine parameter of g. If we put

$$\varphi = \varphi_a \frac{dx^a}{d\sigma}\,,$$

then along g the function φ is a solution of the following differential equation

$$\frac{d\varphi}{d\sigma} = \varphi^2\,,$$

because the tangent vector $dx^i/d\sigma$ of g is parallel along g and φ_j satisfies (3.2).

We assume for a moment that the function φ does not vanish on g. Then, integrating the above differential equation, we have

$$\varphi = -\frac{1}{\sigma - A}$$

on g, where A is a constant. Since M is complete, there exists on g a point p such that $\sigma = A$ at p. Thus the function φ has a singularity at the point p. This contradicts the assumption of the lemma. Consequently, φ must vanish identically on g. Since the path g is arbitrarily chosen, it is easily seen that the given convariant vector field φ_j vanishes identically in M. This proves the lemma.

Lemma 4 implies the following theorem, since the covariant vector field corresponding to a projective transformation is regularly distributed in the manifold.

THEOREM 7. *If M is a complete affinely connected manifold, then $P^*(M) = A(M)$. If, moreover, the Ricci tensor of M vanishes identically, then $P(M) = A(M)$.*

It is easily seen that a Riemannian manifold is complete, if it is compact. We have thereby the following

COROLLARY. *If a Riemannian manifold M is compact, then $P^*(M) = A(M)$. If, moreover, the Ricci tensor of M vanishes identically in M, then $P(M) = A(M)$.*

We shall show by an example that the completeness of M in Theorem 7 is necessary. In an n-dimensional Euclidean space, let Σ be the set of all points such that $\sum_{i=1}^{n}(x^i)^2 < 1$, where (x^i) is a system of rectangular coordinates. It is obvious that Σ is not complete. Denote by G the group of all projective transformations leaving Σ invariant in the Euclidean space. Then we have $G = P^*(\Sigma)$, since Σ is locally flat. It is easily seen that any element of G is not affine, unless it is an isometry. It follows thus $P^*(\Sigma) \neq A(\Sigma)$, since $\dim G = n(n+1)/2$ and $\dim I(\Sigma) = n(n-1)/2$.

§ 4. Groups of projective transformations leaving the Ricci tensor invariant in a Riemannian manifold.

Recently, several interesting theorems have been proved concerning infinitesimal transformations in a Riemannian manifold.

Theorem A. *In a compact orientable Riemannian space with positive definite metric, there exists no vector* ξ^i *which defines a one-parameter group of conformal transformations and satisfies the relation*

$$R_{jk}\xi^j\xi^k \leqq 0 \,,$$

unless we have $\xi_{j;k}=0$. *If the space has negative Ricci curvature everywhere, then the exceptional case cannot arise* [1, 19, 20].

Theorem B. *In a compact orientable Riemannian space with positive definite metric, any one-parameter group of affine collineations must be that of motions* [3, 19, 20].

Theorem C. *Let M be a complete Riemannian manifold whose restricted homogeneous holonomy group has no invariant vector. Then any infinitesimal affine transformation in M is a Killing vector field* [6, 3].

It might be interesting to attack analogous problems regarding projective or conformal transformations preserving the Ricci tensor. First, we have the following theorem, as an immediate consequence of Theorems 6 and C.

Theorem 8. *If the restricted homogeneous holonomy group of a complete Riemannian manifold M has no invariant vector, then* $P_0^*(M) = I_0(M)$. *If, moreover, the Ricci tensor of M vanishes identically, then* $P_0(M)=I_0(M)$.

As has been proved recently [3, 6, 8, 11], *in a complete irreducible Riemannian manifold*[13] *any affine transformation is necessarily an isometry,* Thus, as a consequence of Theorem 6, we have the following proposition:

If M is complete irreducible Riemannian manifold, then $P^*(M) = I(M)$. *If, moreover, the Ricci tensor of M vanishes identically, then* $P(M)=I(M)$.

Further, we have the following theorem as a consequence of Theorem B and the corollary to Theorem 7.

Theorem 9. *If M is a compact Riemannian manifold, then* $P_0^*(M) = I_0(M)$.

As a corollary to Theorem 9, we have the following proposition:

In a compact Riemannian manifold with vanishing Ricci tensor any vector field defining an infinitesimal projective transformation is necessarily a Killing vector field.

13) If the restricted homogeneous holonomy group at a point of a Riemannian manifold M is irreducible in the tangent space of M at the point, M is said to be irreducible. If dim $M=1$, we say that M is not irreducible.

