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# Remark on my paper: On Skolem's theorem.

By Gaisi TAKEUTI

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It was proved in my paper [1] that if the system of axioms of Fraenkel-von Neumann of the set-theory is consistent, then system remains consistent after addition of the axiom that every set is a univalent image of $\omega$. This result was established by Theorems 1, 2 of that paper. I should like to remark now that from the proof of these theorems follows immediately also the following result:

"Let $\Gamma_1$ be any consistent system of axioms in Gentzen's *LK*, representing mathematically a certain domain $D$ of elements. Then $\Gamma_1$ remains consistent after addition of the system of axioms of the theory of natural numbers, and the axiom that every element of $D$ is a univalent image of a natural number.

I shall formulate this result more precisely in the following lines, and indicate how to prove it.

We begin with $\Gamma_a$, the system of axioms of "arithmetic" consisting of axioms of the theory of natural numbers except the axiom of mathematical induction. In this paper, $\Gamma_a$ means the following axioms:

$\forall x(x=x)$

$\forall A\forall x\forall y(x=y\vdash(A(x)\leftrightarrow A(y)))$ (See [3], § 1 for the notation $\forall A$.)

$\forall x\forall y\forall z(x<y\wedge y<z\vdash x<z)$

$\forall x\forall y\neg(x=y\wedge x<y)$

$\forall x\forall y(x<y\vdash x'<y\vee x'=y)$

$\forall x(x<x')$

$\forall x(0<x\vee 0=x)$

$\forall x(x+0=x)$

$\forall x\forall y(x+y'=(x+y)')$

$\forall x\forall y(x+y=y+x)$

$\forall x\forall y\forall z((x+y)+z=x+(y+z))$

$\forall x\forall y(x<y\leftrightarrow\exists z(0<z\wedge x+z=y))$

$\forall x\forall y(x\cdot y=y\cdot x)$

$\forall x\forall y\forall z((x+y)\cdot z=x\cdot z+y\cdot z)$

$\forall x(x\cdot 0'=x)$
$\forall x\forall y\forall z((x\cdot y)\cdot z=x\cdot(y\cdot z))$
$\forall x(j(g_1(x), g_2(x))=x)$
$\forall x\forall y(g_1(j(x, y))=x\wedge g_2(j(x, y))=y)$
$\forall x(0<x\vdash g_2(x)<x)$
$\forall x(0'<x\vdash g_1(x)<x)$
$\forall x\forall y(y<x\vdash j(x, y)=x^2+y)$
$\forall x\forall y(x\leqq y\vdash j(x, y)=y^2+y+x)$

Then, we have the following theorem.

THEOREM. *Let $\Gamma_1$ be consistent axioms in LK and satisfy the equality axioms with regard to $=$. (See* [1] *for equality axioms) Moreover, we assume that none of special variables, functions and predicates other than $=$ is contained in $\Gamma_1$ and $\Gamma_a$ at the same time. Then the following axioms are consistent in LK.*

$\Gamma_1^{e_1(\,)}$ *(See* [2], §7 *for the notation $\Gamma_1^{e_1(\,)}$) where $e_1(\,)$ is a predicate not contained in $\Gamma_1$ nor $\Gamma_a$.*

$e_1(s)$ *for every special variable $s$ in $\Gamma_1$.*

$\forall x_1\cdots\forall x_k e_1(f(x_1,\cdots, x_k))$ *for every function $f$ in $\Gamma_1$.*

$\forall x(x=x)$

$\forall A\forall x\forall y(x=y\vdash(A(x)\dashv\vdash A(y)))$

$\forall x\neg(e_1(x)\wedge n(x))$, *where $n(\,)$ is a predicate not contained in $\Gamma_1$ nor $\Gamma_a$.*

$n(0)$

$\forall x(n(x)\vdash n(x')\wedge n(g_1(x))\wedge n(g_2(x)))$

$\forall x\forall y(n(x)\wedge n(y)\vdash n(x+y)\wedge n(x\cdot y)\wedge n(j(x, y)))$

$\Gamma_a^{n(\,)}$

$\forall A\forall x(A(0)\wedge\forall x(A(x)\vdash A(x'))\wedge n(x)\vdash A(x))$

$\forall x\exists y(n(y)\wedge x=f_0(y))$, *where $f_0$ is a function not contained in $\Gamma_1$ nor $\Gamma_a$.*

For the proof of this theorem we use the following three lemmas.