This implies together with Theorem A the following

COROLLARY. *In a compact Riemannian manifold with vanishing Ricci tensor any vector field defining an infinitesimal projective transformation is parallel.*

Now, we shall give a proof of the corollary to Theorem 7, when the manifold M is Riemannian. Let s be a projective transformation in a compact Riemannian manifold M and φ_j be the covariant vector field defined by (1.2) corresponding to s. Suppose that s preserves the Ricci tensor. Then, if M is orientable, φ_j satisfying (3.2), it follows $\varphi^a_{;a}=\varphi^a\varphi_a$ $(\varphi^i=g^{ia}\varphi_a)$ and then

$$\int\varphi^a\varphi_a dv=\int\varphi^a_{;a}dv=0\,,$$

where dv is the volume element of M and the integral is extended over the whole manifold M. Hence, we find $\varphi_j=0$ because of $\varphi^a\varphi_a\geqq 0$. This shows that the given transformation s is affine in M. When M is not orientable, there exists a compact orientable covering manifold $\tilde{M}$ which covers M twice. Let $\tilde{\varphi}_j$ be the vector field in $\tilde{M}$ induced from φ_j. Then, we find $\tilde{\varphi}_{j;k}=\tilde{\varphi}_j\tilde{\varphi}_k$ as a consequence of (3.2). We can therefore conclude $\tilde{\varphi}_j=0$ just as in the above discussions. Consequently, it follows $\varphi_j=0$, which proves the first part of the corollary to Theorem 7 for a Riemannian manifold M. The second part of the corollary is an immediate consequence of the first.

§ 5. Groups of conformal transformations leaving the Ricci tensor invariant.

As is well known, the problems of transformations are reduced to those of infinitesimal transformations, as far as connected groups are concerned. We shall give some well known formulas concerning infinitesimal conformal transformations for the sake of completeness, according to [17]. Let M be a Riemannian manifold with the metric tensor g_{ij}. Denote by X the Lie derivation with respect to an infinitesimal transformation ξ^i in M. Then, if for an infinitesimal transformation in M the relation

$$Xg_{ij}=2\phi g_{ij} \tag{5.1}$$

holds with a scalar field ϕ, ξ^i is called an *infinitesimal conformal transformation* in M [17]. If ϕ is a constant, ξ^i is called an *infini-*

tesimal homothetic transformation in M [18]. If ϕ is equal to zero identically, then ξ^i is nothing but a Killing vector field. It is known that an infinitesimal homothetic transformation is affine as well as conformal [18].

For an infinitesimal conformal transformation ξ^i we find [17]

$$XR^i_{jkl} = -\delta^i_l\phi_{j;k} + \delta^i_k\phi_{j;l} - g_{jk}\phi^i_{;l} + g_{jl}\phi^i_{;k}, \tag{5.2}$$

where we have put

$$\phi_j = \frac{\partial\phi}{\partial x^j} \qquad \text{and} \qquad \phi^i = g^{ia}\phi_a.$$

Contracting indices i and l in (5.2), we find

$$XR_{jk} = -(n-2)\phi_{j;k} - g_{jk}\phi^a_{;a}, \tag{5.3}$$

where $R_{jk} = R^a_{jka}$. We have here the following proposition:

Let M be a Riemannian manifold of n dimensions. If an infinitesimal conformal transformation ξ^i in M preserves the Ricci tensor,[14] *i. e. if $XR_{jk}=0$, then the field $\phi_j = \partial\phi/\partial x^j$ is parallel for $n>2$, where the function ϕ is defined by* (5.1) *corresponding to ξ^i.*

In fact, from (5.3) it follows

$$-(n-2)\phi_{j;k} - g_{jk}\phi^a_{;a} = 0$$

because of $XR_{jk}=0$. Thus, multiplying g^{jk} and contracting, we find

$$-2(n-1)\phi^a_{;a} = 0$$

and hence $\phi^a_{;a}=0$. We have therefore

$$-(n-2)\phi_{j;k} = 0$$

and, consequently

$$\phi_{j;k} = 0,$$

which proves the proposition, since $n>2$.

By means of the above proposition, we can conclude that *an infinitesimal conformal transformation preserves the Ricci tensor if and only if it leaves the curvature tensor invariant.* This fact holds also for a conformal transformation.

We have here the following proposition:

In an n-dimensional Riemannian manifold M, if the homogeneous holonomy group has no invariant vector, then $C^*_0(M) = H_0(M)$ for $n>2$.

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14) If the Lie derivative of a tensor with respect to an infinitesimal transformation ξ^i vanishes identically, then we say that ξ^i preserves the tensor.

If, moreover, the Ricci tensor of M vanishes identically, then $C_0(M) = H_0(M)$.

The first part is an immediate consequence of the above proposition. If $n>2$, the second part follows from the first. When $n=2$, there exists no Riemannian manifold satisfying the second conditions. In fact, as is well known, we have

$$R_{11} = -g^{22}R_{1212}, \quad R_{12}=R_{21}=g^{12}R_{1212}, \quad R_{22}=-g^{11}R_{1212}.$$

Thus, $R_{jk}=0$ implies $R^i_{jkl}=0$, i. e. M is locally flat. This shows that any two-dimensional Riemannian manifold with vanishing Ricci tensor is locally flat.

Let s be a conformal transformation in an n-dimensional Riemannian manifold M with the metric tensor g_{ij} and $\rho^2(\rho>0)$ be the function given in (2.1) corresponding to s, i. e.

$$\bar{g}_{ij}=\rho^2 g_{ij},$$

where $\bar{g}_{ij}$ is the tensor field induced from g_{ij} by s. Denoting by $\bar{R}^i_{jkl}$ the curvature tensor of $\bar{g}_{ij}$, then we find [14]

$$\bar{R}^i_{jkl}=R^i_{jkl}-\rho_{jk}\delta^i_l+\rho_{jl}\delta^i_k-g_{jk}\rho^i_{\cdot l}+g_{jl}\rho^i_{\cdot k},$$

where $\rho_j=\partial \log\rho/\partial x^i$ and

$$\rho_{jk}=\rho_{j;k}-\rho_j\rho_k+\frac{1}{2}g^{ab}\rho_a\rho_b g_{jk},$$

$\rho^i_{\cdot j}$ being defined by $\rho^i_{\cdot j}=g^{ia}\rho_{aj}$. Contracting indices i and l, we have

$$\bar{R}_{jk}=R_{jk}-(n-2)\rho_{jk}-g^{ab}\rho_{ab}g_{jk},$$

where $\bar{R}_{jk}=\bar{R}^a_{jka}$. We now suppose that s leaves the Ricci tensor invariant, i. e. $\bar{R}_{jk}=R_{jk}$. Then we find

$$(n-2)\rho_{jk}+g^{ab}\rho_{ab}g_{jk}=0. \tag{5.4}$$

Multiplying g^{jk} and contracting, we obtain from (5.4)

$$2(n-1)g^{ab}\rho_{ab}=0.$$

Since $n\neq 1$, it follows

$$g^{ab}\rho_{ab}=0,$$

i. e.

$$\rho^a_{;a}+\frac{n-2}{2}\rho^a\rho_a=0, \tag{5.5}$$

where $\rho^i = g^{ia}\rho_a$. If we substitute (5.5) in (5.4), we find

$$(n-2)\rho_{jk}=0 .$$

Supposed $n>2$, we find thus $\rho_{jk}=0$, i.e.

$$\rho_{j;k}=\rho_j\rho_k-\frac{1}{2}\rho^a\rho_a g_{jk} .$$

Then we have obtained the following proposition:

If a conformal transformation s preserves the Ricci tensor in an n-dimensional Riemannian manifold, then the function ρ *corresponding to s satisfies* (5.5). *If, moreover,* $n>2$, *the equation* (5.6) *holds.*

We have next the following lemma.

LEMMA 5. *In a complete Riemannian manifold, if a vector field* ρ_j *satisfying*

$$\rho_{j;k}=\rho_j\rho_k-\frac{1}{2}\rho^a\rho_a g_{jk} \tag{5.6}$$

has no singularity at any point, then the field ρ_j *vanishes identically.*

PROOF. We suppose for a moment that the vector field ρ_j does not vanish identically. A curve $x^i=x^i(t)$ defined by

$$\frac{dx^i}{dt}=\rho^i$$

is called a ρ-curve. Multiplying ρ^k and contracting in (5.6), we find

$$\rho_{j;a}\rho^a=\frac{1}{2}(\rho^a\rho_a)\rho_j .$$

This shows that any ρ-curve is a geodesic. Multiplying $\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}$ and contracting in (5.6), we find that along a ρ-curve $x^i=x^i(\sigma)$ the relation

$$\rho_{a;b}\frac{dx^a}{d\sigma}\frac{dx^b}{d\sigma}=\frac{1}{2}\rho^a\rho_a$$

holds, where σ denotes the arc-length of the ρ-curve. Since any ρ-curve is a geodesic, its unit tangent vector $\frac{dx^i}{d\sigma}$ is parallel along the ρ-curve itself. The above relation thus implies

$$\frac{d\lambda}{d\sigma}=\frac{1}{2}\lambda^2 ,$$

where we have defined λ by $\lambda=\sqrt{\rho^a\rho_a}$. Now we suppose that the function λ does not vanishes identically along the ρ-curve. Then, integrating the above differential equation, we have

$$\lambda=-\frac{2}{\sigma-A} \tag{5.7}$$

along the ρ-curve, where A is a constant.