LEMMA 1. *Let $\Gamma_1$ and $\Gamma_2$ be two consistent systems of axioms and let $\Gamma_i$ $(i=1,2)$ satisfy the equality axioms with regard to $\overset{i}{=}$. Moreover, we assume that none of special variables, functions and predicates is contained in $\Gamma_1$ and $\Gamma_2$ at the same time. Let $e_1(\,)$, $e_2(\,)$ be two predicates not contained in $\Gamma_1$ nor $\Gamma_2$. Then the following system of axioms $\tilde{\Gamma}$ is consistent.*

$\Gamma_1^{e_1(\,)}$

$\Gamma_2^{e_2(\,)}$

$e_1(s^1)$ *for every special variable $s^1$ contained in $\Gamma_1$.*

*$e_2(s^2)$ for every special variable $s^2$ contained in $\Gamma_2$.*

*$\forall x_1 \cdots \forall x_k e_1(f^1(x_1, \cdots, x_k))$ for every function $f^1$ contained in $\Gamma_1$.*

*$\forall x_1 \cdots \forall x_k e_2(f^2(x_1, \cdots, x_k))$ for every function $f^2$ contained in $\Gamma_2$.*

*$\forall x_1 \cdots \forall x_k (e_2(x_1) \vee \cdots \vee e_2(x_k) \vdash f^1(x_1, \cdots, x_k) = s^1{}_0)$ for every function $f^1$ contained in $\Gamma_1$, where $s^1{}_0$ is a fixed special variable contained in $\Gamma_1$.*

*$\forall x_1 \cdots \forall x_k (e_1(x_1) \vee \cdots \vee e_1(x_k) \vdash f^2(x_1, \cdots, x_k) = s^2{}_0)$ for every function $f^2$ contained in $\Gamma_2$, where $s^2{}_0$ is a fixed special variable contained in $\Gamma_0$.*

*$\forall x_1 \cdots \forall x_i (p^1(x_1, \cdots, x_i) \vdash e_1(x_1) \wedge \cdots \wedge e_1(x_i))$ for every predicate $p^1$ contained in $\Gamma_1$.*

*$\forall x_1 \cdots \forall x_i (p^2(x_1, \cdots, x_i) \vdash e_2(x_1) \wedge \cdots \wedge e_2(x_i))$ for every predicate $p^2$ contained in $\Gamma_2$.*

*$\forall x (e_1(x) \vee e_2(x))$*

*$\forall x \neg (e_1(x) \wedge e_2(x))$.*

LEMMA 2. *Under the same hypothesis as in Lemma 1, $\tilde{\Gamma}$ satisfies the equality axioms with regard to $=$, provided that $a = b$ is defined to be $(e_1(a) \wedge e_1(b) \wedge a \overset{1}{=} b) \vee (e_2(a) \wedge e_2(b) \wedge a \overset{2}{=} b)$.*

LEMMA 3. *$\Gamma_a$ is consistent in LK.*

We need not dwell upon the proof of these lemmas which are immediate. Our Theorem is deduced as follows.

In virtue of Lemma 3, $\Gamma_a$ can be used as $\Gamma_2$ in Lemma 1; we use $\Gamma_1$ in our Theorem as $\Gamma_1$ in Lemma 1. Then we can follow, in virtue of Lemmas 1, 2, the proof of Theorems 1, 2 in [1] in regarding $\Gamma_0$ in [1] as $\tilde{\Gamma}$ and $e(\ )$ in [1] as $e_2(\ )$. We obtain thus our Theorem, in considering $n(a)$ in [1] as $n(a)$ in our Theorem.

## References


[1] G. Takeuti: On Skolem's theorem. J. Math. Soc. Japan, 9 (1957).

[2] ———: On a generalized logic calculus. Jap. J. Math., 23 (1953), pp. 39–96; Errata to 'On a generalized logic calculus'. Jap. J. Math., 24 (1954), pp. 149–156.

[3] ———: A metamathematical theorem on functions. J. Math. Soc. Japan, 8 (1956) pp. 65–78.