Since any ρ-curve is a geodesic and the given Riemannian manifold is complete, on the ρ-curve there exists a point p such that $\sigma=A$ at p. Thus the function λ has a singularity at the point p by means of (5.7). This contradicts the assumption of the lemma. Consequently, the function λ must be identically zero on the ρ-curve. This implies that the vector field ρ_j vanishes identically in the given manifold. The proof of Lemma 5 is thereby completed.

Lemma 5 implies the following lemma, since the function ρ^2 defined by (1.2) corresponding to a conformal transformation is regularly distributed in the manifold.

LEMMA 6. *If M is a complete Riemannian manifold of n dimensions, then $C^*(M)=H(M)$ for $n>2$. If, moreover, the Ricci tensor of M vanishes identically, then $C(M)=H(M)$ for $n>2$.*

We note here the following lemma [5].

LEMMA 7. *Let M be a complete, connected Riemannian manifold which is not locally flat, then $H(M)=I(M)$.*

Lemmas 6 and 7 imply the following

THEOREM 10. *If M is a complete, non-flat Riemannian manifold of n dimensions, then $C^*(M)=I(M)$ for $n>2$. If, moreover, the Ricci tensor of M vanishes identically, then $C(M)=I(M)$.*

We shall show by an example that the completeness of M in Lemma 6 is necessary. In a Euclidean space of dimension $n>2$, we denote by Σ the set of all interior points of the unit sphere as in the example of § 3. Obviously, the set Σ is not complete. Let G be the group of all conformal transformations leaving Σ invariant in the Euclidean space. We have then $G=C^*(\Sigma)$, since Σ is locally flat. But it is easily seen that any element of G is homothetic if and only if it is an isometry. It follows thereby $C^*(\Sigma)\neq H(\Sigma)$, since $\dim G=n(n+1)/2$ and $\dim I(\Sigma)=n(n-1)/2$.

LEMMA 8. *If M is a compact Riemannian manifold, then $H(M)=I(M)$.*

PROOF. Let s be a homothetic transformation in M; then the function defined by (2.1) corresponding to s is a constant $A^2(A>0)$, i. e.

$$\bar{g}_{ij}=A^2 g_{ij}\,,$$

where $\bar{g}_{ij}$ is the tensor field induced from g_{ij} by s. Suppose that M is orientable. If we denote by V and $\bar{V}$ the total volume measured by g_{ij} and $\bar{g}_{ij}$ respectively, then we have

$$\bar{V}=A^2 V\,,$$

since A is a constant. However, it is obvious $\bar{V}=V$. We have consequently $A=1$. This means $s\in I(M)$, i. e. $H(M)=I(M)$. The lemma is thus proved, if M is orientable.

When M is not orientable, there exists a compact orientable covering manifold $\tilde{M}$ which covers M twice. Let s be a homothetic transformation of M, then there exists a transformation $\tilde{s}$ in $\tilde{M}$ such that $s\circ f=f\circ\tilde{s}$, where $f:\tilde{M}\rightarrow M$ is the projection of the covering. It is easily seen that $\tilde{s}$ is homothetic in $\tilde{M}$ with respect to the metric tensor $\tilde{g}_{ij}$ in $\tilde{M}$ which is induced from g_{ij} by f. From the above discussions it follows thus that $\tilde{s}$ is an isometry with respect to $\tilde{g}_{ij}$. This implies that s is an isometry in M. The lemma is thereby proved completely.

Finally we have the following

THEOREM 11. *If M is a compact Riemannian manifold, then $C^*(M)=I(M)$. If, moreover, the Ricci tensor vanishes identically, then $C(M)=I(M)$.*

PROOF. First, we assume that M is orientable. If s is an element of $C^*(M)$, then the function $\rho^2(\rho>0)$ corresponding to s satisfies (5.5). Then it follows

$$\frac{n-2}{2}\int\rho^a\rho_a dv=-\int\rho^a_{;a}dv=0\,,$$

where dv is the volume element of M and the integral is extended over the whole manifold M. We assume moreover $n>2$. Thus the above relation implies

$$\int\rho^a\rho_a dv=0\,.$$

Consequently, it follows $\rho_j=0$, i. e. that ρ is a constant, since $\rho^a\rho_a\geqq 0$.

Therefore, we have $C^*(M)=H(M)$ for $n>2$. When $n=2$, it follows from (5.5)

$$\rho^a_{;a}=0\,.$$

Then, ρ_j is a harmonic vector field, since it is a gradient. We see thus $\rho_j=0$, because in a compact orientable Riemannian manifold any harmonic vector field which is a gradient vanishes identically. We have hence $C^*(M)=H(M)$ also for $n=2$, if M is orientable.

When M is not orientable, denoting by $\tilde{\rho}^2$ the function in $\tilde{M}$ induced from ρ^2, we see that $\tilde{\rho}_j$ satisfies (5.5), where $\tilde{M}$ is a compact orientable covering manifold which covers M twice. Thus, by means of the above discussions, it follows that $\tilde{\rho}$ is a constant in $\tilde{M}$, and hence that ρ is so in M. This proves $C^*(M)=H(M)$.

Summing up, we have $C^*(M)=H(M)$ for a compact Riemannian manifold M. This implies, by virtue of Lemma 8, $C^*(M)=I(M)$. The first part of Theorem 11 is thus proved. The second part is an immediate consequence of the first. Thus Theorem 11 is completely proved.

§ 6. The proof of Lemma 2.[15]

In the present section we shall prove Lemma 2 given in § 1. For this purpose, some lemmas are required. Keeping notations and assumptions as in Lemma 2, we first prove

LEMMA 6.1. *The restriction of the homomorphism β to Γ is an isomorphism of Γ onto $\tilde{\Gamma}$.*

PROOF. Since the kernel $N\cap\Gamma$ of β in Γ is a discrete normal subgroup of Γ, $N\cap\Gamma$ is contained in the center of Γ. Let τ and σ be any elements of $N\cap\Gamma$ and Γ respectively. Then, by virtue of $\tau\sigma=\sigma\tau$, we find

$$a^a_{\cdot j}(\sigma)a^0_{\cdot a}(\tau)=a^0_{\cdot j}(\tau)\,. \tag{6.1}$$

The matrix σ being arbitrary, this shows that the covariant vector $(a^0_{\cdot 1}(\tau), a^0_{\cdot 2}(\tau),\cdots, a^0_{\cdot n}(\tau))$ is invariant under $\tilde{\Gamma}$ and also under the identity component $\tilde{\Gamma}_0$ of $\tilde{\Gamma}$. Hence, $\tilde{\Gamma}_0$ being irreducible, the covariant vector vanishes and so does each of the n numbers $a^0_{\cdot j}(\tau)$. Therefore, the

15) The author wishes to express his thanks to M. Obata who has given the author valuable advices concerning the proof of Lemma 2.

matrix τ is reduced to the unit matrix, i.e. the kernel $N \cap \Gamma$ is the trivial group $\{e\}$. This proves Lemma 6.1.

Let Σ be a set of (n, n)-matrices. Then Σ is regarded naturally as a set of endomorphisms of a vector space V of n dimensions. Suppose that, for any subspace P of V invariant under Σ, there exists a subspace Q of V invariant under Σ such that $V = P + Q$ (direct sum). Then Σ is said to be *semi-simple*. If there exists a non-trivial subspace invariant under Σ, we say that Σ is reducible, otherwise, it is said to be irreducible. Let G be a group and ρ a representation of G. If the image $\rho(G)$ of G by ρ is semi-simple as a set of matrices, the representation ρ is called semi-simple. Thus, it is well known that any representation of a compact group is semi-simple. Further, any representation of semi-simple Lie algebra is always semi-simple [15].

Let $\mathfrak{L}$ be the Lie algebra of all complex (real) matrices of the form

$$A = \begin{pmatrix} 0 & \alpha^{0}_{\cdot j}(A) \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix} \tag{6.2}$$

and $\tilde{\mathfrak{L}}$ be the Lie algebra of all complex (real) (n, n)-matrices. Then, there exists a homomorphism ${}^*\beta : \mathfrak{L} \to \tilde{\mathfrak{L}}$ such that

$${}^*\beta(A) = (\alpha^{i}_{\cdot j}(A))$$

for any matrix A of $\mathfrak{L}$ represented by (6.2).

LEMMA 6.2. *Let Σ be a semi-simple subset of $\mathfrak{L}$ and $\tilde{\Sigma}$ the subset of $\tilde{\mathfrak{L}}$ defined by $\tilde{\Sigma} = {}^*\beta(\Sigma)$. If $\tilde{\Sigma}$ is irreducible, then there exists a matrix T of the form* (1.9) *such that*

$$TAT^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix} \tag{6.3}$$

for any matrix A of Σ.

PROOF. Σ being semi-simple, there exists a matrix

$$T = \begin{pmatrix} t & t_j \\ t^i & t^{i}_{\cdot j} \end{pmatrix} \tag{6.4}$$

such that

$$TAT^{-1}=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^i_{\cdot j}(A) \end{pmatrix}={}'A \tag{6.5}$$

for any A of Σ. Now, we can assume without loss of generality that $t=1, t^i_{\cdot j}=\delta^i_j$. In fact, if $t\neq 1$, or, if $t^i_{\cdot j}\neq\delta^i_j$, we put

$$S=\begin{pmatrix} t^{-1} & 0 \\ 0 & s^i_{\cdot j} \end{pmatrix},$$

where $s^i_{\cdot a}t^a_{\cdot j}=\delta^i_j$. Then, denoting by $'T$ the product ST, we have

$$'T=\begin{pmatrix} 1 & u_j \\ u^i & \delta^i_j \end{pmatrix}.$$

Further, we find

$$'TA=STA=S'AT=S'AS^{-1}\,'T$$

and hence

$$'TA'T^{-1}=S'AS^{-1},$$

where $S'AS^{-1}$ has the same form as that of $'A$. Thus, from the beginning we assume that $t=1, t^i_{\cdot j}=\delta^i_j$ in (6.4).

Under this assumptions, we find that

$$\alpha^i_{\cdot a}(A)t^a=0$$

holds for any $A\in\Sigma$, because $TA='AT$. This means that the vector $(t^1, t^2, \cdots, t^n)$ is invariant under $\tilde{\Sigma}$. The set $\tilde{\Sigma}$ of matrices being irreducible, the vector vanishes and so also does each of n numbers t^i. This proves Lemma 6.2.

Now, we shall introduce some notations and terminologies. Let V be an n-dimensional vector space over the field of real numbers and $\{e_1, e_2, \cdots, e_n\}$ a basis of V. The set of all linear combinations $\sum_{i=1}^n c_ie_i$ with complex coefficients c_i forms a vector space V^c over the field of complex numbers, which is called the *complexification* of V. Obviously, V is naturally contained in V^c. Let $\mathfrak{G}$ be a Lie algebra over the field of real numbers and $\{X_1, X_2, \cdots, X_r\}$ a basis of $\mathfrak{G}$. The set of all linear combinations $\sum_{\alpha=1}^r c_\alpha X_\alpha$ with complex coefficients c_α forms a Lie algebra $\mathfrak{G}^c$ over the field of complex numbers, which is called the *complexification* of $\mathfrak{G}$. It is easily seen that $\mathfrak{G}$ is naturally contained in $\mathfrak{G}^c$. Now, concerning the Lie algebra of matrices of the form (6.2), we have the following lemma corresponding to Lemma 2.

LEMMA 6.3. *Let* $\mathfrak{G}$ *be a Lie algebra of matrices of the form* (6.2). *If the restriction of the homomorphism* $*\beta$ *to* $\mathfrak{G}$ *is an isomorphism, and, if* $\tilde{\mathfrak{G}}={}^*\beta(\mathfrak{G})$ *is irreducible, then there exists a matrix* T *of the form* (1.9) *such that*

$$TAT^{-1}=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix}$$

for any matrix A *of* $\mathfrak{G}$.

PROOF. Let $\mathfrak{G}^c$ and $\tilde{\mathfrak{G}}^c$ be the complexifications of $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ respectively. $\tilde{\mathfrak{G}}$ is regarded naturally as a Lie algebra of endomorphisms of a vector space V over the field of real numbers, and then $\tilde{\mathfrak{G}}^c$ is also considered as that of the complexification V^c of V. Here, we discuss the following two possible cases; (i) $\tilde{\mathfrak{G}}$ is irreducible in V^c; (ii) $\tilde{\mathfrak{G}}$ is reducible in V^c.

(i) In this case the complexification $\tilde{\mathfrak{G}}^c$ of $\tilde{\mathfrak{G}}$ is also irreducible in V^c. Being irreducible, as is well known, $\tilde{\mathfrak{G}}^c$ is the direct product of the Lie algebra of homothetic endomorphisms of V^c and a semi-simple Lie algebra $\mathfrak{S}$ of endomorphisms of V^c, or $\tilde{\mathfrak{G}}^c$ itself is a semi-simple Lie algebra $\mathfrak{S}$ of endomorphisms of V^c, where $\mathfrak{S}$ is irreducible in V^c [16]. Denote by $\tilde{\mathfrak{A}}$ the Lie algebra of homothetic endomorphisms of V^c, when $\tilde{\mathfrak{G}}^c$ is not semi-simple. Let $\tilde{\mathfrak{A}}$ be the trivial Lie algebra $\{0\}$, when $\tilde{\mathfrak{G}}^c$ is semi-simple. We extend naturally the isomorphism $*\beta:\mathfrak{G}\to\tilde{\mathfrak{G}}$ to that of $\mathfrak{G}^c$ onto $\tilde{\mathfrak{G}}^c$. Denote by $\mathfrak{A}$ and $\mathfrak{S}$ the inverse images ${}^*\beta^{-1}(\tilde{\mathfrak{A}})$ and ${}^*\beta^{-1}(\tilde{\mathfrak{S}})$ respectively. Then $\mathfrak{G}^c$ is the direct product of $\mathfrak{A}$ and $\mathfrak{S}$.

By virtue of Lemma 6.2, it follows that there exists a matrix T of the form (1.9) such that (6.3) holds for any matrix A of $\mathfrak{S}$, where the coefficients ξ_j of T are complex numbers in general. On the other hand, we may assume that $\mathfrak{A}$ is generated by a matrix

$$B=\begin{pmatrix} 0 & b_j \\ 0 & \delta^i_j \end{pmatrix},$$

since $\tilde{\mathfrak{A}}$ is homothetic in V^c. Here, we note $B=TBT^{-1}$. Since BA

16) See p. 147 in [2].

$=AB$ for any matrix A of $\mathfrak{S}$, we have then $B'A={}'AB$, where $'A=TAT^{-1}$ is the matrix given by (6.5). Therefore we have

$$\alpha^{a}_{\cdot j}(A)b_a=0$$

for any A of $\mathfrak{S}$, because $B'A={}'AB$. This means that the covariant vector $(b_1, b_2, \cdots, b_n)$ of V^c is invariant under $\tilde{\mathfrak{S}}$. Since $\tilde{\mathfrak{S}}$ is irreducible, the vector is zero and so also is each of the n numbers b_j. Therefore, (6.3) holds for the matrix B and also for any matrix of $\mathfrak{A}$. Summing up, we can conclude that (6.3) holds for any matrix of $\mathfrak{G}^c$.

The given Lie algebra $\mathfrak{G}$ is a subset of $\mathfrak{G}^c$. Then we have

$$TA=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix}T, \qquad A\in\mathfrak{G},$$

from which we find by conjugation

$$\overline{T}A=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix}\overline{T},$$

since any matrix A of $\mathfrak{G}$ has real coefficients $\alpha^{0}_{\cdot j}(A)$ and $\alpha^{i}_{\cdot j}(A)$. Thus, if we put

$$U=\frac{1}{2}(T+\overline{T}),$$

U is a real matrix of the form (1.9) and UAU^{-1} has the form (6.3) for any matrix A of $\mathfrak{G}$. Thus, Lemma 6.3 is proved in the case (i).

(ii) We discuss the case where the Lie algebra $\tilde{\mathfrak{G}}$ is reducible in V^c. Since $\tilde{\mathfrak{G}}$ is irreducible in V, n is even, say $n=2m$, and V^c is the direct product of two subspaces P and $\overline{P}$ of dimension m.[17] Moreover, we can find a basis $\{e_1, e_2, \cdots, e_m; \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_m\}$ of V^c such that $\{e_1, e_2, \cdots, e_m\}$ and $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_m\}$ span respectively the subspaces P and $\overline{P}$, where $\bar{e}_\alpha$ is the complex conjugate of e_α $(1\leqq\alpha\leqq m)$. Any element $*\beta(A)$ of $\tilde{\mathfrak{G}}$, $A\in\mathfrak{G}$, is expressed by a matrix

$$\begin{pmatrix} \phi^{\alpha}_{\cdot\beta}(A) & 0 \\ 0 & \overline{\phi}^{\alpha}_{\cdot\beta}(A) \end{pmatrix} \qquad (1\leqq\alpha, \beta\leqq m)$$

17) See the appendix in [12].

with respect to the bases $\{e_\alpha, \bar{e}_\alpha\}$, i.e. there exists a matrix

$$\tilde{S}=(s^{i}_{\cdot j}) \tag{6.6}$$

such that $\tilde{S}^{-1*}\beta(A)\tilde{S}$ is the above matrix for any matrix A of $\mathfrak{G}$, where $\overline{\phi^{\alpha}_{\cdot\beta}}(A)$ is the complex conjugate of $\phi^{\alpha}_{\cdot\beta}(A)$.

The set

$$\tilde{\mathfrak{G}}'=\{(\phi^{\alpha}_{\cdot\beta}(A))\,|\,A\in\mathfrak{G}\} \qquad (1\leqq\alpha,\beta\leqq m)$$

of matrices forms an irreducible Lie algebra of endomorphisms of the m-dimensional vector space P over the field of complex numbers. Corresponding to this decomposition of $\tilde{\mathfrak{G}}$, we find

$$SAS^{-1}=\begin{pmatrix} 0 & \phi^{0}_{\cdot\beta}(A) & \overline{\phi^{0}_{\cdot\beta}}(A) \\ 0 & \phi^{\alpha}_{\cdot\beta}(A) & 0 \\ 0 & 0 & \overline{\phi^{\alpha}_{\cdot\beta}}(A) \end{pmatrix} \qquad (1\leqq\alpha,\beta\leqq m), \tag{6.7}$$

$\overline{\phi^{0}_{\cdot\beta}}(A)$ being the complex conjugate of $\phi^{0}_{\cdot\beta}(A)$ for any matrix A of $\mathfrak{G}$, where S is the following matrix

$$S=\begin{pmatrix} 1 & 0 \\ 0 & \tilde{S} \end{pmatrix}$$

and $\tilde{S}$ is the matrix given by (6.6).

The set $\mathfrak{G}'$ of all matrices

$$\begin{pmatrix} 0 & \phi^{0}_{\cdot\beta}(A) \\ 0 & \phi^{\alpha}_{\cdot\beta}(A) \end{pmatrix} \qquad (1\leqq\alpha,\beta\leqq m)$$

forms a Lie algebra isomorphic to the $\mathfrak{G}$, where A runs over $\mathfrak{G}$. Thus, $\mathfrak{G}'$ and $\tilde{\mathfrak{G}}'$ are isomorphic to each other. Since $\tilde{\mathfrak{G}}'$ is irreducible in P, as was proved in (i), there exists a matrix

$$'T=\begin{pmatrix} 1 & \xi_\beta \\ 0 & \delta^{\alpha}_{\beta} \end{pmatrix} \qquad (1\leqq\alpha,\beta\leqq m)$$

such that

$$'T\begin{pmatrix} 0 & \phi^{0}_{\cdot\beta}(A) \\ 0 & \phi^{\alpha}_{\cdot\beta}(A) \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & \phi^{\alpha}_{\cdot\beta}(A) \end{pmatrix}{}'T$$

for any matrix A of $\mathfrak{G}$. Putting

$$''T=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 'T & 0 \\ 0 & 0 & '\overline{T} \end{pmatrix},$$

by virtue of (6.7), we find

$$TA=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix} T$$

for any A of $\mathfrak{G}$, where $T={}''TS$. By conjugation and summation we have

$$UA=\begin{pmatrix} 0 & 0 \\ 0 & \alpha^{i}_{\cdot j}(A) \end{pmatrix} U, \qquad U=\frac{1}{2}(T+\overline{T}),$$

since the coefficients of $A\in\mathfrak{G}$ are real numbers. The real matrix U thus obtained is the required one. Therefore, the proof of Lemma 6.3 is completed.

Summing up the lemmas above proved, we can prove Lemma 2 as follows. Keeping notations as in Lemma 2, by virtue of Lemma 6.1 we see that Γ is isomorphic to $\tilde{\Gamma}$. Denote by Γ_0 and $\tilde{\Gamma}_0$ the identity components of Γ and $\tilde{\Gamma}$ respectively. Then, Lemma 6.3 implies that Lemma 2 holds good for the group Γ_0. Thus, we suppose any matrix τ of Γ_0 has the form (1.10). Let σ be a matrix of Γ. Since $\sigma\tau\sigma^{-1}=\tau'\in\Gamma_0$ for any matrix τ of Γ_0, we have

$$\begin{pmatrix} 1 & a^{0}_{\cdot j}(\sigma) \\ 0 & a^{i}_{\cdot j}(\sigma) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a^{i}_{\cdot j}(\tau) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & a^{i}_{\cdot j}(\tau') \end{pmatrix}\begin{pmatrix} 1 & a^{0}_{\cdot j}(\sigma) \\ 0 & a^{i}_{\cdot j}(\sigma) \end{pmatrix}$$

and hence

$$a^{a}_{\cdot j}(\tau)\, a^{0}_{\cdot a}(\sigma)=a^{0}_{\cdot j}(\sigma)$$

The matrix τ being arbitrary in Γ_0, this equation means that the covariant vector $(a^{0}_{\cdot 1}(\sigma), a^{0}_{\cdot 2}(\sigma),\cdots, a^{0}_{\cdot n}(\sigma))$ is invariant under Γ_0. Since Γ_0 is irreducible, the vector is thus zero and so also is each of the n numbers $a^{0}_{\cdot j}(\sigma)$. Therefore, the matrix σ has the form (1.10). Consequently, Lemma 2 is proved completely.

REMARK. The dual form of Lemma 2 implies immediately the following theorem concerning groups of affine motions in the ordinary affine space E_n. Let G be a group of affine motions σ:

$$\bar{x}^i = a^i_{\cdot a}(\sigma)x^a + a^i(\sigma),$$

where (x^i) are Cartesian coordinates of a point in E_n. The correspondence $\sigma \rightarrow (a^i_{\cdot j}(\sigma))$ defines a homomorphism γ of G onto a linear group

$$\tilde{G} = \{a^i_{\cdot j}(\sigma) \mid \sigma \in G\}.$$

The formalization of the theorem is as follows:

If the kernel of $\gamma: G \rightarrow \tilde{G}$ is discrete and the identity component of $\tilde{G}$ is irreducible, then there exists a point in E_n which is fixed by all elements of G.

The theorem is proved by S. Sasaki and M. Goto, when the group G is compact[18]. However, for the compact group G the theorem can be proved by virtue of Lemma 6.2.

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18) See the proof of Theorem 4 in [13].

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